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About one direct initial-boundary value problem for nonlinear one-dimensional poroelasticity equations

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Abstract. We consider a one-dimensional direct initial-boundary value problem for a nonlinear system of the poroelasticity equations. The theorem of local solvability of the classical solution to the problem is proved. The Frechet differentiability of the problem operator is proved, too.

Keywords: hyperbolic system, direct problem, Volterra's equation, porous medium, friction coefficient.

Let us consider the one-dimensional nonlinear system of equations of poroelasticity

$$\rho_s u_{tt} = (\mu(u_x)u_x)_x - \rho_l^2((u-v)\chi(u-v))_t, \quad x \in (0,L), \quad t \in (0,T), \tag{1}$$

$$\rho_l v_t = \rho_l^2(u-v)\chi(u-v), \quad x \in (0,L), \quad t \in (0,T), \tag{2}$$

with the initial conditions

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad v|_{t=0} = 0, \quad x \in (0,L), \tag{3}$$

and the boundary conditions

$$\mu(u_x)u_x|_{x=L} = f(t), \quad u|_{x=0} = 0, \quad t \in (0,T). \tag{4}$$

Here u and v are the velocities of elastic porous body with a constant partial density $\rho_s = \rho_s^f(1-d_0)$ and of the fluid with a constant partial density $\rho_l = \rho_l^f d_0$, respectively, d_0 is porosity, $u_t = \dfrac{\partial u}{\partial t}$, $f : [0,T] \to R$, $u_0 : [0,L] \to R$, $u_1 : [0,L] \to R$, ρ_s^f and ρ_l^f are the physical density of elastic porous body and the fluid, respectively, $\mu(\nu)$ is a three times continuously differentiable positive function, $\chi(\nu)$ is a two times continuously differentiable positive function.

The nonlinear wave equation of the form of (1) ($\chi \equiv 0$ in a reversible approximation) arises in many problems. For example, when the string vibrations for the elastic coefficient depend on deformation. Many mechanics models of porous media, taking into account energy dissipation, the friction coefficient (permeability) are the function of the velocity difference [1, 2].

Next, we are interested in the classical solution of the initial-boundary value problem (1)–(4), i.e., $u \in C^{2,2}([0,L]\times[0,T])$, $v \in C^{0,1}([0,L]\times[0,T])$, where $C^{k,m}([0,L]\times[0,T])$ is the space of k times continuously differentiable functions with respect to x, and m times continuously differentiable functions with respect to t.

In this paper, using the ideas of [3], we study the direct problem for one-dimensional dynamic system of equations of porous media.

The statement of the problem and formulation of results. The problem of the definition of u and v from (1)–(4) with given μ, χ, ρ_s, ρ_l will be called a one-dimensional direct dynamic problem for porous media. The one-dimensional inverse dynamic problem for porous media by the definition u, v, μ from (1)–(4) (with given χ, ρ_s, ρ_l) for additional information $\tilde{u} := u(L,\cdot)$ will be considered separately.

We introduce the functions $\tilde{\mu}(s) = s\mu(s)$, $\tilde{\chi}(s) = s\chi(s)$. To study the properties of our mathematical model, we consider the operator F, that maps the function $\tilde{\mu}$ on to the given $\tilde{u} := u(L,\cdot)$ which is a restriction of the solution u for the following initial boundary value problem

$$\begin{aligned} \rho_s u_{tt} &= (\tilde{\mu}(u_x))_x - \rho_l^2(\tilde{\chi}(u-v))_t, \\ v_t &= \rho_l\tilde{\chi}(u-v), \qquad x\in(0,L), \quad t\in(0,T), \end{aligned} \tag{5}$$

with the initial conditions

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad v|_{t=0} = 0, \quad x\in(0,L), \tag{6}$$

and the boundary conditions

$$\tilde{\mu}(u_x)|_{x=L} = f(t), \quad u|_{x=0} = 0, \quad t\in(0,T). \tag{7}$$

Then the function $\tilde{\mu}$ can be found from the solution of the operator equation

$$F(\tilde{\mu}) = \tilde{u}. \tag{8}$$

The derivative of the operator F in some direction $\delta\tilde{\mu}$ is calculated in the following

$$F'(\tilde{\mu})[\delta\tilde{\mu}] = \widehat{u}(L,\,\cdot\,), \tag{9}$$

where the functions $\widehat{u}$ and $\widehat{v}$ are the solution of the initial-boundary value problem

$$\begin{aligned} \rho_s\widehat{u}_{tt} &= (\tilde{\mu}'(u_x)\widehat{u}_x)_x - \rho_l^2(\tilde{\chi}'(u-v)(\widehat{u}-\widehat{v}))_t + (\delta\tilde{\mu}(u_x))_x, \\ \widehat{v}_t &= \rho_l\tilde{\chi}'(u-v)(\widehat{u}-\widehat{v}), \qquad x\in(0,L), \quad t\in(0,T), \end{aligned} \tag{10}$$

with the initial conditions

$$\widehat{u}|_{t=0} = 0, \quad \widehat{u}_t|_{t=0} = 0, \quad \widehat{v}|_{t=0} = 0, \quad x \in (0, L), \tag{11}$$

and the boundary conditions

$$\widehat{u}|_{x=0} = 0, \quad \tilde{\mu}'(u_x)\widehat{u}_x + \delta\tilde{\mu}(u_x)|_{x=L} = 0, \quad t \in (0, T). \tag{12}$$

In formulas (10)–(12) the functions u, v are the solution of the initial-boundary value problem (5)–(7).

Suppose that the following conditions are valid

$$u_0 \in C^3(0, L), \quad u_1 \in C^2(0, L), \quad f \in C^2(0, T), \tag{13}$$

and the compatibility conditions

$$\begin{gathered} (\tilde{\mu}^{-1} \circ f)(0) = u_0'(L), \quad (\tilde{\mu}^{-1} \circ f)'(0) = u_1'(L), \\ \rho_s(\tilde{\mu}^{-1} \circ f)''(0) = (\tilde{\mu}(u_0'))''(L) - \rho_l^2[(u_1 - \rho_l\tilde{\chi}(u_0))\tilde{\chi}'(u_0)]'(L) \end{gathered} \tag{14}$$

on the right boundary and

$$u_0(0) = u_1(0) = u_0''(0) = u_1''(0) = 0 \tag{15}$$

on the left boundary.

Assume that the functions $\tilde{\mu}$, $\tilde{\chi}$ belong to the set

$$D(F) = \left\{ \begin{array}{l} (\tilde{\mu}, \tilde{\chi}) \in X \mid \tilde{\mu}'(s) \geq \mu_0, \ \tilde{\mu}''(s) \leq C, \ \tilde{\chi}''(s) \leq C \\ \text{for any } s \in [0, S], \text{ and condition (14) is fulfilled} \end{array} \right\}, \tag{16}$$

for some positive constants μ_0, C. Further, we denote by C a positive constant that is greater than the previous C,

$$X = \bigl\{(\tilde{\mu}, \tilde{\chi}) \in C^3(0, S) \times C^2(0, S) \mid \tilde{\mu}(0) = 0, \ \tilde{\chi}(0) = 0\bigr\}, \tag{17}$$

where $S > 0$. Note that in applications we often set parameters to be strictly monotonously increasing and smooth functions. These conditions are satisfied in the domain of definition $D(F)$ and the space X, defined by above formulas (16), (17). The assumption of smoothness parameters in our model is also important for the effective solution of the original initial-boundary value problem. If $\tilde{\mu}$, $\tilde{\chi}$ are sufficiently smooth, then to determine the parameters of a medium, we can apply methods of Newton type with quadratic convergence.

Theorem. *Suppose that T is sufficiently small, S sufficiently large, condition* (13) *and $D(F)$ is defined by formula* (16).

Then, for any $(\tilde{\mu}, \tilde{\chi}) \in D(F)$, *the problem* (5)–(7) *has a unique solution* $u \in C^{3,2}([0, L] \times [0, T])$, $v \in C^{0,1}([0, L] \times [0, T])$. *Consequently, the operator of the direct problem*

$$F : D(F) \subseteq X \to C^2(0, T), \quad \tilde{\mu} \mapsto u(L, \cdot),$$

where u, v *is a solution of* (5)–(7), *is well defined. Furthermore, for*

$$X' := X \cap C^4(0, s) \times C^4(0, s), \quad D'(F) := D(F) \cap C^4(0, s) \times C^4(0, s),$$

$$F : D'(F) \subseteq X' \to C^2(0, T)$$

is continuously differentiable according to Frechet, and derivatives are calculated by formulas (9)–(12).

Remark. A similar result holds for the homogeneous Neumann conditions instead of the Dirichlet boundary conditions on the left boundary $x = 0$, which corresponds to the case free from tension on the left boundary. In the proof of correctness (see below) of the boundary conditions for $w = u_x$, $\tilde{w} = v_x$ we have

$$\tilde{\mu}(w)|_{x=L} = f(t), \quad w|_{x=0} = 0, \quad t \in (0, T).$$

Here the functions u, v are defined in terms of the functions w, $\tilde{w}$ by the formulas

$$\begin{aligned} u(x,t) = & \int_0^x w(\xi, t)\, d\xi + \frac{1}{\rho_s} \int_0^t \int_0^\tau (\tilde{\mu}(w))_x(0, \eta)\, d\eta\, d\tau + \\ & \rho_l^2 \frac{\rho_l}{\rho_s} \int_0^t \int_0^\tau [\tilde{\chi}(u - v)\tilde{\chi}'(u - v)](0, \eta)\, d\eta\, d\tau, \\ v(x,t) = & \int_0^x \tilde{w}(\xi, t)\, d\xi + \rho_l \int_0^t (\tilde{\chi}(u - v))(0, \eta)\, d\eta. \end{aligned}$$

This system is at fixed x closed system of the nonlinear Volterra integral equations of the second kind with respect to t.

Proof. Let us introduce the functions $\breve{u} = u_x$, $\breve{v} = v_x$. We differentiate both sides of system (5) with respect to x and relative to $\breve{u}$, $\breve{v}$ obtain the system of equations

$$\begin{aligned} \rho_s \breve{u}_{tt} = & (\tilde{\mu}'(\breve{u})\breve{u}_x)_x - \rho_l^2 \tilde{\chi}'(u - v)\breve{u}_t - \\ & \rho_l^2 \{[\tilde{\chi}'(u - v)]_t - \rho_l [\tilde{\chi}'(u - v)]^2\}(\breve{u} - \breve{v}), \end{aligned} \tag{18}$$

$$\breve{v}_t = \rho_l \tilde{\chi}'(u - v)(\breve{u} - \breve{v}), \qquad x \in (0, L), \quad t \in (0, T), \tag{19}$$

Suppose that on the left boundary, given the homogeneous Neumann condition (see the Remark) and using condition (4), we obtain

$$\tilde{\mu}'(\breve{u})\breve{u}_x|_{x=0} = 0, \quad \tilde{\mu}(\breve{u})|_{x=L} = f(t), \quad t \in (0,T),$$

or

$$\breve{u}_x|_{x=0} = 0, \quad \breve{u}|_{x=L} = \tilde{\mu}^{-1}(f(t)), \quad t \in (0,T). \tag{20}$$

The initial conditions are of the form

$$\breve{u}|_{t=0} = u_{0x}(x), \quad \breve{u}'_t|_{t=0} = u_{1x}(x), \quad \breve{v}|_{t=0} = 0, \quad x \in (0,L). \tag{21}$$

To show the existence of solutions of the nonlinear initial-boundary value problem, we reduce equation (18) to a standard form with a smooth transformation of variables [4], using the characteristic curves

$$\frac{dx(t)}{dt} = \pm\sqrt{\tilde{\mu}'(\breve{u})(x(t),t)/\rho_s}.$$

We introduce a new function [4]

$$U(\varphi(x,t)+\psi(x,t), \varphi(x,t)-\psi(x,t)) = \widehat{u}(x,t), \tag{22}$$

where $\varphi(x,t)$, $\psi(x,t)$ satisfy the system

$$\varphi_t + \sqrt{\tilde{\mu}'(u_x)/\rho_s}\varphi_x = 0, \quad \psi_t - \sqrt{\tilde{\mu}'(u_x)/\rho_s}\psi_x = 0. \tag{23}$$

After simple transformations with respect to $\breve{v}$ and U, equations (18) and (19) take the form

$$\breve{v}_t = \rho_l \tilde{\chi}'(u-v)(U-\widehat{v}), \tag{24}$$

$$U_{\zeta\zeta} - U_{\eta\eta} = \frac{1}{8}\left(\frac{\tilde{b}_x}{\tilde{b}} + \frac{\tilde{b}_t}{\tilde{b}\sqrt{\tilde{b}}} + \frac{\rho_l^2}{\rho_s}\frac{\chi_1}{\sqrt{\tilde{b}}}\right)\frac{1}{\psi_x}(U_\eta + U_\zeta) +$$
$$\frac{1}{8}\left(\frac{\tilde{b}_x}{\tilde{b}} - \frac{\tilde{b}_t}{\tilde{b}\sqrt{\tilde{b}}} - \frac{\rho_l^2}{\rho_s}\frac{\chi_1}{\sqrt{\tilde{b}}}\right)\frac{1}{\varphi_x}(U_\eta - U_\zeta) + \frac{\rho_l^2}{\rho_s}\frac{\chi_2}{4\tilde{b}\varphi_x\psi_x}(U-\breve{v}). \tag{25}$$

where $\tilde{b} = \tilde{\mu}'(\breve{u})/\rho_s$, $\chi_1 = \tilde{\chi}'(U-v)$, $\chi_2 = [\tilde{\chi}'(u-v)]_t - \rho_l[\tilde{\chi}'(u-v)]^2$.

To show that this is a regular transformation of the variables and it is from the class C^2 on a sufficiently small interval $(0,\bar{t})$, provided $\widehat{u} \in C^2$, $\widehat{v} \in C^1$, consider the determinant of the Jacobian matrix

$$\det\begin{pmatrix} \varphi_x+\psi_x & \varphi_t+\psi_t \\ \varphi_x-\psi_x & \varphi_t-\psi_t \end{pmatrix} = \det\begin{pmatrix} \varphi_x+\psi_x & -\sqrt{\tilde{\mu}'(\breve{u})/\rho_s}(\varphi_x-\psi_x) \\ \varphi_x-\psi_x & -\sqrt{\tilde{\mu}'(\breve{u})/\rho_s}(\varphi_x+\psi_x) \end{pmatrix}$$
$$= -4\sqrt{\tilde{\mu}'(\breve{u})/\rho_s}\varphi_x\,\psi_x. \tag{26}$$

We show that the determinant of the Jacobian matrix is different from zero, i.e. $\psi_x \neq 0$ and $\varphi_x \neq 0$.

From characteristics of the ordinary differential equation (23) for ψ

$$
\begin{aligned}
&t_\tau(\tau,\xi) = 1, && t(0,\xi) = \xi, \\
&x_\tau(\tau,\xi) = -\sqrt{\tilde{\mu}'(\breve{u}(x(\tau,\xi),t(\tau,\xi))/\rho_s}, && x(0,\xi) = 0, \\
&\psi_\tau(\tau,\xi) = 0, && \psi(0,\xi) = \xi,
\end{aligned} \tag{27}
$$

we obtain

$$
t(\tau,\xi) = \tau + \xi, \quad \psi(\tau,\xi) = \xi, \tag{28}
$$

for $\tau \geq 0$, $\xi = t(0,\xi) \geq 0$.

After differentiating the second relation of (28), we obtain

$$
1 = \psi_\xi = \psi_x x_\xi + \psi_t t_\xi = \psi_x \cdot \Big(x_\xi + \sqrt{\tilde{\mu}'(\breve{u})/\rho_s}\Big). \tag{29}
$$

Hence, we obtain $\psi_x \neq 0$. Similarly we can prove that $\varphi_x \neq 0$. Consequently, the determinant of the Jacobian matrix is nonzero. On the other hand, the limit of $|\psi_x|$ is performed as long as

$$
x_\xi(\tau,\xi) \neq -\sqrt{\tilde{\mu}'(\breve{u}(x(\tau,\xi),t(\tau,\tau+\xi))/\rho_s}.
$$

It is true for all t, that $\tau = t - \xi$ is less than $\tilde{t} > 0$, which can depend only on $\|\breve{u}\|_{C^1}$.

We differentiate the second relation of (27) with respect to ξ. Relative to x_ξ we obtain the ordinary differential equation of the first order with zero Cauchy data. Let us reduce this problem to the solution of the integral equation. Using obvious estimates and Gronwall's inequality, we obtain the inequality

$$
|x_\xi(\tau,\xi)| \leq e^{\frac{C\|u\|_{C^2}}{2\sqrt{c\rho_s}}} - 1 < \sqrt{c/\rho_s} \leq \sqrt{\tilde{\mu}'(\breve{u}(x(\tau,\xi),\tau+\xi))/\rho_s}, \tag{30}
$$

which is valid for

$$
\tau \leq \tilde{t} = \frac{2\sqrt{c\rho_s}\ln(\sqrt{c/\rho_s}+1)}{C\|\breve{u}\|_{C^1}}.
$$

Similarly, we can prove the boundedness of all derivatives of the functions φ and ψ up to the second order [3].

According to the theorem about the inverse function, the existence of the solution $\breve{u} \in C^{2,2}$, $\breve{v} \in C^{0,1}$ of problem (18)–(21) follows from the existence of the solution $U \in C^{2,2}$, $\breve{v} \in C^{0,1}$ of problem (24), (25) transformed with initial and boundary conditions. To prove the existence of solutions of system (24) and (25) we use the Banach theorem [5]. Namely, we define a fixed point of the operator $M = (M_1, M_2)$, mapping the functions $u \in C^{2,2}$, $\breve{v} \in C^{0,1}$ onto the solutions $M_1(U,\breve{v}) = Y$, $M_2(U,\breve{v}) = y$ from

$$Y_{\zeta\zeta} - Y_{\eta\eta} = \frac{1}{8}\left(\frac{\tilde{b}_x}{\tilde{b}} + \frac{\tilde{b}_t}{\tilde{b}\sqrt{\tilde{b}}} + \frac{\rho_l^2}{\rho_s}\frac{\chi_1}{\sqrt{\tilde{b}}}\right)\frac{1}{\psi_x}(U_\eta + U_\zeta) +$$

$$\frac{1}{8}\left(\frac{\tilde{b}_x}{\tilde{b}} - \frac{\tilde{b}_t}{\tilde{b}\sqrt{\tilde{b}}} - \frac{\rho_l^2}{\rho_s}\frac{\chi_1}{\sqrt{\tilde{b}}}\right)\frac{1}{\varphi_x}(U_\eta - U_\zeta) + \frac{\rho_l^2}{\rho_s}\frac{\chi_2}{4\tilde{b}\varphi_x\psi_x}(U - \breve{v}), \tag{31}$$

$$y_t = \rho_l\tilde{\chi}'(u - v)(U - \breve{v}), \tag{32}$$

transformed with initial and boundary conditions. Note that the right-hand sides depend on U, $\widehat{v}$ not only linearly relative to $U_\eta + U_\zeta$, $U_\eta - U_\zeta$, but also nonlinear through $\tilde{b} = \tilde{\mu}'(\breve{u})/\rho_s$, $\chi_1 = \tilde{\chi}'(U - v)$ and φ, ψ. The operator M is a contraction for small values $\bar{t}$ $(0 < t \leq \bar{t})$ due to the limited right-hand sides of (31), (32) in the norm $\|U\|_{C^{2,2}}$, $\|\breve{v}\|_{C^{0,1}}$. This implies the existence of the solution $u \in C^{3,2}$, $v \in C^{1,1}$ nonlinear initial-boundary value problem (1)–(4) and, consequently, according to

$$u(x,t) = \int_0^x \breve{u}(\xi,t)\,d\xi, \quad v(x,t) = \int_0^x \breve{v}(\xi,t)\,d\xi + \rho_l\int_0^t [\tilde{\chi}(u-v)](0,\tau)\,d\tau,$$

there is a solution $\breve{u} \in C^{2,2}$, $\breve{v} \in C^{0,1}$ to the initial-boundary value problem (18)–(21). The uniqueness is proved in a standard way [3], applying Theorem 4 from [4] for the system

$$\rho_s\hat{u}_{tt} = \left(\int_0^1 \tilde{\mu}(u_x^2 + (u_x^1 - u_x^2)\theta)\,d\theta\,\widehat{u}_x\right)_x -$$

$$\rho_l^2\left(\int_0^1 \tilde{\chi}(u^2 - v^2 + (u^1 - v^1 - u^2 + v^2)\theta)\,d\theta\right)_t,$$

$$\hat{v}_t = \rho_l\int_0^1 \tilde{\chi}(u^2 - v^2 + (u^1 - v^1 - u^2 + v^2)\theta)d\theta,$$

with homogeneous initial and boundary conditions for a difference of solutions u^1, v^1 and u^2, v^2.

Similarly, we can show the continuous dependence of the solution $u \in C^{2,2}$, $v \in C^{0,1}$ on the parameter $\tilde{\mu} \in C^3$.

To prove the Frechet differentiability of the operator of the direct problem F let us note that $F(\tilde{\mu}+\delta\tilde{\mu}) = \bar{u}(L,\cdot)$, where the functions $\bar{u}, \bar{v}$ is the solution of the initial-boundary value problem (1)-(4) with $\tilde{\mu} = \tilde{\mu} + \delta\tilde{\mu}$, the functions $w := \bar{u} - u - u$, $\omega := \bar{v} - v - v$ are the solution of the initial-boundary value problem

$$\rho_s w_{tt} = (\tilde{\mu}'(u_x)w_x)_x - \rho_l^2\tilde{\chi}'(u-v)w_t + G_x -$$

$$\rho_l^2[[\tilde{\chi}'(u-v)]_t - \rho_l[\tilde{\chi}'(u-v)]^2](w - \omega),$$

$$\omega_t = \rho_l\tilde{\chi}'(u-v)(w - \omega),$$

with zero initial conditions and the boundary conditions

$$\tilde{\mu}'(u_x)w_x|_{x=L} = G(L,t), \quad w|_{x=0} = 0, \quad t \in (0,T),$$

where

$$G = \tilde{\mu}(\bar{u}_x) - \tilde{\mu}(u_x) - \tilde{\mu}'(u_x)(\bar{u}_x - u_x) + \delta\tilde{\mu}(\bar{u}_x) - \delta\tilde{\mu}(u_x).$$

Therefore, applying the conversion characteristic of the class C^2, known results for the wave equation and Gronwall's inequality, we find that the norm $\|w\|_{C^2}$ can be estimated through $\|\delta\mu\|_{C^3}$ [3]. This means the continuity of F.

Similarly, we prove the continuity of the Frechet derivative of the operator F. The theorem is proved. □

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