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Maxwell's equations for the Earth's electromagnetic fields of force and without it

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Electromagnetic fields that are observed on the Earth with the worldwide network of magnetic observatories as well as regional means when surveying the Earth's interior are conventionally considered to be force and one-mode fields [1, 2].

For the numerical modeling of electromagnetic fields observed on the Earth, a force component of Maxwell's equation is commonly used. This is because of its low-frequency oscillations. However for the numerical modeling of the major geomagnetic field (MGF), the potential of a force modification (mode) of a magnetic field in the air [1, 2] is employed.

A somewhat different situation exists in the theory of the MGF generation, where along with a force mode of the electromagnetic field its non-force part, which is called a Toroidal magnetic and Poloidal electric field, is used. This theory of the MGF generation as opposed to the other was called a theory of dynamic excitation of the MGF. Its main feature is the following: a non-force mode is not to appear in the Earth's atmosphere due to its strong tension (200–500 Hs) in the Earth's interior that is necessary for the MGF generation [3, 4]. The author shows that this is not so [5]. A non-force electromagnetic field (a Toroidal magnetic and a Poloidal electric fields) is measured in the Earth's atmosphere along with a force component when the electromagnetic field tension is immediately fixed with an instrument [5, 6]. A magnetic field is measured with a magnetometer, an electric field—with an electrometer. Thus, a non-force electromagnetic field is an insurmountable obstacle in interpreting observations with a one-mode scheme [5].

It is natural that changing from the one-mode to the two-mode interpretation demands the development of the original equations for a complete description of electromagnetic fields observed on the Earth (in the atmosphere) in terms of a force and a non-force modification. In [5, 7, 8], a force and a non-force part of the observed electromagnetic field are expressed via concepts of Toroidal and Poloidal electromagnetic fields. In terms of such concepts Maxwell's force equations in a low-frequency domain can be written down in the following manner [5]:

$$\operatorname{rot} \boldsymbol{H}_P = \sigma \boldsymbol{E}_T; \quad \operatorname{rot} \boldsymbol{E}_T = -\frac{\partial \boldsymbol{H}_P}{\partial t}; \quad \operatorname{div} \big(\boldsymbol{H}_P, \boldsymbol{H}_T, \boldsymbol{E}_T\big) = 0. \qquad (1)$$

A non-force electromagnetic field with a force part is introduced in a somewhat different way [5]:

$$\boldsymbol{H} = \boldsymbol{H}_T + \boldsymbol{H}_P; \quad \boldsymbol{H}_T = \operatorname{rot}(Q\boldsymbol{r}); \quad \boldsymbol{H}_P = \operatorname{rot}\operatorname{rot}(Q\boldsymbol{r}). \tag{2}$$

The spherical coordinates (r, θ, φ, t) with the origin in the Earth's center are used.

In formulas (1) and (2), $\boldsymbol{H}_P$ is a force Poloidal magnetic field, $\boldsymbol{H}_T$ and $\boldsymbol{E}_T$ are a non-force Toroidal magnetic and a force Toroidal electric fields, respectively, σ is conductivity, μ is magnetic permeability of the medium, Q is an arbitrary scalar mathematical function of three spherical variables and time (r, θ, φ, t) of the class C^∞. Equations in (1) and (2) and further are written down in spherical coordinates and are applicable only in spherical domains. This remark is essential because the three so-called anti-dynamo theorems [3, 9, 10] forbid from occurrence of a non-force mode in planar and cylindrical domains due to the effect of "reciprocal" symmetry [8].

Relations (2) enable us to obtain equations, which we have conventionally called equations of dynamic excitation of a MGF and its variations [7]:

$$\begin{aligned} &\operatorname{rot}\boldsymbol{H}_T = \operatorname{rot}\operatorname{rot}\big(Q\boldsymbol{r}\big) = \boldsymbol{H}_P \quad \Rightarrow \quad \operatorname{rot}\boldsymbol{H}_T = \boldsymbol{H}_P; \\ &\operatorname{rot}\boldsymbol{H}_P = \operatorname{rot}\operatorname{rot}\operatorname{rot}\big(Q\boldsymbol{r}\big) = -\operatorname{rot}(\Delta Q\boldsymbol{r}\big) = \chi\boldsymbol{H}_T \;\; \Rightarrow \;\; \operatorname{rot}\boldsymbol{H}_P = \chi\boldsymbol{H}_T. \end{aligned} \tag{3}$$

Here

$$\Delta Q = -\chi Q = \begin{cases} -\dfrac{\gamma}{\eta}Q, & t = 0, \\ -(i\omega\mu\sigma)^{1/2}Q, & t > 0; \end{cases}$$

γ is diffusion velocity of the field, $\eta = \dfrac{1}{\sigma\mu}$ is magnetic viscosity, ω is a circular frequency.

Similar to relations (1) for a non-force mode in a low-frequency domain we can write down the equations

$$\operatorname{rot}\boldsymbol{H}_T = \boldsymbol{H}_P; \quad \operatorname{rot}\boldsymbol{E}_P = 0; \quad \operatorname{div}\big(\boldsymbol{H}_T, \boldsymbol{H}_P, \boldsymbol{E}_P\big) = 0. \tag{4}$$

Relations (4) include an important feature of a Toroidal magnetic field, i.e., its vortices do not generate electric current in any medium (including that in the Earth's atmosphere) but excite a force magnetic field. In this connection standard boundary conditions for a full magnetic field, including that on the Earth's surface, can be written down in the following way:

$$\boldsymbol{H}_{T1} - \boldsymbol{H}_{T2} = 0; \qquad \boldsymbol{H}_{P1} - \boldsymbol{H}_{P2} = 0. \tag{5}$$

Here 1 and 2 denote different sides of the spherical surface. Relations (5) indicate to the fact that a non-force Toroidal magnetic field is continuously transferred through any surfaces (including thin screens) without a jump, because its vortices do not generate an electric current. Therefore it takes essentially more time for a non-force magnetic field to penetrate into a medium than for a force one [11]. The penetration of a force field into a medium is restricted to a skin effect. A non-force magnetic field does not generate the skin effect, because its vortices do not excite electric current (rot $\boldsymbol{H}_T = \boldsymbol{H}_P$). The attenuation $\boldsymbol{H}_T$ is controlled only by a geometric factor, i.e., a decrease from a source according to the law $1/r^3$.

Now it appears possible to combine force and non-force electromagnetic fields in the generalized Maxwell's equations. A complete force mode with allowance for the second dynamo-excitation equation from (3) has the form

$$\begin{gathered} \operatorname{rot} \boldsymbol{H}_P = \sigma \boldsymbol{E}_T + \chi \boldsymbol{H}_T; \quad \operatorname{rot} \boldsymbol{E}_T = -\frac{\partial \boldsymbol{B}_P}{\partial t}; \\ \operatorname{div}\big(\boldsymbol{H}_P, \boldsymbol{H}_T, \boldsymbol{E}_T\big) = 0; \qquad \boldsymbol{B}_P = \mu \boldsymbol{H}_P. \end{gathered} \tag{6}$$

In this case, an additional electric current, for example,

$$\frac{\gamma}{\eta}\boldsymbol{H}_T = \sigma(\mu\gamma\boldsymbol{H}_T) = \sigma\boldsymbol{E}'_T,$$

which arises due to spherical properties of a source, is the source of non-force Toroidal magnetic field, penetrating into the Earth's atmosphere as well employing boundary conditions (5).

The non-force mode can also be characterized by its equations

$$\begin{gathered} \operatorname{rot} \boldsymbol{H}_T = \boldsymbol{H}_P; \quad \operatorname{rot} \boldsymbol{E}_P = 0; \quad \frac{\partial \boldsymbol{D}_P}{\partial t} = -\frac{\gamma}{\eta}\boldsymbol{H}'_T; \\ \operatorname{div}(\boldsymbol{H}_T, \boldsymbol{H}_P) = 0; \quad \operatorname{div} \boldsymbol{D}_P = \rho; \quad \boldsymbol{D}_P = \varepsilon \boldsymbol{E}_P; \\ \boldsymbol{B}_{P,T} = \mu \boldsymbol{H}_{P,T}; \qquad \boldsymbol{D}_{P,T} = \varepsilon \boldsymbol{E}_{P,T}. \end{gathered} \tag{7}$$

Here ρ is the electric charge density, ε is dielectric permeability. A specific feature of equations (7) is the fact that they reflect the occurrence of excitation of a Toroidal magnetic field $\boldsymbol{H}'_T$ by a rapidly varying in time electric induction. In this case, the equation for a temporal derivative of electric induction is a symmetric reflection of the second equation from (5). Actually, if we assume rot $\sim \frac{1}{L}$, where L is a characteristic size of a local domain with a magnetic field, then $\frac{\partial \boldsymbol{B}_P}{\partial t} \sim -\frac{1}{L}\boldsymbol{E}_T$. The coefficient $\frac{\gamma}{\eta}$ from (7) has also the dimension of a characteristic length $\frac{\gamma}{\eta} \sim \frac{1}{L}$ [7], i.e., $\frac{\partial \boldsymbol{D}_P}{\partial t} \sim -\frac{1}{L}\boldsymbol{H}_T$. Thus, a force and a non-force electromagnetic fields in a temporal domain are described by equations of a certain symmetry. This fact seems not to have been observed in Physics before.

Combining force and non-force electromagnetic fields into a single system of equations, it appears possible to write down their full modification, called the generalized Maxwell's equations for the force and non-force fields observed on the Earth:

$$\begin{gathered}
\operatorname{rot}\boldsymbol{H}_P = \sigma\boldsymbol{E}_T + \frac{\partial \boldsymbol{D}_P}{\partial t} + \chi\boldsymbol{H}_T + \boldsymbol{J}^{CT}; \qquad \operatorname{rot}\boldsymbol{H}_T = \boldsymbol{H}_P;\\
\operatorname{rot}\boldsymbol{E}_T = -\mu\frac{\partial \boldsymbol{H}_P}{\partial t}; \qquad \operatorname{rot}\boldsymbol{E}_P = 0; \qquad \frac{\partial \boldsymbol{D}_P}{\partial t} = -\chi\boldsymbol{H}'_T;\\
\operatorname{div}\big(\boldsymbol{H}_P, \boldsymbol{H}_T, \boldsymbol{E}_T\big) = 0; \qquad \operatorname{div}\boldsymbol{D}_P = \rho'; \qquad \operatorname{div}\boldsymbol{D}_T = \rho;\\
\boldsymbol{B}_{P,T} = \mu\boldsymbol{H}_{P,T}; \qquad \boldsymbol{D}_{P,T} = \varepsilon\boldsymbol{E}_{P,T}.
\end{gathered} \tag{8}$$

It is natural that equations (8) are valid only in spherical domains and for spherical sources. In ordinary (laboratory) conditions with $\boldsymbol{H}_T = 0$ and $\boldsymbol{E}_P = 0$, equations (8) are automatically transformed to standard Maxwell's equations.

In a constant electromagnetic field, similar equations reduce to

$$\begin{gathered}
\operatorname{rot}\boldsymbol{H}_P = \sigma\boldsymbol{E}_T + \frac{\gamma}{\eta}\boldsymbol{H}_T + \boldsymbol{J}^{CT}; \quad \operatorname{rot}\boldsymbol{H}_T = \boldsymbol{H}_P; \quad \operatorname{rot}\boldsymbol{E}_{P,T} = 0;\\
\operatorname{div}(\boldsymbol{H}_P, \boldsymbol{H}_T, \boldsymbol{E}_T) = 0; \qquad \operatorname{div}\boldsymbol{D}_P = \rho';\\
\boldsymbol{B}_{P,T} = \mu\boldsymbol{H}_{P,T}; \qquad \boldsymbol{D}_{P,T} = \varepsilon\boldsymbol{E}_{P,T}.
\end{gathered} \tag{9}$$

Equations (8) and (9) allow another formulation of the potentiality principle of the electromagnetic field in the Earth's atmosphere, i.e.,

$$\oint (\boldsymbol{H}_T \cdot \boldsymbol{dl}) = \int (\operatorname{rot}\boldsymbol{H}_T \cdot \boldsymbol{ds}) = \int \boldsymbol{H}_{Pr}\, ds \neq 0. \tag{10}$$

Wherever the force magnetic field $\boldsymbol{H}_P$ ($\boldsymbol{H}_{Pr} \neq 0$) exists, there can also exist a non-force magnetic field $\boldsymbol{H}_T$, in the event that $\boldsymbol{H}_{Pr}$ is not equal to zero. In other words, the potentiality condition of a force magnetic field in the atmosphere ($\operatorname{rot}\boldsymbol{H}_P = 0$) in the non-force magnetic field $\boldsymbol{H}_T$ is not valid, because $\operatorname{rot}\boldsymbol{H}_T = \boldsymbol{H}_P$. Therefore, experiments and their interpretation declaring the presence of a non-potential magnetic field in the Earth's atmosphere [6, 12, 13] are conceptually true. Criticism of these works is collapsing due to the occurrence of a non-force electromagnetic field in the Earth's atmosphere in the context of the impact of the Earth's spherical property on sources of the natural electromagnetic field located both in the ionosphere and in the Earth's layers.

Efficient algorithms for interpreting observations on the worldwide network of magnetic observatories and interpretation of the observed electromagnetic field were developed using equations (8) and (9), i.e., with allowance for a non-force modification of the Earth's electromagnetic field [5]. In [5], sources of a non-force electromagnetic field, observed in the Earth's atmosphere were also found there. These sources are spherical electric conductivity currents, which circulate in spherical layers of the Earth as well as electric currents in the ionosphere and exciting variations of the MGF.

A group of the Lorenz transforms for systems (8) and (9) is obviously the same as that for the force part of Maxwell's equations.

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