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Uniform in ε convergence of the standard Galerkin finite element method in L_2-norm for the singularly-perturbed elliptic problems on a priori chosen meshes*

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Convection-diffusion problems are models for describing the transportation of matters in a diffusive medium. They are also a part of more general equations, such as the Navier–Stokes equations appearing in fluid flow problems.

We consider the linear convection-diffusion problem: seek u such that

$$L_\varepsilon u_\varepsilon \equiv -\varepsilon \Delta u_\varepsilon + \boldsymbol{b} \cdot \nabla u_\varepsilon + c u_\varepsilon = f \quad \text{in} \quad \Omega, \tag{1}$$

$u_\varepsilon = 0$ on Γ_- and $\frac{1}{\sigma}\varepsilon \nabla u_\varepsilon \cdot \boldsymbol{n} + u_\varepsilon - g = 0$ on $\Gamma_1 = \Gamma \setminus \Gamma_-$, whose solution is driven by the given vector-function $\boldsymbol{b}$, and $\Gamma_- = \{x \in \Gamma,\ \boldsymbol{b} \cdot \boldsymbol{n} < 0\}$ is the inflow part of Γ, the boundary of Ω. Similarly, we denote by $\Gamma_+ = \{x \in \Gamma,\ \boldsymbol{b} \cdot \boldsymbol{n} > 0\}$ the outflow part of Γ and by $\Gamma_0 = \{\boldsymbol{x} \in \Gamma,\ \boldsymbol{b} \cdot \boldsymbol{n} = 0\}$, the characteristic line boundary.

Here Ω is a bounded simply connected polygonal domain in $\boldsymbol{R}^2$, and $\boldsymbol{b}$, c, f are given bounded sufficiently smooth functions in Ω. The positive constant $\varepsilon \le 1$ is used to measure the relative amount of diffusion to convection and σ, g are given functions on Γ_1. Further, $\boldsymbol{n}$ is the outward pointing unit normal vector on Γ. When $\sigma \equiv \infty$ on Γ_1 we have the Dirichlet boundary conditions on the whole boundary Γ of Ω.

We assume also that

$$\min_{x \in \Omega}\Big(c - \frac{1}{2}\nabla \cdot \boldsymbol{b}\Big) = c_0 > 0, \quad \min_{\Gamma_1} \sigma \ge 0. \tag{2}$$

Under these conditions if meas(Γ_-) it can be shown that (1) has a unique solution in $H_0^1(\Omega)$, the first order Sobolev space with the trace $u_\varepsilon = 0$ on Γ_-.

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1. General estimates

Now we prove the lemmas similar to those in [2] for the problems with the inhomogeneous Dirichlet boundary conditions that we shall need in our further investigations.

Lemma 1. *Let* (2) *hold and* u_ε *be the solution to* (1) *with* $u_\varepsilon = g$ *on* Γ, *where* $g \in H^{1/2}(\Gamma)$. *Then*

(a) $\varepsilon^{1/2}\|u_\varepsilon\|_1 \le C\left(\|f\| + \|g\|_{L_2(\Gamma)}\right)$,

and if Ω *and the boundary conditions are such that the inequality*

$$\|u_\varepsilon\|_2 \le C\left(\|\Delta u_\varepsilon\| + \|u_\varepsilon\|\right) \tag{3}$$

holds (see Remark 1), then

(b) $\varepsilon^{3/2}\|u_\varepsilon\|_2 + \varepsilon^{1/2}\|u_\varepsilon\|_1 \le C\left(\|f\| + \|g\|_{L_2(\Gamma)}\right)$.

Remark 1. It is known that if Ω is either convex polygon or smooth, then (3) holds.

Remark 2. The powers $\varepsilon^{3/2}$ and $\varepsilon^{1/2}$ occurring in part (*b*) of Lemma 1 can be readily shown to be sharp by considering simple examples in one dimension.

Lemma 2. *Assume that* $f \in H^{k-2}(\Omega)$, $k \ge 2$, *and* Ω *is such, that* (3) *holds. If* $g \in H^{k-1/2}(\Gamma)$, *then solution to* (1) *satisfies* $u_\varepsilon \in H^k(\Omega)$, $u_\varepsilon = g$ *on* Γ *and*

$$\varepsilon^{k-1/2}\|u_\varepsilon\|_k \le C\left(\|f\| + \varepsilon^{3/2}\|f\|_1 + \ldots + \varepsilon^{k-2+1/2}\|f\|_{k-2} + \|g\|_{L_2(\Gamma)}\right).$$

2. Splitting of the solution into regular and layer parts

Consider now the homogeneous Dirichlet problem (1) in $\Omega = (0,1)^2$. Let us assume that the velocity field $\boldsymbol{b} \ge \boldsymbol{b}_0$, where $\boldsymbol{b}_0$ is the vector with constant and positive components. Under this assumption, two exponential boundary layers occur at the outflow boundary Γ_+ and there will be no internal and parabolic boundary layers. To avoid also layers caused by data incompatibility at the corner points of the unit square the appropriate compatibility conditions must be imposed.

The derivatives of the exact solution u_ε are in general unbounded for $\varepsilon \to 0$. This unboundedness occurs in general only in the layer region. This can be shown, in particular, using asymptotic expansion of the solution and its derivatives. Various forms of such expansions have appeared in the literature, for a survey see [4], for example. Estimates of the higher order derivatives are also provided in papers. Alternative approach for the outflow layers analysis is given in [3].

To cope with this, when computing a numerical solution, we must properly adapt the finite element mesh to the solution [5]. For this purpose we should first closely consider the behavior of the solution in the layers. Let

$$L_\varepsilon u_0 = f \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{at } \Gamma_-, \qquad u_0 = F \quad \text{at } \Gamma \setminus \Gamma_-, \tag{4}$$

where the function F will be specified later.

Let $z = u_\varepsilon - u_0$. It follows that

$$L_\varepsilon z = 0 \quad \text{in } \Omega, \qquad z = 0 \quad \text{at } \Gamma_-, \qquad z = u_\varepsilon - u_0 \quad \text{at } \Gamma \setminus \Gamma_-. \tag{5}$$

Due to the boundary conditions, in this problem there occur no corner singularities and therefore u_0 will be almost free of layers (see Lemma 4), but z captures the layer behavior of $u_\varepsilon = u_0 + z$.

3. The standard Galerkin method

Let us write the variational formulation of (1) for the inhomogeneous Dirichlet boundary conditions, namely $u_\varepsilon = g$ on Γ,

$$a_\varepsilon(u_\varepsilon, w) \equiv \varepsilon a_1(u_\varepsilon, w) + a_0(u_\varepsilon, w) = (f, w), \quad u_\varepsilon \in W, \quad \forall w \in W_0, \tag{6}$$

where $W = \{w \in H^1(\Omega) : \ w = g \text{ on } \Gamma\}$, $W_0 = \{w \in H^1(\Omega) : \ w = 0 \text{ on } \Gamma\}$,

$$a_1(u, w) = \int_\Omega (\nabla u \cdot \nabla w)\, d\Omega, \qquad a_0(u, w) = \int_\Omega (\boldsymbol{b} \cdot \nabla u w + c u w)\, d\Omega,$$

$$(u, v) = \int_\Omega u v \, d\Omega.$$

Then the Galerkin approximation $u_{\varepsilon,h} \in W_h \subset W$ is defined by

$$a_\varepsilon(u_{\varepsilon,h}, w_h) = (f, w_h), \quad \forall w_h \in W_{0,h} \subset W_0.$$

From assumption (2) easy follows that the bilinear form $a_\varepsilon(\cdot, \cdot)$ is coercive over $W_0 \times W_0$, that is,

$$a_\varepsilon(w, w) \geq \rho \|w\|_{1,\varepsilon}^2, \tag{7}$$

where ρ is a constant and

$$\|w\|_{1,\varepsilon}^2 = \varepsilon |w|_1^2 + \|w\|^2. \tag{8}$$

Let u_{ε,I_h} be the interpolant in W_h of u_ε and let

$$\eta_h = u_\varepsilon - u_{\varepsilon,I_h}, \qquad \theta_h = u_{\varepsilon,h} - u_{\varepsilon,I_h}. \tag{9}$$

Then we can easy see that $u_\varepsilon - u_{\varepsilon,h} = \eta_h - \theta_h$ and $\theta_h \in W_{0,h}$. Hence,

$$a_\varepsilon(u_{\varepsilon,h},\theta_h) = a_\varepsilon(u_\varepsilon,\theta_h) \quad \text{and} \quad a_\varepsilon(\theta_h,\theta_h) = a_\varepsilon(\eta_h,\theta_h). \tag{10}$$

Now we are able to prove the following result:

Lemma 3. *Let u_ε be the solution to the variational problem (6). Then the following inequalities hold:*

(a) $\|\theta_h\|_{1,\varepsilon} \le C\left(\delta^{1/2}|\eta_h|_1 + \delta^{-1/2}\|\eta_h\|\right)$, *where $\delta = 1$ or ε,*

(b) $\|u_{\varepsilon,h}\|_{1,\varepsilon} \le C\|u_\varepsilon\|_1$,

(c) $\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon} \le C\left(\delta^{1/2}|\eta_h|_1 + \delta^{-1/2}\|\eta_h\|\right)$, *where $\delta = 1$ or ε.*

4. Splitting of the error

Now, we are going to obtain the uniform in ε estimates of the discretization error. For this aim, we write

$$u_\varepsilon - u_{\varepsilon,h} = (u_0 - u_{0,h}) + (z - z_h), \tag{11}$$

where

$$\begin{aligned} a_\varepsilon(u_{0,h},w_h) &= a_\varepsilon(u_0,w_h), \quad \forall w_h \in W_h, \\ a_\varepsilon(z_h,w_h) &= a_\varepsilon(z,w_h), \quad \forall w_h \in W_h. \end{aligned}$$

Let us consider the sequence of problems:

$$a_0(v_i,w) = -a_1(v_{i-1},w), \quad v_i = 0 \text{ on } \Gamma_-, \qquad i = 0,\dots,r+1, \tag{12}$$

where $a_1(v_{-1},w) \equiv (f,w)$. Now we can define $F = \sum_{i=0}^{r+1} \varepsilon^i v_i$ in (4). Then, for a quasi-uniform mesh we have the following result (see [1] for details).

Lemma 4. *Let u_0 be the solution to (4). Then for the appropriate compatibility conditions the following estimate holds:*

$$\begin{cases} \|u_\varepsilon^{(0)} - u_h^{(0)}\| \le C(\varepsilon^{1/2} + h)h^r, & \text{if } r \text{ is odd}, \\ \|u_\varepsilon^{(0)} - u_h^{(0)}\| \le C(\varepsilon^{1/2} + 1)h^r, & \text{if } r \text{ is even}, \end{cases}$$

where the constant C is independent of h and ε.

In order to estimate the L_2-norm of $(z - z_h)$, we use the Aubin–Nitsche trick. Finally, we prove the following result:

Theorem. *If $u_\varepsilon \in H^{r+2}(\Omega)$, then the discretization error for the Galerkin method on the Shishkin type mesh satisfies*

$$\begin{cases} \|u_\varepsilon - u_{\varepsilon,h}\| \le C\{\varepsilon^{1/2} + h_0\}h_0^r, & \text{if } r \text{ is odd}, \\ \|u_\varepsilon - u_{\varepsilon,h}\| \le C\{\varepsilon^{1/2} + 1\}h_0^r, & \text{if } r \text{ is even}, \end{cases}$$

where the constant C is independent of ε and h_0, but strongly depends on r, where r is a number such that the polynomials of the degree r are contained in the space W_h.

5. Conclusions

A standard Galerkin method for singularly perturbed convection-diffusion problems has been presented, based on a Shishkin mesh which is a priori adapted mesh to the behavior of the solution with the exponential layers. The total number of mesh points are of the same order in the boundary layers as in the layer free part of the domain. This method is stabilized due to the appropriate refinement of the Shishkin mesh in the layer and have a quasi-optimal rate of the convergence in L_2-norm for odd degree polynomials and quasi-uniform meshes uniformly in ε, that was illustrated by numerical examples.

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