Bull. Nov. Comp. Center, Math. Model. in Geoph., 19 (2016), 67–82

The effective coefficients for 2D elastic equations with multiscale isotropic random mass density and elastic parameters

O.N. Soboleva

Abstract. The propagation of elastic waves in strongly heterogeneous media using subgrid modeling approach is studied. The local elastic parameters and the mass density have essential variations in some interval of scales at each spatial point. To approximate a strongly heterogeneous medium, we have started from the modified Kolmogorov theory in terms of the ratios of smoothed fields. The correlated fields of the elastic stiffness and of the mass density have been represented mathematically by the Kolmogorov multiplicative cascades. The wave propagates over a distance that is of the same order as the typical wavelength of a source. The 2D averaged elastic equations are obtained if the wavelength is large as compared with a maximum scale of the medium heterogeneities. If a medium is assumed to satisfy the improved Kolmogorov similarity hypothesis, the expression for the effective elastic parameters is especially simple. It has been shown that small-scale heterogeneities affect the wave propagation.

1. Introduction

The study of wave propagation in heterogeneous media is not only of fundamental scientific interest, but also is of significant practical importance. It is relevant to such important problems as detecting underground nuclear explosions, understanding the large-scale structure of oil, gas, and geothermal reservoirs, gaining insight into what happens at large depths in the oceans [1]. In order to compute the wave propagation in an arbitrary medium, one must numerically solve elastic equations. The large-scale variations of coefficients as compared with the wavelength are taken into account in these models with the help of some boundary conditions. The numerical solution to the problem with variations of parameters at all the scales requires high computational costs. The small-scale heterogeneities are taken into account with the help of effective parameters or additional terms in wave equations like the Frenkel–Biot models [2].

The methods of homogenization or asymptotic methods are often applied to the elasticity equations for such a problem. There are three different wave propagation regimes (waves in a smoothly varying body, coda waves and a homogenized part of the wave field) depending on the ratio of the wave field characteristic scale to the one of the heterogeneities. It was very difficult to find a clear spatial scale delimitation (to apply a homogenization procedure), to catch wave field properties in each of these regimes.

The two-scale homogenization approaches are well known in the solid mechanics community. An example of the two-scale approach for the dynamic case can be found in [3, 4].

In self-consistent methods, complex actual wave fields propagating in heterogeneous media are approximated by simple ones using physically reasonable hypotheses. All the self-consistent methods are based on the two types of such hypotheses. The first one reduces the problem of interactions between many inclusions in the heterogeneous media to a problem for one inclusion (the one-particle problem). The second hypothesis is the condition of self-consistency. For application of these methods, the heterogenous medium should have specific features: a typical element (particle) should exist in the medium. Such a particle may be an inclusion in the matrix-inclusion composites, a grain in random polycrystalline materials, a crack in materials with defects, etc. Application of self-consistent methods to the solving the wave propagation problem is considered in [5]. The effective elastic, electric, dielectric, thermo-conductive and other properties of composite materials reinforced by ellipsoidal and spherical multi-layered inclusions, thin hard and soft inclusions, short fibers and uni-directed multi-layered fibers are discussed in Ch. 6, Vol. 2.

For most cases, when effective medium theory becomes ineffective, many researchers consider a way to address numerical upscaling methods. Various numerical upscaling techniques are applied to geophysical problems (see, for example [6, 7]). Considerable progress has been reached in this direction allowing the calculation of upscaled properties at the expense of a certain amount of CPU-time. In this case, "upscaling" refers to the techniques used to transform a fine-grid model to an applied, coarser one. But solving the equations calls for boundary conditions. Subsequently, arbitrary boundary conditions are selected. They are often uniform or periodic. Clearly, the computed equivalent coefficients can depend on the boundary conditions considered. If the spatial position of the small scale heterogeneities is exactly known, these methods are successfully applied.

Very often the spatial position of small-scale heterogeneities are not exactly known. It is customary to assume these parameters to be random fields. However, it is difficult to measure higher-order statistical moments for the geophysical parameters. At best, only the mean values and the second-order correlation functions are known. Hence, averaged or effective solutions cannot be constructed using only the conventional perturbation theory with a high accuracy. In [8, Ch. 8], some problems of multiple scattering of waves are considered. The problem is solved for the Helmholtz equation in an infinite medium with log-normal probability distribution of the coefficients. The generalized singular approximation method for the elastic modulus evaluation of polycrystals is considered in [9].

The wave propagation in randomly layered media has been studied extensively in [10]. In this book, the authors obtain effective coefficients for layered media with parameters that are rapidly varying around a constant value or piecewise constant values and weakly and strongly heterogeneous white-noise values.

The propagation of elastic waves in strongly heterogeneous elastic media is also studied using the re-normalization group (RG) analysis [11, 12]. In these papers, the authors study the propagation and localization of acoustic and elastic waves in strongly heterogeneous media, which are characterized by a broad distribution of the elastic constants, using the Martin–Siggia–Rose method. The Gaussian-white distributed elastic constants, as well as those with long-range correlations with non-decaying power law correlation functions, are considered. The authors have investigated how the heterogeneities having the scale comparable with the wavelength affect the elastic wave propagation in disordered media.

It has been shown that the irregularity of elastic parameters, density, permeability, porosity, increases as the scale of measurements decreases for some natural media [1, 13]. Many natural media are "scale regular" in the sense that they can be described by fractals and hierarchical cascade models with non-Gaussian distributions [13, 14]. In the present paper, based on this fact we apply the subgrid modeling method to hierarchical cascade models of media with non-Gaussian distributions of parameters. As the first step toward the eventual goal of finding the effective coefficients in the problem of propagation of elastic waves in strongly heterogeneous media, we study the propagation of elastic waves in the media, in which the heterogeneities, represented by the spatial distribution of the local parameters having essential variations of all the scales from a certain interval at each spatial point. We will obtain averaged elastic equations if the wavelength essentially exceeds a maximum scale of heterogeneity (the right boundary of the scales interval). If a propagation distance is of order of the wavelength, these equations are the effective equations because random scattering is weak, and, there is no backscattering [10]. The waves cannot efficiently probe the small scales. At these scales the fluctuations of a medium tend to be averaged by the low sensitivity of the wave [10]. The density of a medium and its elastic stiffness are approximated by a multiplicative cascade with the log-normal joint probability distribution functions. If a medium is assumed to satisfy the improved Kolmogorov similarity hypothesis [15], the effective coefficients take an especially simple form.

2. Statement of the problem

The elastic wave propagation in a heterogeneous medium is described by the equation:

$$\rho\frac{\partial^2 u_{x_1}}{\partial^2 t}-\frac{\partial}{\partial x_1}\left((\lambda+2\mu)\frac{\partial u_{x_1}}{\partial x_1}+\lambda\frac{\partial u_{x_2}}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(\mu\frac{\partial u_{x_1}}{\partial x_2}+\mu\frac{\partial u_{x_2}}{\partial x_1}\right)=f_{x_1},$$
$$\rho\frac{\partial^2 u_{x_2}}{\partial^2 t}-\frac{\partial}{\partial x_1}\left(\mu\frac{\partial u_{x_1}}{\partial x_2}+\mu\frac{\partial u_{x_2}}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(\lambda\frac{\partial u_{x_1}}{\partial x_1}+(\lambda\ +2\mu)\frac{\partial u_{x_2}}{\partial x_2}\right)=f_{x_2}, \tag{1}$$

where t is the time, $\boldsymbol{x}$ is the vector of spatial coordinates; $u_{x_1}(\boldsymbol{x},t)$ and $u_{x_2}(\boldsymbol{x},t)$ are the displacements along the axes x_1, x_2, respectively; $\rho(\boldsymbol{x})$ is the density, $\lambda(\boldsymbol{x})$, $\mu(\boldsymbol{x})$ are the elastic parameters of a medium; $f_{x_1}(\boldsymbol{x},t)$, $f_{x_2}(\boldsymbol{x},t)$ are the components of a source with the dominant frequency ω_0. The wavelength is assumed to be large as compared with the maximum scale of heterogeneities L. As the first step toward the eventual goal of finding the effective coefficients in the problem of elastic waves propagation in strongly heterogeneous media we study the elastic waves propagation, when $\lambda(\boldsymbol{x})$=$\mu(\boldsymbol{x})$. For the approximation of the coefficients $\rho(\boldsymbol{x})$, $\lambda(\boldsymbol{x})$ we use the approach described in [16].

Let, for example, the field $\mu(\boldsymbol{x})$ be known. This means that the field is measured on a small scale l_0 at each point $\boldsymbol{x}$, $\mu_{l_0}(\boldsymbol{x})=\mu(\boldsymbol{x})$. Following Kolmogorov [15], let us consider a dimensionless field ψ, which is equal to the ratio of the two fields obtained by smoothing the field $\mu_{l_0}(\boldsymbol{x})$ at two different scales l,l'. Let $\mu_l(\boldsymbol{x})$ denote the parameter $\mu_{l_0}(\boldsymbol{x})$ smoothed at the scale l. Then $\psi(\boldsymbol{x},l,l')=\mu_{l'}(\boldsymbol{x})/\mu_l(\boldsymbol{x})$, $l'<l$. Expanding the field ψ to a power series in $(l-l')$ and retaining first order terms of the series, at $l'\to l$, we obtain the equation

$$\frac{\partial\ln\mu_l(\boldsymbol{x})}{\partial\ln l}=\varphi(\boldsymbol{x},l), \tag{2}$$

where $\varphi(\boldsymbol{x},l')=\left.\frac{\partial\psi(\boldsymbol{x},l',l'y)}{\partial y}\right|_{y=1}$. The small scale fluctuations of the field φ are observed only in the interval (l_0,L). The solution of equation (2) is

$$\mu_{l_0}(\boldsymbol{x})=\mu_0\exp\left(-\int_{l_0}^{L}\varphi(\boldsymbol{x},l_1)\frac{dl_1}{l_1}\right), \tag{3}$$

where μ_0 is a constant. The field φ determines the statistical properties of the elastic stiffness. According to the central limit theorem for sums of independent random variables [17] if the variance of $\varphi(\boldsymbol{x},l)$ is finite, the integral in (3) tends to a field with a normal distribution as the ratio L/l_0 increases. If the variance of $\varphi(\boldsymbol{x},l)$ is infinite and there exists a non-degenerate limit of the integral in (3), the integral tends to a field with a stable distribution.

In this paper, we assume that the field $\varphi(\boldsymbol{x}, l)$ is statistically homogeneous with normal distribution. The density coefficient $\rho(\boldsymbol{x})$ is constructed by analogy with the elastic stiffness coefficient:

$$\rho_{l_0}(\boldsymbol{x}) = \rho_0 \exp\Big(-\int_{l_0}^{L} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big). \tag{4}$$

The function $\chi(\boldsymbol{x}, l)$ is assumed to have the normal distribution and to be statistically homogeneous. For such a field as the density, the cascade model must be conservative, i.e. the following equality should be satisfied

$$\langle \rho_l(\boldsymbol{x})\rangle = \rho_0, \tag{5}$$

for any scale l, where $\langle\cdot\rangle$ means statistical averaging. Condition (5) follows from the physical essence of the field ρ. The average density is equal to the arithmetical mean of the densities in the volumes l^3, so the average density in the volume does not depend on the scale of measurements. The fluctuations of the fields $\varphi(\boldsymbol{x}, l)$, $\chi(\boldsymbol{x}, l)$ are considered to be statistically independent over the scales and isotropic:

$$\begin{aligned}
\langle \varphi(\boldsymbol{x}, l)\varphi(\boldsymbol{y}, l')\rangle - \langle \varphi(\boldsymbol{x}, l)\rangle\langle \varphi(\boldsymbol{y}, l')\rangle &= \Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l')\delta(\ln l - \ln l'),\\
\langle \chi(\boldsymbol{x}, l)\chi(\boldsymbol{y}, l')\rangle - \langle \chi(\boldsymbol{x}, l)\rangle\langle \chi(\boldsymbol{y}, l')\rangle &= \Phi^{\chi\chi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l')\delta(\ln l - \ln l'),\\
\langle \varphi(\boldsymbol{x}, l)\chi(\boldsymbol{y}, l')\rangle - \langle \varphi(\boldsymbol{x}, l)\rangle\langle \chi(\boldsymbol{y}, l')\rangle &= \Phi^{\varphi\chi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l')\delta(\ln l - \ln l').
\end{aligned} \tag{6}$$

This supposition is usually assumed in the scaling models and reflects the decay of statistical dependence when the scales of fluctuations become different in the order of magnitude. The latter was proposed in [16]. To derive equations for the averaged displacements, this assumption may be ignored. However, this assumption is important for the numerical simulation of the field ρ, μ. If a minimum scale l_0 in formulas (3), (4) tends to zero, the parameters in the equations tend to continuous multifractals. Hence, the parameters in the equations are described by extremely irregular fields that are close to continuous multifractals. If the fields are statistically invariant to the scale transform, the following equality is valid for any positive K:

$$\begin{aligned}
\Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{y}|, l) &= \Phi^{\varphi\varphi}(K|\boldsymbol{x}-\boldsymbol{y}|, Kl),\\
\Phi^{\chi\chi}(|\boldsymbol{x}-\boldsymbol{y}|, l) &= \Phi^{\chi\chi}(K|\boldsymbol{x}-\boldsymbol{y}|, Kl).
\end{aligned}$$

For simplicity we use the same notation Φ in the right-hand side. Choosing $K = 1/l$, we obtain

$$\Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{y}|, l) = \Phi^{\varphi\varphi}\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{l}\Big), \quad \Phi^{\chi\chi}(|\boldsymbol{x}-\boldsymbol{y}|, l) = \Phi^{\chi\chi}\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{l}\Big),$$

when $\boldsymbol{x} = \boldsymbol{y}$ the functions $\Phi^{\varphi\varphi}$, $\Phi^{\chi\chi}$ are equal to constants $\Phi_0^{\varphi\varphi}$, $\Phi_0^{\chi\chi}$. If condition (5) is satisfied in a scale-invariant medium, then $\Phi_0^{\chi\chi} = 2\langle\chi\rangle$. In a scale-invariant medium, the correlation function does not depend on the scale at $\boldsymbol{x} = \boldsymbol{y}$, and the following estimation is obtained [16]:

$$l_0 < l_\varepsilon < r < L, \quad \langle \mu_{l_0}(\boldsymbol{x})\mu_{l_0}(\boldsymbol{x}+\boldsymbol{r})\rangle \sim C\Big(\frac{r}{L}\Big)^{-\Phi_0^{\varphi\varphi}}, \tag{7}$$

where $C = \mu_0^2 e^{-\Phi_0^{\varphi\varphi}\gamma/2}$, γ is the Euler constant. For $r \gg L$, we have

$$\langle \mu_{l_0}(\boldsymbol{x})\mu_{l_0}(\boldsymbol{x}+\boldsymbol{r})\rangle \to \mu_0^2. \tag{8}$$

As the minimum scale l_0 tends to zero, the field μ, described by (3), becomes a multifractal. We obtain an irregular field on a Cantor-type set to be nonzero. The correlation between ρ and μ is determined by the correlation of the fields $\chi(\boldsymbol{x}, l')$ and $\varphi(\boldsymbol{x}, l')$:

$$\langle \varphi(\boldsymbol{x}, l)\chi(\boldsymbol{y}, l')\rangle - \langle \varphi(\boldsymbol{x}, l)\rangle\langle \chi(\boldsymbol{y}, l')\rangle = \Phi^{\varphi\chi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l')\delta(\ln l - \ln l').$$

3. A subgrid model

The density and the elastic stiffness $\rho(\boldsymbol{x}) = \rho_{l_0}(\boldsymbol{x})$ and $\mu(\boldsymbol{x}) = \mu_{l_0}(\boldsymbol{x})$, respectively, are divided into two components with respect to the scale l. The large-scale (ongrid) components $\mu(\boldsymbol{x}, l)$, $\rho(\boldsymbol{x}, l)$ are obtained, respectively, by statistical averaging over all $\varphi(\boldsymbol{x}, l_1)$ and $\chi(\boldsymbol{x}, l_1)$ with $l_0 < l_1 < l$, $l - l_0 = \Delta l$, where Δl is small. The small-scale (subgrid) components are equal to $\rho'(\boldsymbol{x}) = \rho(\boldsymbol{x}) - \rho(\boldsymbol{x}, l)$, $\mu'(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \mu(x, l)$. Applying (3)–(5) yields the formulas:

$$\begin{aligned}
\rho(\boldsymbol{x}, l) &= \rho_0 \exp\Big(-\int_l^L \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big),\\
\rho'(\boldsymbol{x}) &= \rho(\boldsymbol{x}, l)\Big[\exp\Big(-\int_{l_0}^l \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big) - 1\Big], \quad \langle \rho'(\boldsymbol{x})\rangle = 0,\\
\mu(\boldsymbol{x}, l) &= \mu_0 \exp\Big(-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\langle \exp\Big(-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\rangle,\\
\mu'(\boldsymbol{x}) &= \mu(\boldsymbol{x}, l)\left[\frac{\exp\Big(-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)}{\Big\langle \exp\Big(-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}]\Big)\Big\rangle} - 1\right], \quad \langle \mu'(\boldsymbol{x})\rangle = 0.
\end{aligned} \tag{9}$$

From formulas (9) with the second order of accuracy in $\Delta l/l$ follows

$$\begin{aligned}
\rho(\boldsymbol{x}, l) = \rho_l(\boldsymbol{x}), \quad \mu(\boldsymbol{x}, l) &\simeq \Big[1 - \langle\varphi\rangle\frac{\Delta l}{l} + \frac{1}{2}\Phi^{\varphi\varphi}(0, l)\frac{\Delta l}{l}\Big]\mu_l(\boldsymbol{x}),\\
\langle \mu'(\boldsymbol{x})\mu'(\boldsymbol{x}')\rangle &\simeq \Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{x}'|, l)\mu(\boldsymbol{x}, l)^2\frac{\Delta l}{l},\\
\langle \rho'(\boldsymbol{x})\rho'(\boldsymbol{x}')\rangle &\simeq \Phi^{\chi\chi}(|\boldsymbol{x}-\boldsymbol{x}'|, l)\rho(\boldsymbol{x}, l)^2\frac{\Delta l}{l},\\
\langle \rho'(\boldsymbol{x})\mu'(\boldsymbol{x}')\rangle &\simeq \Phi^{\chi\varphi}(|\boldsymbol{x}-\boldsymbol{x}'|, l)\rho(\boldsymbol{x}, l)\mu(\boldsymbol{x}, l)\frac{\Delta l}{l}.
\end{aligned} \tag{10}$$

Consider the temporal Fourier transform of equations (1) in which the large-scale of μ, ρ are fixed and the small components μ', ρ' are random variables. The subgrid components are equal to $u'_{x_i}(\boldsymbol{x}) = u_{x_i}(\omega, \boldsymbol{x}) - u_{x_i}(\omega, \boldsymbol{x}, l)$:

$$\begin{aligned}
-\omega^2 \rho u_{x_1} + \frac{\partial}{\partial x_1}\left(3\mu \frac{\partial u_{x_1}}{\partial x_1} + \mu \frac{\partial u_{x_2}}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial u_{x_1}}{\partial x_2} + \mu \frac{\partial u_{x_2}}{\partial x_1}\right) &= -f_{x_1}, \\
-\omega^2 \rho u_{x_2} + \frac{\partial}{\partial x_1}\left(\mu \frac{\partial u_{x_1}}{\partial x_2} + \mu \frac{\partial u_{x_2}}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial u_{x_1}}{\partial x_1} + 3\mu \frac{\partial u_{x_2}}{\partial x_2}\right) &= -f_{x_2}
\end{aligned} \tag{11}$$

with transformed $f_{x_1}(\omega, \boldsymbol{x})$, $f_{x_2}(\omega, \boldsymbol{x})$. The large-scale (ongrid) components of the displacements $u(\omega, \boldsymbol{x}, l)$ are obtained by averaging the solutions to equations (11):

$$\begin{aligned}
&\omega^2 \rho(\boldsymbol{x}, l) u_{x_1}(\omega, \boldsymbol{x}, l) + \frac{\partial}{\partial x_1}\left(3\mu(\boldsymbol{x}, l) \frac{\partial u_{x_1}(\omega, \boldsymbol{x}, l)}{\partial x_1} + \mu(\boldsymbol{x}, l) \frac{\partial u_{x_2}(\omega, \boldsymbol{x}, l)}{\partial x_2}\right) + \\
&\quad \frac{\partial}{\partial x_2}\left(\mu(\boldsymbol{x}, l) \frac{\partial u_{x_1}(\omega, \boldsymbol{x}, l)}{\partial x_2} + \mu(\boldsymbol{x}, l) \frac{\partial u_{x_2}(\omega, \boldsymbol{x}, l)}{\partial x_1}\right) + \\
&\qquad \omega^2 \langle \rho'(\boldsymbol{x}) u'_{x_1}(\boldsymbol{x}) \rangle + \frac{\partial}{\partial x_1}\left\langle 3\mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1} + \mu'(\boldsymbol{x}) \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\right\rangle + \\
&\qquad\quad \frac{\partial}{\partial x_2}\left\langle \mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_2} + \mu(\boldsymbol{x}') \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_1}\right\rangle = -f_{x_1}(\omega, \boldsymbol{x}), \qquad (12) \\
&\omega^2 \rho(\boldsymbol{x}, l) u_{x_2}(\omega, \boldsymbol{x}, l) + \frac{\partial}{\partial x_1}\left(\mu(\boldsymbol{x}, l) \frac{\partial u_{x_1}(\omega, \boldsymbol{x}, l)}{\partial x_2} + \mu(\boldsymbol{x}, l) \frac{\partial u_{x_2}(\omega, \boldsymbol{x}, l)}{\partial x_1}\right) + \\
&\quad \frac{\partial}{\partial x_2}\left(\mu(\boldsymbol{x}, l) \frac{\partial u_{x_1}(\omega, \boldsymbol{x}, l)}{\partial x_1} + 3\mu(\boldsymbol{x}, l) \frac{\partial u_{x_2}(\omega, \boldsymbol{x}, l)}{\partial x_2}\right) + \\
&\qquad \omega^2 \langle \rho'(\boldsymbol{x}) u'_{x_2}(\boldsymbol{x}) \rangle + \frac{\partial}{\partial x_1}\left\langle \mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_2} + \mu'(\boldsymbol{x}) \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_1}\right\rangle \\
&\qquad\quad \frac{\partial}{\partial x_2}\left\langle \mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1} + 3\mu'(\boldsymbol{x}) \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\right\rangle = -f_{x_2}(\omega, \boldsymbol{x}).
\end{aligned}$$

The subgrid terms

$$\begin{aligned}
S_1 &= \langle \rho'(\boldsymbol{x}) u'_{x_1}(\boldsymbol{x}) \rangle, \quad S_2 = \langle \rho'(\boldsymbol{x}) u'_{x_2}(\boldsymbol{x}) \rangle, \\
S_3 &= \left\langle 3\mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1} + \mu(\boldsymbol{x}') \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\right\rangle, \\
S_4 &= \left\langle \mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_2} + \mu(\boldsymbol{x}') \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_1}\right\rangle, \\
S_5 &= \left\langle \mu'(\boldsymbol{x}) \frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1} + 3\mu'(\boldsymbol{x}) \frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\right\rangle
\end{aligned}$$

in equation (12) are unknown. These terms cannot be neglected without preliminary estimation. The form of these terms in (12) determines a subgrid model. The subgrid terms are estimated using perturbation theory. Subtracting equation (12) from equation (11) and taking into account only the first order terms, the reduced equations for the subgrid displacements are written down as

$$\begin{aligned}
&\omega^2\rho(\boldsymbol{x},l)u'_{x_1}(\boldsymbol{x}) + 3\mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x_1}(\boldsymbol{x})}{\partial x_1^2} + 2\mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x_2}(\boldsymbol{x})}{\partial x_2\partial x_1} + \mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x1(\boldsymbol{x})}}{\partial x_2^2}\\
&\quad = -\omega^2\rho' u_{x_1}(\omega,\boldsymbol{x},l) - \frac{\partial}{\partial x_1}\left(3\mu'(\boldsymbol{x})\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_1} + \mu'(\boldsymbol{x})\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\right) -\\
&\quad\quad \frac{\partial}{\partial x_2}\left(\mu'(\boldsymbol{x})\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_1} + \mu'(\boldsymbol{x})\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_2}\right), \qquad (13)\\
&\omega^2\rho(\boldsymbol{x},l)u'_{x_2}(\boldsymbol{x}) + 2\mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x_1}(\boldsymbol{x})}{\partial x_2\partial x_1} + \mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x_2}(\boldsymbol{x})}{\partial x_1^2} + 3\mu(\boldsymbol{x},l)\frac{\partial^2 u'_{x_2}(\boldsymbol{x})}{\partial x_2^2}\\
&\quad = -\omega^2\rho' u_{x_2}(\omega,\boldsymbol{x},l) - \frac{\partial}{\partial x_1}\left(\mu'(\boldsymbol{x})\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_2} + \mu'(\boldsymbol{x})\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_1}\right) -\\
&\quad\quad \frac{\partial}{\partial x_2}\left(\mu'(\boldsymbol{x})\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_1} + 3\mu'(\boldsymbol{x})\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\right).
\end{aligned}$$

The variable $u(\omega,\boldsymbol{x},l)$ in the right-hand side of (13) is assumed to be known. For the fields, in which a small variation in the scale causes significant fluctuations of the field as it is (this is common to fractal fields) it is possible to consider $\mu(\boldsymbol{x},l)$, $\rho(\boldsymbol{x},l)$, $u(\boldsymbol{x},l)$ and their derivatives varying slower than $\mu(\boldsymbol{x})'$, $\rho(\boldsymbol{x})'$, $u'(\boldsymbol{x})$ and their derivatives. So, the 2D Green function of equations (13) for the points $\boldsymbol{x}$ and $\boldsymbol{x}'$ is

$$\begin{gathered}
G_{ij}(x_1,x_2,x'_1,x'_2) = \frac{1}{8i\rho}(A\delta_{ij} - B(2\gamma_i\gamma_j - \delta_{ij})),\\
A = \frac{1}{\alpha^2}H_0^{(2)}(k_\alpha r) + \frac{1}{\beta^2}H_0^{(2)}(k_\beta r), \quad B = \frac{1}{\alpha^2}H_2^{(2)}(k_\alpha r) - \frac{1}{\beta^2}H_2^{(2)}(k_\beta r),\\
r = \sqrt{(x_1-x'_1)^2 + (x_2-x'_2)^2}, \quad k_\alpha = \omega/\alpha, \quad k_\beta = \omega/\beta,\\
\gamma_i = \frac{x_i - x'_i}{r}, \quad \alpha^2 = 3\beta^2, \quad \beta = \sqrt{\frac{\mu(\boldsymbol{x},l)}{\rho(\boldsymbol{x},l)}}
\end{gathered}$$

with the Hankel function $H_m^{(2)}(\cdot) = J_m(\cdot) - iY(\cdot)$ of the second kind and order m expressed in terms of the Bessel functions of the first and second kind.

Let us denote the right-hand side functions of equations (13) as $\Omega_{x_1}(\omega,\boldsymbol{x},l)$ and $\Omega_{x_2}(\omega,\boldsymbol{x},l)$, respectively. Then the solution to equations (13) takes the form

$$u'_{x_1}(\boldsymbol{x}) = \frac{1}{8i\rho(\boldsymbol{x},l)} \int \bigl[(A - B(2\gamma_1^2 - 1))\Omega_{x_1}(\omega,\boldsymbol{x}',l) - 2\gamma_1\gamma_2 B\Omega_{x_2}(\omega,\boldsymbol{x}',l)\bigr]\,d\boldsymbol{x}',$$
$$u'_{x_2}(\boldsymbol{x}) = \frac{1}{8i\rho(\boldsymbol{x},l)} \int \bigl[(A - B(2\gamma_2^2 - 1))\Omega_{x_2}(\omega,\boldsymbol{x}',l) - 2\gamma_1\gamma_2 B\Omega_{x_1}(\omega,\boldsymbol{x}',l)\bigr]\,d\boldsymbol{x}'.$$

Substituting this solution in the subgrid terms gives

$$\begin{aligned}
S_1 &= \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \rho'(\boldsymbol{x}) \int \Bigl[(A(r) - B(r)(2\gamma_1^2 - 1))\Omega_{x_1}(\omega,\boldsymbol{x}',l) - \\
&\qquad 2\gamma_1\gamma_2 B(r)\Omega_{x_2}(\omega,\boldsymbol{x}',l)\Bigr]\,d\boldsymbol{x}'\Bigr\rangle, \\
S_2 &= \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \rho'(\boldsymbol{x}) \int \Bigl[(A(r) - B(r)(2\gamma_2^2 - 1))\Omega_{x_2}(\omega,\boldsymbol{x}',l) - \\
&\qquad 2\gamma_1\gamma_2 B(r)\Omega_{x_1}(\omega,\boldsymbol{x}',l)\Bigr]\,d\boldsymbol{x}'\Bigr\rangle, \\
S_3 &= \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[3\frac{\partial}{\partial x_1}(A(r) - B(r)(2\gamma_1^2 - 1)) - \\
&\qquad 2\frac{\partial}{\partial x_2}\gamma_1\gamma_2 B(r)\Bigr]\Omega_{x_1}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle + \\
&\quad \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[\frac{\partial}{\partial x_2}(A(r) - B(r)(2\gamma_2^2 - 1)) - \\
&\qquad 6\frac{\partial}{\partial x_1}\gamma_1\gamma_2 B(r)\Bigr]\Omega_{x_2}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle, \qquad (14) \\
S_4 &= \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[\frac{\partial}{\partial x_2}(A(r) - B(r)(2\gamma_1^2 - 1)) - \\
&\qquad 2\frac{\partial}{\partial x_1}\gamma_1\gamma_2 B(r)\Bigr]\Omega_{x_1}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle + \\
&\quad \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[\frac{\partial}{\partial x_1}(A(r) - B(r)(2\gamma_2^2 - 1)) - \\
&\qquad 2\frac{\partial}{\partial x_2}\gamma_1\gamma_2 B(r)\Bigr]\Omega_{x_2}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle, \\
S_5 &= \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[\frac{\partial}{\partial x_1}(A - B(2\gamma_1^2 - 1)) - \\
&\qquad 6\frac{\partial}{\partial x_2}\gamma_1\gamma_2 B\Bigr]\Omega_{x_1}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle + \\
&\quad \frac{1}{8i\rho(\boldsymbol{x},l)} \Bigl\langle \mu'(\boldsymbol{x}) \int \Bigl[3\frac{\partial}{\partial x_2}(A - B(2\gamma_2^2 - 1)) - \\
&\qquad 2\gamma_1\gamma_2\frac{\partial}{\partial x_1}B\Bigr]\Omega_{x_2}(\omega,\boldsymbol{x}',l)\,d\boldsymbol{x}'\Bigr\rangle,
\end{aligned}$$

Again, treating the terms with lower derivatives of the large-scale field $u_{x_1}(\omega, \boldsymbol{x}, l)$, $u_{x_2}(\omega, \boldsymbol{x}, l)$, $\mu(\boldsymbol{x}, l)$, $\rho(\boldsymbol{x}, l)$ as constants in the integrand, taking into account $\frac{\partial}{\partial x_i} r = -\frac{\partial}{\partial x_i'} r$, the correlation radius much less than the wavelength and (10), after integrating by parts we can write down

$$
\begin{aligned}
S_1 = {} & \frac{1}{8i}\int (A(r) - B(r)(2\gamma_1^2 - 1))(-\omega^2\Phi^{\chi\chi}(r,l))\,d\boldsymbol{x}'\,\rho(\boldsymbol{x},l)u_{x_1}(\omega,\boldsymbol{x},l)\frac{\Delta l}{l} + \\
& \frac{1}{8i}\int \frac{\partial}{\partial x_1'}(A(r) - B(r)(2\gamma_1^2 - 1))\Phi^{\chi\varphi}(r,l)\,d\boldsymbol{x}' \times \\
& \qquad \mu(\boldsymbol{x},l)\Big(3\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_1} + \frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\Big)\frac{\Delta l}{l} + \\
& \frac{1}{8i}\int \frac{\partial}{\partial x_2'}(A(r) - B(r)(2\gamma_1^2 - 1))\Phi^{\chi\varphi}(r,l)\,d\boldsymbol{x}' \times \\
& \qquad \mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_1} + \frac{\partial u_{x_1}(\omega,\boldsymbol{x}',l)}{\partial x_2}\Big)\frac{\Delta l}{l} - \\
& \frac{1}{4i}\int \gamma_1\gamma_2 B(r)(-\omega^2\Phi^{\chi\chi}(r,l))\,d\boldsymbol{x}'\,\rho(\boldsymbol{x},l)u_{x_2}(\omega,\boldsymbol{x},l)\frac{\Delta l}{l} - \\
& \frac{1}{4i}\int \frac{\partial}{\partial x_1}(\gamma_1\gamma_2 B(r))\Phi^{\chi\varphi}(r,l)\,d\boldsymbol{x}' \times \\
& \qquad \mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_2} + \frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_1}\Big)\frac{\Delta l}{l} - \\
& \frac{1}{4i}\int \frac{\partial}{\partial x_2}(\gamma_1\gamma_2 B(r))\Phi^{\chi\varphi}(r,l)\,d\boldsymbol{x}' \times \\
& \qquad \mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_1} + 3\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\Big)\frac{\Delta l}{l}.
\end{aligned}
\tag{15}
$$

The functions $\Phi^{\varphi\varphi}$, $\Phi^{\chi\chi}$, $\Phi^{\chi\varphi}$ depend only on r and l. These functions and the Green functions are the even functions, but the partial derivatives of A, B, and Φ with respect to x_i or x_j are the odd functions. Hence, the integrals with the partial derivatives in (15) are equal to zero. Passing to the polar coordinates and using the formula $\int_0^{2\pi}(2\gamma_j^2 - 1)\,d\theta = 0$, $\int_0^{2\pi}\gamma_1\gamma_2\,d\theta = 0$, $\int_0^{2\pi}\gamma_2(\gamma_2^2 - \gamma_1^2)\,d\theta = 0$, where $\gamma_1 = \cos\theta$, $\gamma_2 = \sin\theta$, we obtain

$$
S_1 \approx -\rho(\boldsymbol{x},l)u_{x_1}(\omega,\boldsymbol{x},l)\frac{\Delta l}{l}I, \quad S_2 \approx -\rho(\boldsymbol{x},l)u_{x_2}(\omega,\boldsymbol{x},l)\frac{\Delta l}{l}I, \tag{16}
$$

$$
\begin{aligned}
I = {} & \frac{\omega^2\pi}{4i\alpha^2(\boldsymbol{x},l)}\int_0^\infty rH_0^{(2)}(k_\alpha r)\Phi^{\chi\chi}(r,l)\,dr + \\
& \frac{\omega^2\pi}{4i\beta^2(\boldsymbol{x},l)}\int_0^\infty rH_0^{(2)}(k_\beta r)\Phi^{\chi\chi}(r,l)\,dr.
\end{aligned}
$$

Substituting the formulas from (10) into (14) yields:

$$
\begin{aligned}
S_3 = {} & -\frac{\omega^2\mu(\boldsymbol{x},l)}{8i}\int\Big(3\frac{\partial}{\partial x_1'}(A-B(2\gamma_1^2-1))+2\frac{\partial}{\partial x_2'}(\gamma_1\gamma_2 B)\Big)\times \\
& \qquad \Phi^{\varphi\chi}(r,l)\,d\boldsymbol{x}'\,u_{x_1}(\boldsymbol{x},l)\frac{\Delta l}{l}+ \\
& \frac{\omega^2\mu(\boldsymbol{x},l)}{8i}\int\Big(6\frac{\partial}{\partial x_1'}(\gamma_1\gamma_2 B(r))-\frac{\partial}{\partial x_2'}(A(r)-B(r)(2\gamma_2^2-1))\Big)\times \\
& \qquad \Phi^{\varphi\chi}(r,l)\,d\boldsymbol{x}'\,u_{x_2}(\boldsymbol{x},l)\frac{\Delta l}{l}- \\
& \frac{\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\Big(9\frac{\partial^2}{\partial x_1'^2}A(r)+\frac{\partial^2}{\partial x_2'^2}A(r)\Big)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}+ \\
& \frac{\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\Big(9\frac{\partial^2}{\partial x_1'^2}(B(2\gamma_1^2-1))+\frac{\partial^2}{\partial x_2'^2}(B(2\gamma_2^2-1))\Big)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}+ \\
& \frac{12\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\frac{\partial^2}{\partial x_2'\partial x_1'}(\gamma_1\gamma_2 B(r))\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\,\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}- \\
& \frac{3\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\Big(\frac{\partial^2}{\partial x_1'^2}A(r)+\frac{\partial^2}{\partial x_2'^2}A(r)\Big)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}+ \\
& \frac{3\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\Big(\frac{\partial^2}{\partial x_1'^2}(B(2\gamma_1^2-1))+\frac{\partial^2}{\partial x_2'^2}(B(2\gamma_2^2-1)\Big)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}+ \\
& \frac{20\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\frac{\partial^2}{\partial x_2'\partial x_1'}(\gamma_1\gamma_2 B(r))\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\,\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}- \\
& \frac{4\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\frac{\partial^2}{\partial x_2'\partial x_1'}A(r)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_2}+\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_1}\Big)\frac{\Delta l}{l}+ \\
& \frac{2\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\frac{\partial^2}{\partial x_2'\partial x_1'}(B(r)(2\gamma_1^2-1))\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_2}+\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_1}\Big)\frac{\Delta l}{l}+ \\
& \frac{2\mu(\boldsymbol{x},l)}{8i\rho(\boldsymbol{x},l)}\int\Big(3\frac{\partial^2}{\partial x_1'^2}(\gamma_1\gamma_2 B)+\frac{\partial^2}{\partial x_2'^2}(\gamma_1\gamma_2 B)\Big)\Phi^{\varphi\varphi}(r,l)\,d\boldsymbol{x}'\times \\
& \qquad \Big(\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_2}+\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_1}\Big)\frac{\Delta l}{l}.
\end{aligned}
\tag{17}
$$

The first and second integrals in (17) are equal to zero by similar arguments as for formula (15). The third integral with allowance for the formula

$$\frac{\partial^2}{\partial x_i'^2}\Big(\frac{1}{\alpha^2}H_0^{(2)}(k_\alpha r)+\frac{1}{\beta^2}H_0^{(2)}(k_\beta r)\Big)= \frac{8i\delta(r)}{3\beta^2}-\frac{\omega^2}{2\beta^4}\Big(\frac{1}{9}H_0^{(2)}(k_\alpha r)+H_0^{(2)}(k_\beta r)\Big) \tag{18}$$

passing to the polar coordinates becomes equal to

$$I_3=-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)3\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}- \frac{5\omega^2\pi}{12i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{9}rH_0^{(2)}(k_\alpha r)+rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times 3\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}. \tag{19}$$

The rest of the integrals in (17) are equal

$$I_4+I_5=\frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{9}rH_0^{(2)}(k_\alpha r)-rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times \mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l},$$

$$I_6=-2\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}+ \frac{3\omega^2\pi}{4i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{9}rH_0^{(2)}(k_\alpha r)+rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times \mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l},$$

$$I_7+I_8=\frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{9}rH_0^{(2)}(k_\alpha r)-rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times \mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l},$$

$$I_9=I_{10}=I_{11}=0.$$

Hence,

$$S_3=-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)3\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}-2\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}- \frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{36}rH_0^{(2)}(k_\alpha r)+\frac{9}{4}rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times \mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}+$$

$$\frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{7}{36}rH_0^{(2)}(k_\alpha r)-\frac{1}{4}rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times$$
$$\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}. \tag{20}$$

In the same way we evaluate S_4 and S_5:

$$S_4\approx-\frac{1}{3}\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_1}+\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_2}\Big)\frac{\Delta l}{l}, \tag{21}$$

$$S_5=-2\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)3\mu(\boldsymbol{x},l)\frac{\Delta l}{l}\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}+$$
$$\frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{7}{36}rH_0^{(2)}(k_\alpha r)-\frac{1}{4}rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times$$
$$\frac{\Delta l}{l}\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}-$$
$$\frac{\pi\omega^2}{i\beta^2(\boldsymbol{x},l)}\int_0^\infty\Big(\frac{1}{36}rH_0^{(2)}(k_\alpha r)+\frac{9}{4}rH_0^{(2)}(k_\beta r)\Big)\Phi^{\varphi\varphi}(r,l)\,dr\times$$
$$\mu(\boldsymbol{x},l)\frac{\Delta l}{l}\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}. \tag{22}$$

The correlation radii of φ, χ (hence, the correlation radii of ρ, μ) are much smaller than the wavelength, since the correlation radius and scales of inhomogeneities are, approximately, of the same order of magnitude [8], [18]. Thus, the integrals in (16), (20)–(22) are of second order in L. The maximum scale of inhomogeneities L is much smaller than the wavelength. If the following inequalities hold $L^2\omega^2\rho(\boldsymbol{x},l)/\mu(\boldsymbol{x},l)\ll 1$, the integral terms in (16), (20), (22) may be discarded. Hence, we obtain

$$\begin{aligned}
&\langle\rho'(\boldsymbol{x})u'_{x_1}(\boldsymbol{x})\rangle\approx 0,\qquad\langle\rho'(\boldsymbol{x})u'_{x_2}(\boldsymbol{x})\rangle\approx 0,\\
&\Big\langle 3\mu'(\boldsymbol{x})\frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1}+\mu(\boldsymbol{x}')\frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\Big\rangle\approx\\
&\qquad-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)3\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}-2\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l},\\
&\Big\langle\mu'(\boldsymbol{x})\frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_2}+\mu(\boldsymbol{x}')\frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_1}\Big\rangle\approx\\
&\qquad-\frac{1}{3}\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\Big(\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_1}+\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_2}\Big)\frac{\Delta l}{l},\\
&\Big\langle\mu'(\boldsymbol{x})\frac{\partial u'_{x_1}(\boldsymbol{x})}{\partial x_1}+3\mu'(\boldsymbol{x})\frac{\partial u'_{x_2}(\boldsymbol{x})}{\partial x_2}\Big\rangle\approx\\
&\qquad-2\Phi^{\varphi\varphi}(0,l)\mu(\boldsymbol{x},l)\frac{\partial u_{x_1}(\boldsymbol{x},l)}{\partial x_1}\frac{\Delta l}{l}-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)3\mu(\boldsymbol{x},l)\frac{\partial u_{x_2}(\boldsymbol{x},l)}{\partial x_2}\frac{\Delta l}{l}.
\end{aligned}\tag{23}$$

Substituting formulas (10) and (23) in the ongrid equations (12) gives

$$
\begin{aligned}
&\omega^2\rho(\boldsymbol{x},l)u_{x_1}(\omega,\boldsymbol{x},l)+\\
&\frac{\partial}{\partial x_1}\left(\exp\Big(-\int_l^L\varphi(\boldsymbol{x},l_1)\frac{dl_1}{l_1}\Big)\Big(3\mu_1(l)\frac{\partial u_x(\omega,\boldsymbol{x},l)}{\partial x_1}+\mu_2(l)\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\Big)\right)+\\
&\frac{\partial}{\partial x_2}\left(\mu_3(l)\exp\Big(-\int_l^L\varphi(\boldsymbol{x},l_1)\frac{dl_1}{l_1}\Big)\Big(\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_2}+\frac{\partial u_{x_2}(\omega,\boldsymbol{x_1},l)}{\partial x_1}\Big)\right)\\
&\qquad\quad=-f_{x_1}(\omega,\boldsymbol{x}),\\
&\omega^2\rho(\boldsymbol{x},l)u_{x_2}(\omega,\boldsymbol{x},l)+\\
&\frac{\partial}{\partial x_1}\left(\mu_3(l)\exp\Big(-\int_l^L\varphi(\boldsymbol{x},l_1)\frac{dl_1}{l_1}\Big)\Big(\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_2}+\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_1}\Big)\right)+\\
&\frac{\partial}{\partial x_2}\left(\exp\Big(-\int_l^L\varphi(\boldsymbol{x},l_{(1)})\frac{dl_1}{l_1}\Big)\Big(\mu_2(l)\frac{\partial u_{x_1}(\omega,\boldsymbol{x},l)}{\partial x_1}+3\mu_1(l)\frac{\partial u_{x_2}(\omega,\boldsymbol{x},l)}{\partial x_2}\Big)\right)\\
&\qquad\quad=-f_{x_2}(\omega,\boldsymbol{x}),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mu_1(l)&=\Big(1-\frac{10}{9}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\Big(1-\langle\varphi\rangle\frac{\Delta l}{l}+\frac{1}{2}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\mu_0,\\
\mu_2(l)&=\Big(1-2\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\Big(1-\langle\varphi\rangle\frac{\Delta l}{l}+\frac{1}{2}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\mu_0,\\
\mu_3(l)&=\Big(1-\frac{1}{3}\Phi^{\boldsymbol{\varphi\varphi}}(0,l)\frac{\Delta l}{l}\Big)\Big(1-\langle\varphi\rangle\frac{\Delta l}{l}+\frac{1}{2}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\mu_0.
\end{aligned}
$$

With the second order of accuracy in $\Delta l/l$ the coefficients $\mu_j(l)$ satisfy the equations

$$
\begin{aligned}
\mu_1(l)&=\Big(1-\langle\varphi\rangle\frac{\Delta l}{l}-\frac{11}{18}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\mu_0,\\
\mu_2(l)&=\Big(1-\frac{3}{2}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}-\langle\varphi\rangle\frac{\Delta l}{l}\Big)\mu_0,\\
\mu_3(l)&=\Big(1-\langle\varphi\rangle\frac{\Delta l}{l}+\frac{1}{6}\Phi^{\varphi\varphi}(0,l)\frac{\Delta l}{l}\Big)\mu_0.
\end{aligned}
$$

As $\Delta l\to 0$, the effective coefficients $\mu_j(l)$ and $\rho(\boldsymbol{x},l)$ become as follows:

$$
\begin{gathered}
\rho(\boldsymbol{x},l)=\rho_0,\quad \mu_1(l_0)=\mu_2(l_0)=\mu_3(l_0)=\mu_0,\\
\frac{d\ln\mu_1(l)}{d\ln l}=-\frac{11}{18}\Phi^{\varphi\varphi}(0,l)-\langle\varphi\rangle,\quad \frac{d\ln\mu_2(l)}{d\ln l}=-\frac{3}{2}\Phi^{\varphi\varphi}(0,l)-\langle\varphi\rangle,\\
\frac{d\ln\mu_3(l)}{d\ln l}=\frac{1}{6}\Phi^{\varphi\varphi}(0,l)-\langle\varphi\rangle.
\end{gathered}
\tag{25}
$$

In the scale-invariant media, solution of equations (25) has a simple form:

$$
\begin{gathered}
\rho(\boldsymbol{x}, l) = \rho_0, \\
\mu_1(l) = \mu_0\Big(\frac{l}{l_0}\Big)^{-\frac{11}{18}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle}, \quad \mu_2(l) = \mu_0\Big(\frac{l}{l_0}\Big)^{-\frac{3}{2}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle}, \\
\mu_3(l) = \mu_0\Big(\frac{l}{l_0}\Big)^{\frac{1}{6}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle}.
\end{gathered}
\tag{26}
$$

By virtue of formulas (25), the form of the correlation functions does not affect the effective coefficients.

4. Conclusion

We have proposed the effective coefficients (25) for the elastic equations if parameters in equations are described by extremely irregular fields which are close to multifractals. We obtain multifractals if the minimum scale l_0 in formulas (3), (4) tend to zero. As the minimum scale is finite, any singularities are absent, therefore we use only the theory of differential equations and the theory of stochastic processes. For a scale-invariant medium, effective coefficients have the power dependence on the scale of smoothing. The exponents of the power dependencies have been calculated as (26).

References

[1] Sahimi M. Flow phenomena in rocks: from continuum models, to fractals, percolation, cellular automata, and simulated annealing // Reviews Modern Physics. — 1993. — Vol. 65. — P. 1393–1534.

[2] Imomnazarov Kh. Kh., Mikhailov A.A. Application of a spectral method for numerical modeling of propagation of seismic waves in porous media for dissipative case // Sib. J. Vychisl. Mat. — 2014. — Vol. 17. — P. 139–147.

[3] Capdeville Y., Guillot L., Marigo J.-J. Second order homogenization of the elastic wave equation for non-periodic layered media // Geophysical J. International. — 2007. — Vol. 170. — P. 823–838.

[4] Capdeville Y., Guillot L., Marigo J.-J. 2-D non periodic homogenization of the elastic wave equation: SH case // Geophysical J. International. — 2010. — Vol. 182. — P. 897–910.

[5] Kanaun S.K., Levin V. Self-Consistent Methods for Composites. — Vol. 1, 2. — Berlin, Heidelberg: Springer, 2008.

[6] Vdovina T., Minkoff S.E. An apriority error analysis of operator upscaling for the acoustic wave equation // International J. Num. Anal. and Modeling. — 2008. — Vol. 5. — P. 543–569.

[7] Richard L., Gibson Jr., Kai G., et al. Multiscale modeling of acoustic wave propagation in 2D media // Geophysics. — 2014. — Vol. 79. — P. 161–175.

[8] Rvtov S.M., Kravtsov Yu.A., Tatarskii V.I. Introduction to Statistical Radiophysics. — Part 2. Random Fields. — Moscow: Nauka, 1978 (In Russian).

[9] Shermergor T.D. Theory of Elasticity of Microscopically Inhomogeneous Media. — Moscow: Nauka, 1977 (In Russian).

[10] Fouque J.P., Garnnier J., Papanicolaou G., Solna K. Wave Propagation and Time Reversal in Randomly Layered Media. — Berlin, Heidelberg: Springer, 2008. — (Stochastic Modelling and Applied Probability; 56).

[11] Bahraminasab A., Mehdi V.A.S., Shahbazi F., et al. Renormalization group analysis and numerical simulation of propagation and localization of acoustic waves in heterogeneous media // Phys. Rev. B. — 2007. — 75, 64301.

[12] Sepehrinia R., Bahraminasab A., Sahimi M., Reza R.T.M. Dynamic renormalization group analysis of propagation of elastic wave in two-dimensional heterogeneous media // Phys. Rev. B. — 2008. — 77, 014203.

[13] Koohi lai Z., Vasheghani F.S., Jafari G.R. Non-gaussianity effect of petrophysical quantities by using q-entropy and multifractal random walk // Physica A. — 2013. — Vol. 392. — P. 3039–3044.

[14] Shiri Y., Tokhmechi B., Zarei Z., Koneshloo M. Multifractality nature and multifractal detrended fluctuation analysis of well logging data // Int. Res. J. Geol. and Min. (IRJGM). — 2012. — Vol. 2. — P. 148–154.

[15] Kolmogorov A.N. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number // J. Fluid Mech. — 1962. — Vol. 13. — P. 82–85.

[16] Kuz'min G.A., Soboleva O.N. Subgrid modeling of filtration in porous self-similar media // J. Appl. Mech. Tech. Phys. — 2002. — Vol. 43. — P. 583–592.

[17] Gnedenko B.V., Kolmogorov A.N. Limit Distributions for Sums of Independent Random Variables / English trans. K.L. Chung. — Cambridge: Addison Wesley, 1954.

[18] Dagan G. Flow and Transport in Porous Formation. — Berlin: Springer, 1989.