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Inverse problems of plane wave scattering by 1D inhomogeneous layers

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Abstract. Inverse problems of the plane waves scattering of inclined incidence of the SH type by an inhomogeneous half-space (in particular, by a transition layer) or by an inhomogeneous layer with a free boundary are considered. The characteristics of an inhomogeneous elastic medium, i.e., the wave propagation velocity $v(z)$ and the density $\rho(z)$, are functions of depth z and should be determined in the inverse problems. The following are considered: the inverse problems with data for a fixed angle of incidence θ_0 of a plane wave (the angle between the vector of the normal to the plane wave front and z-axis) when the shape $\varphi_0(\xi, \theta_0)$ of an incident wave is known; the inverse problems with data for a family of angles θ_0 both for known and unknown shapes of the incident wave. The functions $\varphi_0(\xi, \theta_0)$ and $\varphi_1(\xi, \theta_0)$ (the shapes of incident and reflected waves), the functions $\varphi_0(\xi, \theta_0)$ and $u(H, \xi, \theta_0)$ (the shapes of the incident wave and the free boundary oscillation field), only the function $u(H, \xi, \theta_0)$ (in the inverse problem with the unknown function $\varphi_0(\xi, \theta_0)$), or other data are given as data corresponding to any fixed value of θ_0. Possible application areas of the inverse problems under consideration are specified.

This paper is essentially a review. A new result presented here has been obtained for inverse problems with the data $\{\varphi_0(\xi, \theta_0), \varphi_1(\xi, \theta_0)\}$ or $\{\varphi(\xi, \theta_0), u(H, \xi, \theta_0)\}$ for a set of angles θ_0. The results [9] (the uniqueness theorem, the solution method) are extended to the case when the limiting point $\bar{\theta}_0$ of this set is zero. Also, a new explicit formula (formula (17) from Section 5), which makes it possible to find the functions $v(z)$ and $\rho(z)$ using the data of these inverse problems for $\bar{\theta}_0 = 0$, has been obtained.

1. Introduction

The inverse problems of plane wave scattering by inhomogeneous layers belong to inverse dynamic problems of wave propagation theory [1, 2]. They consist, first of all, in determining the characteristics of an inhomogeneous elastic medium filling the layer (the wave propagation velocity $v(z)$ and the density $\rho(z)$ as functions of depth z). This is done with the information about the oscillation field (measured by devices or obtained by solving the direct problem) at some point located outside the layer or at one of its boundaries. An inhomogeneous layer can be both a transition one (an inhomogeneous half-space) and the one with a free or a fixed boundary. In both cases, a plane wave of the shape $\varphi_0(\xi, \theta_0)$ is incident at an angle θ_0 from a homogeneous half-space with characteristics v_0 and ρ_0 on the inhomogeneous layer (half-space). This plane wave generates a reflected wave of the shape $\varphi_1(\xi, \theta_0)$ in the homogeneous half-space and an oscillation field $u(z, \xi, \theta_0)$ in the inhomogeneous layer.

Papers [3–12] have shown that the statements of these inverse problems can be rather manifold. This is caused, first, by the fact that in the inverse problems one can specify information of different character and volume. If the shape $\varphi_0(\xi, \theta_0)$ is known, then as initial data one can specify either the reflected wave shape $\varphi_1(\xi, \theta_0)$ or the oscillation field u of a fixed medium point at one of the boundaries, for instance, at the free boundary [3–7, 9–12]. In other statements [8], the shape $\varphi_0(\xi, \theta_0)$ of the incident wave is unknown and subject to determination as well as the medium's characteristics $v(z)$ and $\rho(z)$. The above-mentioned initial data can be specified both for one fixed value of the parameter θ_0 (the angle of incidence) [3–7, 10–12] and for a set of angles θ_0 [8, 9].

Second, as shown in [3], depending on the angle of incidence θ_0 and the characteristics v_0 and $v(z)$ of a medium, a differential equation in a reduced (to a less number of independent variables) problem of the form

$$\frac{\partial u^2}{\partial z^2} + \frac{d}{dz}\{\ln \mu(z)\}\frac{\partial u}{\partial z} = \left\{\frac{1}{v^2(z)} - \frac{\sin^2\theta_0}{v_0^2}\right\}\frac{\partial^2 u}{\partial \xi^2} \tag{1}$$

can be of different types: hyperbolic, elliptic, and mixed depending on the sign of the coefficient in the right-hand side. The variable ξ has the dimensionality of the time t, and if $\theta_0 = 0$ (normal incidence) coincides with t. In the general case of arbitrary θ_0, $\xi = t - x\sin\theta_0/v_0$, where x is the "horizontal" spatial variable.

The case of inclined incidence of a plane wave and inverse problems, where the information about the field is specified for a series of angles θ_0, should be investigated for the following reason. Let us consider a well-known case of normal incidence ($\theta_0 = 0$) [1, 2, 13–17], from which the investigation of the inverse problems of plane wave scattering started. In the inverse problem, neither the functions $v(z)$ or $\rho(z)$, but only some auxiliary intermediate function $\sigma_0(x)$ is uniquely reconstructed. This intermediate function is obtained from the function $\sigma_0(z) = \rho(z)v(z)$ (acoustic stiffness) by the substitution of variables,

$$x = \int_0^z \frac{dz}{v(z)}, \qquad \sigma_0(x) = \sigma_0(z). \tag{2}$$

Thus, using the information specified in the inverse problem for the case $\theta_0 = 0$, we can determine only the acoustic stiffness $\sigma_0(x)$ as function of the wave travel time x along the section $[0, z]$. The travel time $x(z)$ and the properties $\rho(z)$ and $v(z)$ as functions of depth remain unknown.

In this connection, the following questions are of interest:

1. A more complete determination of elastic properties of the inhomogeneous layer, for example, the functions $v(z)$ and $\rho(z)$;
2. Investigation of the general case of an arbitrary inclined incidence of the plane wave.

A series of papers [3–12] concern the solution to these two problems for the case when the incident wave is an elastic transverse wave of the SH type. Let us clarify the origin of equation (1). The initial Lamé system of equations of elasticity theory is reduced for SH waves to a scalar wave equation with independent variables x, z, and t. In turn, this wave equation can be reduced to an equation of the form of (1) with two independent variables z and $\xi = t - x \sin\theta_0/v_0$. The results of these papers are summarized in [23, 24] and [25].

In the case that $v(z) < v_0/\sin\theta_0$ (the velocity in the inhomogeneous layer is smaller than the apparent velocity, that is, the angle of incidence is subcritical), differential equation (1) in the reduced problem is of the hyperbolic type, being a string equation with the velocity depending on the parameter θ_0. This (hyperbolic) case is an analog to the case of supersonic gas flows in gas dynamics. The above-mentioned case of normal incidence of a plane wave ($\theta_0 = 0$) is a hyperbolic one. The hyperbolic case belongs to such a well-developed direction in the theory of inverse problems as those inverse problems for the Sturm–Liouville equation, the string equation and for hyperbolic equations. These are publications by V.A. Marchenko, M.G. Krein, I.M. Gelfand–B.M. Levitan, A.S. Alekseev, M.M. Lavrentiev, V.G. Romanov, A.S. Blagoveshchenskii, S.I. Kabanikhin, and their co-workers. Bibliography for this direction can be found in [1, 2, 18–26].

At inclined incidence ($\theta_0 \neq 0$), in the hyperbolic case only some auxiliary intermediate function $\sigma(x, \theta_0)$ is uniquely reconstructed (as shown in [3, 4, 6]) with the use of the information given for one fixed angle θ_0. This function coincides with the function $\sigma_0(x)$ at $\theta_0 = 0$. The functions $v(z)$ and $\rho(z)$ are not determined in the general case [3, 4, 6] (these results were formulated in Section 3). Therefore, inverse problems of the plane wave scattering with a greater volume of information for a set of different angles θ_0 were stated in [8, 9]. In [8, 9], the sequence Θ_0 of the angles θ_0 with a limiting point was taken as a set providing the uniqueness of determination of $v(z)$ and $\rho(z)$. The set of functions $\big\{\varphi_0(\xi, \theta_0), \varphi_1(\xi, \theta_0)\big\}$ (or $\big\{\varphi(\xi, \theta_0), u(H, \xi, \theta_0)\big\}$, or $\{u(H, \xi, \theta_0)\}$, or other information) for any $\theta_0 \in \Theta_0$ serves as initial data in the inverse problem. Such inverse problems were formulated in Section 4. In [9], an explicit formula, which makes it possible to calculate the function $v(z)$ in terms of the function $\sigma(x, \theta_0)$ and its first-order partial derivatives with respect to x and θ_0, was obtained. This formula (formula (16)) is given in Section 5. A method and a numerical algorithm to reconstruct $v(z)$ and $\rho(z)$ as functions of the depth z, based on this explicit formula, are proposed in [9]. A family of plane waves incident to the inhomogeneous layer and reflected from it at different angles θ_0 is used in the reconstruction. The results of numerical experiments with this algorithm (for the transition layer) are also considered and analyzed there. This method is briefly described in Sections 6 and 7.

It should be noted that when a difference analog of the above-mentioned explicit formula is used in the numerical solution, it is sufficient to specify the information $\{\varphi_0(\xi,\theta_0),\varphi_1(\xi,\theta_0)\}$ or other data that provide the uniqueness of determining the function $\sigma(x)=\sigma(x,\theta_0)$ at a fixed θ_0, for instance, the function $\{\varphi_0(\xi,\theta_0),u(H,\xi,\theta_0)\}$ ($u(H,\xi,\theta_0)$ is the field of displacements on the free boundary $z=H$), only for two or three angles θ_0 [9, 23, 25]. Therefore, one of the authors of this paper suggested that for the uniqueness of the solution to the inverse problem of determining the medium characteristics $v(z)$ and $\rho(z)$ it is sufficient to know the function $\sigma(x,\theta_0)$ (that is, the data of the inverse problem with a fixed angle) for two different angles θ_0.

Such an inverse problem was formulated and offered to V.A. Gorbunov for further investigation. It turned out that the functions $v(z)$ and $\rho(z)$ are actually determined uniquely with the use of the functions $\sigma(x,\theta_0^1)$ and $\sigma(x,\theta_0^2)$ (and hence, with a set $\{\varphi_0(\xi,\theta_0),\varphi_1(\xi,\theta_0)\}$ or other information that provides the uniqueness of determining the function $\sigma(x,\theta_0)$ at a fixed θ_0 given for two different angles, θ_0^1 and θ_0^2). This result was obtained for the analytical functions $v(z)$ by V.I. Dobrinsky and V.A. Gorbunov in [26].

Statements of inverse problems for an inhomogeneous layer with a free (or fixed) boundary, where the shape $\varphi_0(\xi,\theta_0)$ of the incident wave and the characteristics $v(z)$ and $\rho(z)$ are unknown and reconstructed in solution to inverse problems are considered in [8]. In this case, certain restriction is imposed on the Fourier transform of the function $\varphi_0(\xi,\theta_0)$. Also, we make use of analytic features of some functions with respect to the parameter θ_0 in terms of which the solution $u(z,\xi,\theta_0)$ to the direct problem is expressed. These results are briefly described in Section 8.

An inverse problem of the plane wave scattering with a hyperbolic equation of the general form that generalizes the wave equation considered in [3–12] is studied by A.S. Blagoveshchensky and K.E. Voyevodsky in [27].

Inverse problems of the plane wave scattering by an inhomogeneous layer with a free boundary in the hyperbolic case when the medium characteristics v and ρ are assumed to be continuous at the contact of the inhomogeneous layer with the homogeneous half-space were investigated by A.S. Alekseev and V.S. Belonosov [28, 29]. In this case, the equivalence of different statements of the inverse problems of the plane wave scattering was established.

The case $v(z) > v_0/\sin\theta_0$ (the velocity in the inhomogeneous layer is higher than the apparent velocity, that is, the angle of incidence is supercritical) corresponds to the elliptic type equation (of the form (1)) in the reduced problem. The direct and inverse problems corresponding to this (elliptic) case are considered in [7–10, 23, 25]. The hyperbolic case generalizes the problem of normal incidence. The elliptic case qualitatively differs from the hyperbolic one, and corresponds to total internal reflection. It is an analog to the case of subsonic flows in gas dynamics.

The direct and the inverse problems of the plane wave scattering by inhomogeneous layers for the case of the function with an alternating sign $\{v(z)-\sin\theta_0/v_0\}$ (the mixed case) corresponding to the mixed-type equation of form (1) in the reduced problem were first considered in [11, 12]. Further results for this case were obtained by M.I. Belishev [30]. The mixed case is an analog to the case of transonic flows in gas dynamics. The mixed-type equation that corresponds to equation (1) is known in gas dynamics as the Chaplygin equation for the stream function, as noted in [24]. However, direct problems, and, besides, for a limited domain of a special kind (such as Trikomi, Frankl, and other problems), have been conventionally studied in gas dynamics and theory of mixed type equations.

In all the above-mentioned works dealt with inverse problems of the plane wave scattering, except for [8], the shape $\varphi_0(\xi,\theta_0)$ of the incident wave is assumed to be known. It should be noted that the 1D inverse problem for the string equation (when the velocity $v(z)$ does not depend on the parameter) with an unknown source of the form $f(t)$ was investigated by M.L. Gerver [31]. In this case, the uniqueness of the solution to the inverse problem is provided by the following condition for the Fourier transform $\tilde{f}(k)$ of the function $f(t)$: the totality of squares of all zeroes of $\tilde{f}(k)$ and the spectrum of the corresponding boundary problem must not intersect. A.S. Blagoveshchensky [32] considers an inverse problem for the wave equation with an unknown source where two moments of the function $u|_{z=0} = f(x,t)$ are specified (u is the solution to the direct problem) with the "horizontal" variable x of the form $\int_{-\infty}^{\infty} f(x,t)\,dx$ and $\int_{-\infty}^{\infty} x^2 f(x,t)\,dx$.

In this paper, the results of [9] (the uniqueness theorem, the solution method) for inverse problems with the information $\{\varphi_0(\xi,\theta_0),\ \varphi_1(\xi,\theta_0)\}$ or $\{\varphi_0(\xi,\theta_0),\ u(H,\xi,\theta_0)\}$ given for the set Θ_0 of the angles θ_0 are extended to the case when the limiting point $\bar\theta_0$ of the set Θ_0 of angles can be a point $\theta_0 = 0$. Also, a new explicit formula (formula (17) in Section 5) has been obtained. This formula makes it possible to find the functions $v(z)$ and $\rho(z)$ using the data of these inverse problems in the case that $\bar\theta_0 = 0$. (In the case that $\bar\theta_0 \neq 0$, a similar formula (formula (16)) was obtained in [9]). Possible application areas of the inverse problems under discussion are considered in Section 9.

2. Direct problems

The consideration below is constructed so that it simultaneously covers both the case of a scattering inhomogeneous half-space and the case of a scattering inhomogeneous layer with a free boundary. In [3, 4], it is shown that the problem for the inhomogeneous half-space is reduced in the variables z and ξ to the problem for the half-plane $0 \le z < \infty$, $-\infty < \xi < \infty$; this problem will be called the HS ("half-space") problem. The problem of an inhomogeneous layer with a free boundary, as shown in [6], is reduced to the problem in the band $0 \le z \le H$, $-\infty < \xi < \infty$; this problem will be called the FB ("free boundary") problem.

Let D denote a domain on the plane z, ξ of one of the two kinds: the half-plane $0 < z < \infty$, $-\infty < \xi < \infty$ in the HS problem and the band $0 < z < H$, $-\infty < \xi < \infty$ in the FB problem.

Direct HS and FB problems. Let there be given the equation of the hyperbolic type

$$\frac{\partial^2 u}{\partial z^2} + \frac{d}{dz}\{\ln \mu(z)\}\frac{\partial u}{\partial z} = \frac{1}{\bar{v}^2(z,\theta_0)}\,\frac{\partial^2 u}{\partial \xi^2}, \quad (z,\xi) \in D \tag{3}$$

in the domain D, where

$$\frac{1}{\bar{v}^2(z,\theta_0)} = \frac{1}{v^2(z)} - \frac{\sin^2\theta_0}{v_0^2}; \tag{4}$$

the boundary condition with an inclined derivative

$$\frac{\partial u}{\partial z}(0,\xi,\theta_0) - \varkappa(\theta_0)\,\frac{\partial u}{\partial \xi}(0,\xi,\theta_0) = \nu(\theta_0)\,\frac{\partial \varphi_0(\xi,\theta_0)}{\partial \xi} \tag{5}$$

at $z = 0$, $-\infty < \xi < \infty$, where

$$\varkappa(\theta_0) = \frac{\delta(\theta_0)}{\mu(0)}, \quad \delta(\theta_0) = \sigma^0\cos\theta_0, \quad \sigma^0 = \frac{\mu_0}{v_0}, \quad \nu(\theta_0) = -2\varkappa(\theta_0); \tag{6}$$

and the initial condition

$$u(z,\xi,\theta_0)\big|_{\xi<0} \equiv 0, \qquad z \ge 0. \tag{7}$$

Conditions (3)–(7) are specified in both HS and FB problems.

In addition, the following boundary condition is specified in the FB problem:

$$\frac{\partial u}{\partial z}\Big|_{z=H} = 0, \qquad -\infty < \xi < \infty. \tag{8}$$

The constants v_0, μ_0 and the functions $v(z)$ and $\mu(z)$ are strictly positive; Any of the conditions ($v(0) = v_0$, $v(0) \neq v_0$, $\mu(0) = \mu_0$, or $\mu(0) \neq \mu_0$) is allowed. That is, a discontinuity of the characteristics v and μ of the medium at the interface $z = 0$ of the inhomogeneous layer and the homogeneous half-space is allowed. (The interface $z = 0$ is assumed to be rigid, i.e., displacements and stresses are continuous when passing through the boundary $z = 0$; hence, conditions (5) and (9) follow.) The function φ_0 can depend on θ_0: $\varphi_0 = \varphi_0(\xi, \theta_0)$ and has the property $\varphi_0(\xi, \theta_0)|_{\xi<0} = 0$.

Let the numbers θ_0, v_0, and μ_0 and the functions $v(z)$, $\mu(z)$, and $\varphi_0(\xi, \theta_0)$ be given. It is necessary to find the function $u(z, \xi, \theta_0)$ satisfying conditions (3)–(7) in the HS problem and conditions (3)–(8) in the FB problem. This problem is called the direct HS or FB problem, respectively.

It is shown in [3, 4, 6] that the problem of the plane waves scattering of inclined incidence of the SH type by an inhomogeneous elastic half-space (and by an inhomogeneous layer with a free boundary) is reduced to the HS problem with conditions (3)–(7) (and to the FB problem with conditions (3)–(8)) in the following case:

$$v(z) < v_0/\sin\theta_0, \qquad z \geq 0. \tag{H}$$

We call this case *hyperbolic*. Here $v(z)$ and $\mu(z)$ are the wave propagation velocity and the shear modulus in the inhomogeneous medium–half-space $z \geq 0$ or the layer $0 \leq z \leq H$, and v_0 and μ_0 are the wave propagation velocity and shear modulus in the homogeneous half-space $z < 0$ from which a plane wave of the shape $\varphi_0(\xi, \theta_0)$ $(-\infty < \xi < \infty)$ is incident; θ_0 $(0 \leq \theta_0 < \pi/2)$ is its angle of incidence; $u(z, \xi, \theta_0) = u(x, z, t, \theta_0)$, where $u(x, z, t, \theta_0)$, is the full field of displacements in the half–space $z \geq 0$ or in the layer $0 \leq z \leq H$; $\xi = t - x\sin\theta_0/v_0$; t is the physical time, and x and z are the spatial variables. The condition (H) is always satisfied at sufficiently small angles θ_0; in particular, at $\theta_0 = 0$ (the normal incidence). With SH-waves the vector of displacements $\bar{u}$ is perpendicular to the normal to the incident wave front and parallel to the boundary $z = 0$ between the inhomogeneous medium and homogeneous half-space (Figures 1 and 2). That is, it is directed along the axis Oy. The choice of a system of coordinates x, z, t is clear from Figure 1.

The shape $\varphi_1(\xi, \theta_0)$ of the plane wave reflected from the inhomogeneous half-space is unknown and must be determined both in the direct HS and FB problem. The function $\varphi_1(\xi, \theta_0)$ is associated with the solution $u(z, \xi, \theta_0)$ to the direct HS and FB problem by the boundary condition

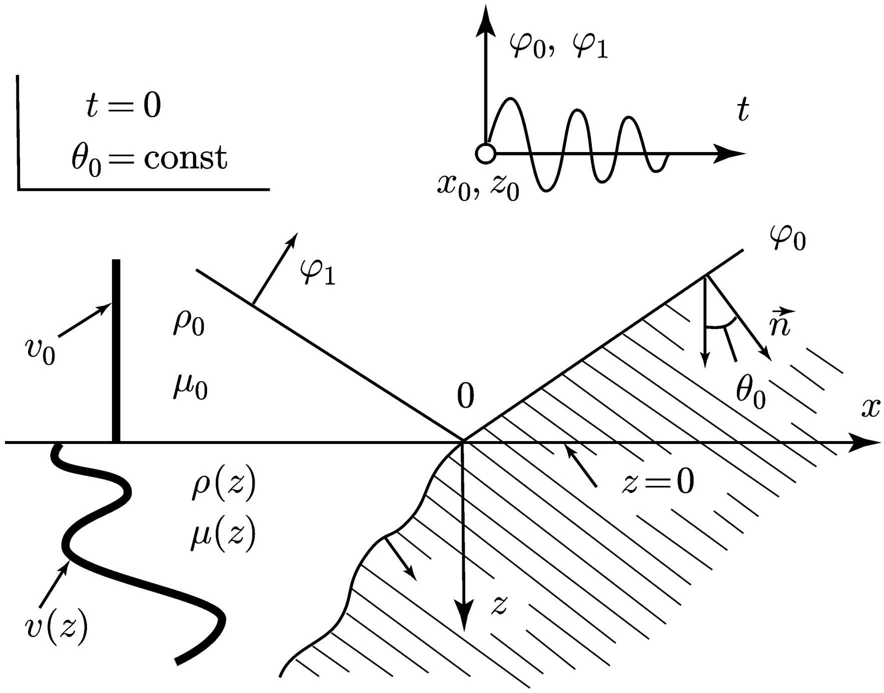

Figure 1. The problem of the plane waves scattering on an inhomogeneous elastic half-space (in particular, on the transition layer) in the coordinates x, z, t

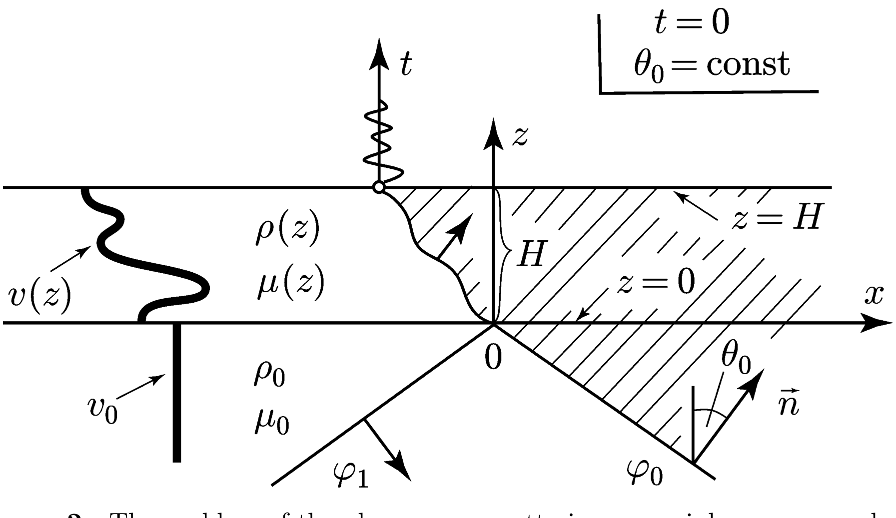

Figure 2. The problem of the plane waves scattering on an inhomogeneous layer with a free or fixed boundary in the coordinates x, z, t

$$u(0,\xi,\theta_0) = \varphi_0(\xi,\theta_0) + \varphi_1(\xi,\theta_0). \tag{9}$$

The following equalities are valid:

$$\rho(z) = \mu(z)/v^2(z), \qquad \sigma_0(z) = \rho(z)v(z),$$

where $\rho(z)$ is the density and $\sigma_0(z)$ is the acoustic stiffness of the medium.

Uniqueness theorems were proved for the direct HS problem in [4] and for the direct FB problem in [6].

3. Inverse problems 1 (with a fixed value of θ_0)

Inverse 1HS and 1FB problems. Let the functions $\varphi_0(\xi,\theta_0)$ and $\varphi_1(\xi,\theta_0)$ (the shapes of the incident and reflected waves) be specified at $0 \le \xi < \infty$ and at some fixed (generally speaking, unknown) value $\theta_0 \in [0,\pi/2)$ satisfying the condition (H) in the inverse 1HS problem. Also, let the functions $\varphi_0(\xi,\theta_0), \varphi_1(\xi,\theta_0)$ or $\varphi_0(\xi,\theta_0), u(H,\xi,\theta_0)$ (the incident wave shapes and oscillations of the free boundary $z = H$) be given in the inverse 1FB problem. Thus, two variants of specifying information in the inverse 1FB problem are possible. The function $\varphi_1(\xi,\theta_0)$ is associated, by condition (9), with the solution $u(z,\xi,\theta_0)$ to the direct HS or FB problem with some unknown (at $0 \le z < \infty$ or $0 \le z \le H$, respectively) functions $v(z)$ and $\mu(z)$ and the unknown numbers θ_0, v_0, and μ_0. The number $\delta = \sigma^0 \cos\theta_0$, where $\sigma^0 = \mu_0/v_0$, is also specified. To know the number δ, it is sufficient, for instance, to specify the numbers $v^* = v_0/\sin\theta_0$ and μ_0 (v^* is the front velocity along the boundary $z = 0$ or $z = H$). We shall obtain such information about the functions $\varphi_0(\xi,\theta_0)$, $\varphi_1(\xi,\theta_0)$, or $u(H,\xi,\theta_0)$ if we measure the incident and the reflected waves as functions of time $t \in (-\infty,\infty)$ at some fixed point (x_0,y_0,z_0) of the domain $z \le 0$ or the field of oscillations as a function t at a fixed point (x_0,y_0,H) of the free boundary. These data will be called the data of the inverse 1HS and 1FB problem, respectively; they relate to a fixed value of the parameter θ_0.

It is necessary to determine, in the general case, the function $\sigma(x) = \sigma(x,\theta_0)$ for the considered fixed θ_0 at $0 \le x < \infty$ in the inverse 1HS problem and at $0 \le x \le x_H$ in the inverse 1FB problem. The number x_H in the 1FB problem must also be determined. By definition, the function $\sigma(x,\theta_0)$ is obtained from the function

$$\sigma(z,\theta_0) = \frac{\mu(z)}{\bar{v}(z,\theta_0)} \tag{10}$$

by changing the variables

$$x = \int_0^z \frac{dz}{\bar{v}(z,\theta_0)} \equiv x(z,\theta_0) \quad \Rightarrow \quad z = \int_0^x \bar{v}(x,\theta_0)\,dx \equiv z(x,\theta_0), \tag{11}$$

$$\sigma(x,\theta_0) = \sigma(z,\theta_0) \quad \Rightarrow \quad \sigma(z,\theta_0) \equiv \sigma(x(z,\theta_0),\theta_0). \tag{12}$$

In this case (by definition) $x_H = x(H,\theta_0)$,

$$\bar{v}(x,\theta_0) = \bar{v}(z,\theta_0), \qquad \mu(x,\theta_0) = \mu(z) \tag{13}$$

(the symbols of the functions σ, $\bar{v}$, and μ are retained), so that

$$\sigma(x,\theta_0) = \frac{\mu(x,\theta_0)}{\bar{v}(x,\theta_0)}. \tag{14}$$

In particular cases, the statement of the inverse 1HS and 1FB problems changes in the following way. Let Z denote the interval $0 \le z < \infty$ in the HS problem and the interval $0 \le z \le H$ in the FB problem. Let the data of the functions $\varphi_0(\xi,\theta_0)$, $\varphi_1(\xi,\theta_0)$, and $u(H,\xi,\theta_0)$ in the inverse 1HS and 1FB problems be the same as in the general case, and let the number δ be specified.

If $\mu(z) = \text{const} = \mu$ and the numbers μ and v^* are specified, it is necessary to find the function $v(z)$ at $z \in Z$ and, in addition, the number H (the layer thickness) in the inverse 1FB problem. If the number v^* and the function $v(z)$ at $z \in Z$ are specified, it is necessary to find the function $\mu(z)$ at $z \in Z$.

The inverse 1HS problem was formulated and investigated in [4], and the inverse 1FB problem — in [6]. With the help of the representation for the solution $u(z,\xi,\theta_0)$ to the direct HS and FB problems obtained in these papers, the inverse 1HS and 1FB problems are reduced to the inverse problem of determining the regular Sturm–Liouville operator by the spectral function and two spectra, respectively. Combining the uniqueness theorems for solving the inverse problems proved in [4, 6] in one statement, we come to

Theorem 1. *Let us denote*

$$B(x) = \frac{1}{2}\frac{\partial}{\partial x}\{\ln\sigma(x,\theta_0)\}, \qquad q(x) = \frac{\partial B}{\partial x} + B^2(x).$$

Let the following conditions (in both inverse 1HS and 1FB problems) be satisfied:

(A) *the derivatives d^2v/dz^2 and $d^2\mu/dz^2$ are continuous at $z \in Z$;*

(B) *at a given θ_0 the function $\varphi_0(\xi,\theta_0)$ and its derivatives with respect to ξ up to fourth order are continuous and absolutely integrable in the straight line $-\infty < \xi < \infty$.*

Also, let the condition

(C) $\displaystyle\int_0^\infty x|q(x)|\,dx < \infty, \quad \int_0^\infty B^2\,dx < \infty, \quad \int_0^\infty |B|\,dx < \infty,$

be satisfied in the inverse 1HS problem, and the condition

$$\text{(D)} \quad \frac{\partial \sigma(z,\theta_0)}{\partial z}\Big|_{z=H} = 0$$

be satisfied in the inverse 1FB problem.

Then the inverse 1HS problem has the unique solution $\sigma(x,\theta_0)$, and the inverse problem 1FB has the unique solution $\{\sigma(x,\theta_0), x_H\}$ (in the particular cases formulated, the solution $v(z)$ or $\mu(z)$).

Condition (C) is satisfied, for instance, for the functions $v(z)$ and $\mu(z)$, which are constant outside some finite interval $[0,H]$ that physically corresponds to the transition layer with the variable velocity $v(z)$ and the density $\rho(z)$. Condition (D) is satisfied, for instance, if we have $dv/dz = d\mu/dz = 0$ at the point $z = H$. It follows from conditions (A) and (C) that the functions $v(z)$, $\mu(z)$, and $\rho(z)$ can be non-monotone.

Remark 1. In the inverse 1HS problem, the number $\sigma(0,\theta_0)$, and in the inverse problem 1FB, any number ($\sigma(0,\theta_0)$ or $\sigma(H,\theta_0)$) can be given instead of the number δ. In all cases (the general case and particular ones), in both inverse problems (1HS and 1FB), the number $\delta = \cos\theta_0\mu_0/v_0$ must be found and is determined uniquely if it is not specified.

A numerical algorithm to reconstruct the function $\sigma(x,\theta_0)$ in the inverse 1HS problem both when the function $\varphi_1(\xi,\theta_0)$ is specified precisely and in the presence of noise (at different angles θ_0) was proposed and tested in [9].

It should be noted that the function $\sigma(x,\theta_0)$ depends on θ_0 for the following two reasons: 1) $\sigma(z,\theta_0)$ depends on θ_0; and 2) $x(z,\theta_0)$ depends on θ_0. At different $\theta_0^{(1)}$ and $\theta_0^{(2)}$, the functions $\sigma(x,\theta_0^{(1)})$ and $\sigma(x,\theta_0^{(2)})$, as shown by the calculations in [9], can essentially differ for the same pair of functions, $v(z)$ and $\mu(z)$. Notice also that at $\theta_0 = 0$, the function $\sigma(x,\theta_0)$ coincides with the function $\sigma_0(x)$ (determined by equalities (2)) from the problem of normal incidence of the plane wave: $\sigma(x,0) = \sigma_0(x)$. Thus, in the general case in inverse problems 1, only the intermediate function $\sigma(x,\theta_0)$ is determined from the data $\{\varphi_0(\xi,\theta_0), \varphi_1(\xi,\theta_0)\}$ or $\{\varphi_0(\xi,\theta_0), u(H,\xi,\theta_0)\}$ for one value of θ_0. The functions $v(z)$, $\mu(z)$ (and $\rho(z)$) are not determined. In order to determine them, let us formulate other inverse problems, where the volume of specified information is greater than in inverse problems 1.

4. Inverse problems 2 (with data for the family of angles $\boldsymbol{\theta_0}$ and the known function $\boldsymbol{\varphi_0(\xi,\theta_0)}$)

In contrast to inverse problems 1, where formula (4) for the function $\bar{v}(z,\theta_0)$ does not play any role, in inverse problems 2 the dependence of the functions

$\bar{v}(z,\theta_0)$ and $\sigma(z,\theta_0)$ on θ_0 of forms (4) and (10) is essentially used, which takes place in the problem with plane waves.

Inverse 2HS and 2FB problems. Let Θ_0 denote an infinite monotone decreasing (or increasing) sequence of real numbers $\theta_0^{(1)}, \theta_0^{(2)}, \dots, \theta_0^{(n)}, \dots$ such that

1) each $\theta_0^{(n)} \in \Theta_0$ belongs to the interval $[0, \pi/2)$ and satisfies the condition (H);

2) the sequence Θ_0 converges to some limit point $\bar{\theta}_0$: $\lim_{n\to\infty} \theta_0^{(n)} = \bar{\theta}_0$, $\bar{\theta}_0 \in [0, \pi/2)$, and the condition (H) is satisfied at $\theta_0 = \bar{\theta}_0$.

Let the following be specified at $0 \le \xi < \infty$ and for any $\theta_0 \in \Theta_0$: the functions $\varphi_0(\xi,\theta_0)$ and $\varphi_1(\xi,\theta_0)$ (the shapes of incident and reflected waves) in the inverse 2HS problem, and either the functions $\varphi_0(\xi,\theta_0)$, $\varphi_1(\xi,\theta_0)$, or the functions $\varphi_0(\xi,\theta_0)$, $u(H,\xi,\theta_0)$ (where $u(z,\xi,\theta_0)$ is the solution to the direct FB problem) in the inverse problem 2FB. Moreover, it is known to which value of $\theta_0 \in \Theta_0$ the functions $\varphi_0(\xi,\theta_0)$ and $\varphi_1(\xi,\theta_0)$ or $\varphi_0(\xi,\theta_0)$ and $u(H,\xi,\theta_0)$ correspond. The function $\varphi_1(\xi,\theta_0)$ is associated with the solution $u(z,\xi,\theta_0)$ to the direct HS or FB problems, respectively, by the condition (9). The functions $v(z)$ and $\mu(z)$ are unknown at $0 \le z < \infty$ in the inverse problem 2HS and at $0 \le z \le H$ in the inverse 2FB problem, in which the number H is also unknown. The numbers v_0 and μ_0 in both inverse problems, 2HS and 2FB, are also specified. Thus, data for the inverse problem 1HS or 1FB for any $\theta_0 \in \Theta_0$ are specified.

It is necessary to determine the functions $v(z)$ and $\mu(z)$ at $0 \le z < \infty$ or at $0 \le z \le H$ in the inverse 2HS or 2FB problem, respectively. Moreover, the number H must be determined in the inverse 2FB problem.

Inverse problems 2 are formulated in [9] (see also [23, 25]), where the uniqueness theorem is proved and a method to solve them for the case $\bar{\theta}_0 \ne 0$ is proposed. A similar theorem proved in [9] is generalized in the theorem given below in that any of the conditions ($\bar{\theta}_0 \ne 0$ or $\bar{\theta}_0 = 0$) is allowed.

Theorem 2. *Let condition* (A) *of Theorem 1 be satisfied, and let the function $\varphi_0(\xi,\theta_0)$ at any fixed $\theta_0 \in \Theta_0$ satisfy condition* (B) *of Theorem 1. In addition, let, for any $\theta_0 \in \Theta_0$, condition* (C) *of Theorem 1 be satisfied in the inverse 2HS problem, and let its condition* (D) *be satisfied for the inverse 2FB problem. That is, let the conditions of Theorem 1 for any $\theta_0 \in \Theta_0$ be satisfied.*

Then the inverse 2HS problem has the unique solution $\{v(z), \mu(z)\}$, and the inverse problem 2FB has the unique solution $\{v(z), \mu(z), H\}$.

Let us prove Theorem 2 and give a method to solve inverse problem 2, according to the method described in [9]. As in [9], the explicit formulas that express the function $\bar{v}(x, \theta_0)$ in terms of the function $\sigma(x, \theta_0)$ and its partial derivatives with respect to x and θ_0 play an important role in this case.

5. The main identities relating the functions $\bar{v}(x, \theta_0)$, $\sigma(x, \theta_0)$ and partial derivatives of the function $\sigma(x, \theta_0)$ to variables x and θ_0

Lemma 1. *Let $v(z)$ and $\mu(z)$ be continuously differentiable functions, condition* (H) *be satisfied, and the functions $\bar{v}(x, \theta_0)$ and $\sigma(x, \theta_0)$ be determined by formulas* (4) *and* (10)–(13)*. Let us denote*

$$\alpha = \alpha(\theta_0) = -\frac{\sin\theta_0 \cos\theta_0}{v_0^2}.$$

Then the following identities are valid:

$$\alpha(\theta_0)\bar{v}^2(x, \theta_0) = \frac{\partial}{\partial x}\{\ln\sigma(x, \theta_0)\} \cdot \alpha(\theta_0) \int_0^x \bar{v}^2(\xi, \theta_0)\, d\xi + \frac{\partial}{\partial \theta_0}\{\ln\sigma(x, \theta_0)\} \qquad \forall\theta_0 \geq 0 \qquad \forall x \geq 0. \quad (15)$$

This formula can be considered as representation of the derivative $\partial\{\ln\sigma(x, \theta_0)\}/\partial\theta_0$ in terms of the derivative $\partial\{\ln\sigma(x, \theta_0)\}/\partial x$ and the function $\bar{v}(x, \theta_0)$ at any θ_0;

$$\bar{v}^2(x, \theta_0) = \frac{1}{\alpha(\theta_0)}\left\{\frac{\partial}{\partial\theta_0}\{\ln\sigma(x, \theta_0)\} + \frac{\partial\sigma(x, \theta_0)}{\partial x}\int_0^x \frac{\partial\{\ln\sigma(\xi, \theta_0)\}/\partial\theta_0}{\sigma(\xi, \theta_0)}\, d\xi\right\} \qquad \theta_0 \neq 0 \qquad \forall x \geq 0, \quad (16)$$

Formula (16) *expresses, in the explicit form, at any $\theta_0 \neq 0$, the function $\bar{v}(x, \theta_0)$ in terms of the function $\sigma(x, \theta_0)$ and its partial derivatives $\partial\sigma(x, \theta_0)/\partial x$ and $\partial\sigma(x, \theta_0)/\partial\theta_0$;*

$$\bar{v}^2(x, 0) = v^2(x) = -v_0^2\left\{h(x) + \frac{d\sigma_0(x)}{dx}\int_0^x \frac{h(\xi)}{\sigma_0(\xi)}\, d\xi\right\} \quad \forall x \geq 0, \qquad (17)$$

where

$$h(x) = \frac{1}{\sigma_0(x)}\left\{\frac{\partial^2\sigma(x, \theta_0)}{\partial\theta_0^2}\right\}\Big|_{\theta_0=0}. \qquad (18)$$

Here $\sigma_0(x)$ is the function obtained from the function $\sigma_0(z) = \rho(z)v(z)$ with substitution (2)*. Therefore,*

$$\sigma_0(x) \equiv \sigma(x, 0) \equiv \frac{\mu(x)}{v(x)} \equiv \rho(x)v(x). \qquad (19)$$

The functions $v(x) \equiv \bar{v}(x,0)$, $\mu(x)$, and $\rho(x)$ are obtained from the functions $v(z) \equiv \bar{v}(z,0)$, $\mu(z)$, and $\rho(z)$ by the substitution

$$x = \int_0^z \frac{dz}{v(z)} \equiv x(z,0), \quad v(x) = v(z), \quad \mu(x) = \mu(z), \quad \rho(x) = \rho(z) \quad (20)$$

(*the symbols σ_0, v, μ, and ρ with the substitution are retained*).

Proof. The lemma follows directly from definitions (4), (10)–(13) of the functions $\sigma(x,\theta_0)$ and $\bar{v}(x,\theta_0)$. Differentiating the identity

$$\sigma(z,\theta_0) \equiv \sigma(x(z,\theta_0),\theta_0) = \sigma(x,\theta_0)$$

with respect to θ_0 at some point (z,θ_0), we obtain

$$\frac{\partial\sigma(z,\theta_0)}{\partial\theta_0}\Big|_{z,\theta_0} = \frac{\partial\sigma(x,\theta_0)}{\partial x}\Big|_{x=x(z,\theta_0),\theta_0} \cdot \frac{\partial x(z,\theta_0)}{\partial\theta_0}\Big|_{z,\theta_0} + \frac{\partial\sigma(x,\theta_0)}{\partial\theta_0}\Big|_{x=x(z,\theta_0),\theta_0}. \quad (21)$$

It follows from formulas (4) and (11), which define the function $x = x(z,\theta_0)$, that

$$\frac{\partial x}{\partial\theta_0} = \alpha(\theta_0)\int_0^z \bar{v}(z,\theta_0)\,dz. \quad (22)$$

Changing variables (11) and (13) in (22), we obtain

$$\frac{\partial x}{\partial\theta_0} = \alpha(\theta_0)\int_0^x \bar{v}^2(x,\theta_0)\,dx. \quad (23)$$

Then, it follows from formulas (4) and (10) that

$$\frac{1}{\sigma(z,\theta_0)}\,\frac{\partial\sigma(z,\theta_0)}{\partial\theta_0} = \alpha(\theta_0)\bar{v}^2(z,\theta_0) = \alpha(\theta_0)\bar{v}^2(x,\theta_0). \quad (24)$$

Dividing identity (21) into $\sigma(z,\theta_0) = \sigma(x,\theta_0)$ and substituting formulas (23) and (24), we obtain identity (15).

From identity (15), by virtue of $\alpha(0) = 0$, there follows

Corollary. *Let the conditions of Lemma 1 be satisfied. Then the following identity is valid*:

$$\frac{\partial\sigma(x,\theta_0)}{\partial\theta_0}\Big|_{\theta_0=0} = 0 \qquad \forall x \geq 0, \quad (25)$$

whence

$$\frac{\partial^2\sigma(x,\theta_0)}{\partial x\,\partial\theta_0}\Big|_{\theta_0=0} = 0 \qquad \forall x \geq 0.$$

Let us prove formula (16). Let $\theta_0 \neq 0$. Then $\alpha(\theta_0) \neq 0$. Let us denote

$$b(x, \theta_0) = \frac{\partial}{\partial \theta_0} \{\ln \sigma(x, \theta_0)\}. \tag{26}$$

We obtain from identity (15) at $x = 0$: $b(0, \theta_0) = \alpha(\theta_0)\bar{v}^2(0, \theta_0) \neq 0$. By virtue of continuity with respect to x, the function $b(x, \theta_0)$ is not zero in some interval $[0, \varepsilon)$ of the axis x. Therefore, equality (15) is an integral Volterra equation of the second kind

$$y(x) = \int_0^x K(x, \xi) y(\xi)\, d\xi + f(x)$$

for the unknown function $y(x) = \alpha(\theta_0)\bar{v}^2(x, \theta_0)$ with a continuous (degenerate) kernel $K(x, \xi) = \partial\{\ln \sigma(x, \theta_0)\}/\partial x$ and a continuous right-hand side $f(x) = b(x, \theta_0)$. It is known from the theory of integral equations [34, p. 56] that this equation has the unique solution $y(x)$ in any finite interval $[0, a]$ of the axis x.

Moreover, the solutions $\bar{v}^2(x, \theta_0)$ to integral equation (15) can be represented by the following explicit formula:

$$\bar{v}^2(x, \theta_0) = \frac{1}{\alpha(\theta_0)} \left\{ b(x, \theta_0) + \frac{\partial \sigma(x, \theta_0)}{\partial x} \int_0^x \frac{b(x, \theta_0)}{\sigma(x, \theta_0)}\, dx \right\}. \tag{27}$$

This formula can be obtained by substituting the right-hand side of equality (27) into equality (15) and integrating by parts (with respect to the variable x). By virtue of (26), formula (27) coincides with identity (16).

Formula (17) can be obtained in two ways: 1) first we multiply identity (16) by $\alpha(\theta_0)$. The identity obtained is valid at any $\theta_0 \geq 0$. Then we differentiate it with respect to θ_0, assuming $\theta_0 = 0$, and using equalities (25), $\alpha(0) = 0$, and $(d\alpha/d\theta_0)|_{\theta_0=0} = -1/v_0^2$; 2) in formula (16) or (27), we pass to the limit as $\theta_0 \to 0$. In fact, at $\theta_0 \neq 0$

$$\frac{b(x, \theta_0)}{\alpha(\theta_0)} \stackrel{\text{def}}{=} -\frac{v_0^2}{\cos\theta_0 \sigma(x, \theta_0)} \cdot \frac{\dfrac{\partial \sigma(x, \theta_0)}{\partial \theta_0}}{\theta_0 \cdot \dfrac{\sin\theta_0}{\theta_0}}.$$

Therefore, by virtue of (25),

$$\begin{aligned}
\lim_{\theta_0 \to 0} \frac{b(x, \theta_0)}{\alpha(\theta_0)} &= -\frac{v_0^2}{\sigma_0(x)} \lim_{\theta_0 \to 0} \frac{\dfrac{\partial \sigma(x, \theta_0)}{\partial \theta_0} - \dfrac{\partial \sigma(x, 0)}{\partial \theta_0}}{\theta_0 - 0} \cdot \frac{1}{\lim\limits_{\theta_0 \to 0} \dfrac{\sin\theta_0}{\theta_0}} \\
&= -\frac{v_0^2}{\sigma_0(x)} \frac{\partial^2 \sigma(x, \theta_0)}{\partial \theta_0^2}\Big|_{\theta_0=0} \stackrel{\text{def}}{=} -v_0^2 h(x).
\end{aligned}$$

Hence, passing in (16) or (27) to the limit as $\theta_0 \to 0$, we obtain (17). $\square$

It follows from formula (17) (by virtue of $v^2(x) \neq 0$) that, in contrast to the quantity $\left.\dfrac{\partial\sigma(x,\theta_0)}{\partial\theta_0}\right|_{\theta_0=0}$ (see the corollary), $\left.\dfrac{\partial^2\sigma(x,\theta_0)}{\partial\theta_0^2}\right|_{\theta_0=0} \not\equiv 0$ at $x \geq 0$.

Remark 2. Formulas (15) and (16) were obtained in [9]. The identities similar to (15) and(16) take place, as noted in [9, 23, 25], also in the elliptic case $v(z) > v_0/\sin\theta_0$ in problems of the HS or the FB types. For instance, in this case, the formula

$$\overset{+}{v}{}^2(x,\theta_0) = \frac{1}{\overset{+}{\alpha}(\theta_0)}\left\{\overset{+}{b}(x,\theta_0) + \frac{\partial\overset{+}{\sigma}(x,\theta_0)}{\partial x}\int_0^x \frac{\overset{+}{b}(\xi,\theta_0)}{\overset{+}{\sigma}(\xi,\theta_0)}\,d\xi\right\}, \tag{28}$$

plays the role of formula (16), where

$$\overset{+}{b}(x,\theta_0) = \frac{\partial}{\partial\theta_0}\{\ln\overset{+}{\sigma}(x,\theta_0)\}, \qquad \overset{+}{\alpha}(\theta_0) = \frac{\sin\theta_0\cos\theta_0}{v_0^2}$$

(this is formula (3.1) in [23] or formula (3.5.1) in [25]). Formula (28) is obtained from (16) by substituting $\bar{v}(x,\theta_0)$ for $\overset{+}{v}(x,\theta_0)$, $\sigma(x,\theta_0)$ for $\overset{+}{\sigma}(x,\theta_0)$, and $\alpha(\theta_0)$ for $\overset{+}{\alpha} = -\alpha(\theta_0)$.

By definition, the functions $\overset{+}{v}(x,\theta_0)$ and $\overset{+}{\sigma}(x,\theta_0)$ are obtained from

$$\frac{1}{\overset{+}{v}{}^2(z,\theta_0)} = \frac{\sin^2\theta_0}{v_0^2} - \frac{1}{v^2(z)}, \qquad \overset{+}{\sigma}(z,\theta_0) = \frac{\mu(z)}{\overset{+}{v}(z,\theta_0)}$$

by changing the variables

$$x = \int_0^z \frac{dz}{\overset{+}{v}(z,\theta_0)}, \quad \overset{+}{v}(x,\theta_0) = \overset{+}{v}(z,\theta_0), \quad \overset{+}{\sigma}(x,\theta_0) = \overset{+}{\sigma}(z,\theta_0).$$

Remark 3. It is evident that formulas (15)–(17) and their analogs for the elliptic case do not depend on the shape of the domain, where the differential equation is considered and on the boundary conditions. They only express a relation between the sought-for coefficients $v(z)$ and $\mu(z)$ in equation (3) and the function $\sigma(x,\theta_0)$ or $\overset{+}{\sigma}(x,\theta_0)$. Therefore, these formulas can be applied in the same way to the hyberbolic and the elliptic cases both in the direct and the inverse problems of the HS type (for an inhomogeneous half-space) and to the direct and the inverse problems of the FB type (for an inhomogeneous layer with a free boundary).

6. Proof of Theorem 2. Solving 2HS and 2FB inverse problems (a method for reconstructing velocity and density in an inhomogeneous layer as functions of depth on the basis of a family of plane waves reflected from the layer at different angles or a family of the records $\boldsymbol{u(H,\xi,\theta_0)}$ of free boundary oscillations with the given shape $\boldsymbol{\varphi_0(\xi,\theta_0)}$ of incident plane waves)

Let us return to inverse problems 2. First we consider the inverse 2HS problem. Let the functions $v(z)$ and $\mu(z)$ be unknown. Solving the inverse 1HS problem for each value of $\theta_0 \in \Theta_0$ by some method (for instance, by the method from [3, 4]), we uniquely determine (by virtue of Theorem 1) the function $\sigma(x,\theta_0)$ for any $\theta_0 \in \Theta_0$ at $x \geq 0$. Hence, we can uniquely find the function $\sigma(x,\bar{\theta}_0)$ and the partial derivatives

$$\frac{\partial}{\partial x}\{\ln\sigma(x,\bar{\theta}_0)\} \quad \text{and} \quad \frac{\partial}{\partial\theta_0}\{\ln\sigma(x,\theta_0)\}\Big|_{\theta_0=\bar{\theta}_0} \equiv b(x,\bar{\theta}_0).$$

These derivatives are calculated at a limiting point $\bar{\theta}_0$ at $x \geq 0$. They are determined uniquely as well.

Let $\bar{\theta}_0 \neq 0$. First we find the function $\bar{v}(x,\bar{\theta}_0)$ by the explicit formula

$$\bar{v}^2(x,\bar{\theta}_0) = \frac{1}{\bar{\alpha}}\Big\{b(x,\bar{\theta}_0) + \frac{\partial\sigma(x,\bar{\theta}_0)}{\partial x}\int_0^x \frac{b(x,\bar{\theta}_0)}{\sigma(x,\bar{\theta}_0)}\,dx\Big\}, \tag{29}$$

resulting from identity (16) with substitution $\theta_0 = \bar{\theta}_0$. Here $\bar{\alpha} = \alpha(\bar{\theta}_0) = -\sin\bar{\theta}_0\cos\bar{\theta}_0/v_0^2$. Then we find the function $\mu(x,\bar{\theta}_0) = \bar{v}(x,\bar{\theta}_0)\sigma(x,\bar{\theta}_0)$ and the functions $z = z(x,\bar{\theta}_0) = \int_0^x \bar{v}(x,\bar{\theta}_0)\,dx$, $\bar{v}(z,\bar{\theta}_0) = \bar{v}(x,\bar{\theta}_0)$, $\mu(z) = \mu(x,\bar{\theta}_0)$, where $z = z(x,\bar{\theta}_0)$. Finally, with the help of formulas (4) and $\rho(z) = \mu(z)/v^2(z)$, we reconstruct the sought for functions $v(z)$ and $\rho(z)$ at $z \geq 0$.

Let $\bar{\theta}_0 = 0$. In this case, solving inverse problem 1 for every value $\theta_0 \in \Theta_0$, we determine the function $\sigma(x,\theta_0)$ for any $\theta_0 \in \Theta_0$ at $x \geq 0$, and then the functions $\sigma_0(x) = \sigma(x,0)$ and $h(x)$ defined by formulas (2) and (18). Then we find the function $v(x)$ by explicit formula (17), the function $\mu(x) = v(x)\sigma_0(x)$, $z = \int_0^x v(x)\,dx \equiv z(x) \equiv z(x,0)$, and then $v(z)$, $\mu(z)$, $\rho(z)$: $v(z) = v(x)$, and $\mu(z) = \mu(x)$, where $z = z(x)$ and $\rho(z) = \mu(z)/v^2(z)$ at $z \geq 0$.

Inverse 2FB problem, according to Remark 3, is considered in a similar way; the number H is determined by the formula $H = z(x_H,\bar{\theta}_0)$. Theorem 2 is proved.

Remark 4. The shape of an incident pulse in direct HS and FB problems and in inverse 1HS, 1FB, 2HS, and 2FB problems can vary for different values of θ_0: $\varphi_0 = \varphi_0(\xi, \theta_0)$. In the above-considered problems, the functions $\varphi_0(\xi) = \varphi_0(\xi, \theta_0)$ at different fixed θ_0 are not interrelated in any way, because each of these problems is solved independently for different θ_0. Therefore, $\varphi_0(\xi, \theta_0)$ in these problems need not depend on θ_0 smoothly (for instance, continuously); it can be discontinuous with respect to θ_0.

Thus, according to the above method, inverse problems 2 are solved in the following two stages:

1. Solving inverse 1HS or 1FB problem and determining the function $\sigma(x) = \sigma(x, \theta_0)$ for the family of angles θ_0. Various methods can be applied at this stage.
2. Reconstructing the functions $v(z)$ and $\rho(z)$ using the family of functions $\{\sigma(x, \theta_0)\}$ with the help of formulas (16) or (17). At this stage, the solution to inverse 2HS and 2FB problems is uniform.

When we pass to the variables

$$x = \int_0^z \frac{dz}{\bar{v}(z, \theta_0)}, \qquad u(x, \xi, \theta_0) = u(z, \xi, \theta_0),$$

we obtain, instead of the direct HS problem with conditions (3)–(7) or the FB problem with conditions (3)–(8), the following equivalent direct problems. Let X and D_x denote the interval $0 \le x < \infty$ and half-plane $0 < x < \infty$, $-\infty < \xi < \infty$ in the HS problem, and the interval $0 \le x \le x_H$ ($x_H = x(H, \theta_0)$) and the band $0 < x < x_H$, $-\infty < \xi < \infty$, in the FB problem. Let the numbers θ_0 and $\sigma^0 = \mu_0/v_0$ and the functions $\sigma(x, \theta_0)$ at $x \in X$, $\varphi_0(\xi, \theta_0)$ at $-\infty < \xi < \infty$ under the following conditions be given:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\{\ln \sigma(x, \theta_0)\}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial \xi^2}, \qquad (x, \xi) \in D_x, \tag{30}$$

$$\frac{\partial u}{\partial x}(0, \xi, \theta_0) - \beta(\theta_0)\frac{\partial u}{\partial \xi}(0, \xi, \theta_0) = -2\beta(\theta_0)\frac{\varphi_0(\xi, \theta_0)}{\partial \xi}, \tag{31}$$

$$u(x, \xi, \theta_0)\big|_{\xi<0} \equiv 0, \qquad x \ge 0. \tag{32}$$

Moreover, in the FB problem, the following boundary condition is specified:

$$\frac{\partial u}{\partial x}\Big|_{x=x_H} = 0, \qquad -\infty < \xi < \infty. \tag{33}$$

Here $\beta(\theta_0) = \cos\theta_0 \cdot \sigma^0/\sigma(0, \theta_0) = \delta(\theta_0)/\sigma(0, \theta_0)$.

It is necessary to determine, with θ_0 considered, the function $u(x, \xi, \theta_0)$ satisfying conditions (30)–(32) in the HS problem and conditions (30)–(33) in the FB problem, as well as the function $\varphi_1(\xi, \theta_0)$ from the condition

$$u(0, \xi, \theta_0) = \varphi_0(\xi, \theta_0) + \varphi_1(\xi, \theta_0). \tag{34}$$

This form of the direct problem is used in [9] when solving the inverse 1HS and 2HS problems numerically.

7. Solving inverse problems 2 numerically

Inverse problems 2 at $\theta_0 \neq 0$ were formulated and investigated in [9] (see, also, [19, 21]), where an algorithm to solve numerically the inverse 2HS problem is also proposed and tested. This algorithm to determine the functions $v(z)$ and $\rho(z)$ specifies the shapes $\varphi_0(\xi, \theta_0)$ and $\varphi_1(\xi, \theta_0)$ of incident and reflected waves for the finite set Θ of the angles θ_0.

If the initial data have no noise, a set Θ consists of two or three close angles. According to the method considered in Section 6, this algorithm has two stages of the numerical reconstruction of the functions $v(z)$ and $\rho(z)$:

1. Solving the problem of determining the function $\sigma(x) = \sigma(x, \theta_0)$ for a fixed θ_0 by using a pair of functions $\varphi_0(\xi) = \varphi_0(\xi, \theta_0)$ and $\varphi_1(\xi) = \varphi_1(\xi, \theta_0)$ (that is, inverse problem 1).

This inverse problem is solved, for instance, three times for three close angles $\theta - \Delta\theta$, θ, and $\theta + \Delta\theta$, where $\Delta\theta$ is small. As a result, we obtain three functions, namely $\sigma_1(x) = \sigma(x, \theta - \Delta\theta)$, $\sigma_2(x) = \sigma(x, \theta)$, and $\sigma_3(x) = \sigma(x, \theta + \Delta\theta)$.

2 Reconstructing the functions $v(z)$ and $\rho(z)$ by the three known functions $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma_3(x)$ with the help of a difference analog to formula (16).

To numerically solve the inverse 1HS problem of determining the function $\sigma(x) = \sigma(x, \theta_0)$ based on the given functions $\varphi_0(\xi) = \varphi_0(\xi, \theta_0)$ and $\varphi_1(\xi) = \varphi_1(\xi, \theta_0)$, the Antonenko method ([15]) for solving the 1D inverse seismic dynamic problem is used in [9]. In this method, the use of the feedback principle is of interest (in the difference scheme inversion method, which is one of the main methods to solve inverse problems).

It should be noted that the inverse 1HS problem with conditions (30)–(32) and (34) differs from that from [15] only by boundary condition (31), which specifies not the normal derivative $\partial u/\partial x(0, \xi)$, but an inclined derivative. The method used for solving inverse problem 1 in [9] is a modification of the method from [15] for problem (30)–(32), (34).

In the presence of noise in the initial data, a set Θ consists of a greater number of values $\theta_0^{(j)} = \theta + j\Delta\theta$, $j = -J, \ldots, -1, 0, 1, \ldots, J$; $\Delta\theta > 0$, $J > 1$ of the parameter θ. The higher is the noise level, the larger number J is chosen; for instance, $J = 15$. In this case, additional operations are carried out. These include smoothing of the functions $\varphi_1^{(k)} = \varphi_1(\xi, \theta_0^{(k)})$ with respect to ξ and reconstructing the values σ_i^r of the functions $\sigma_i(x) = \sigma(x, \theta_0^{(i)})$ with respect to x and θ_0. Numerical experiments with this algorithm were performed and analyzed in [9](see, also, [19, 21]).

In the case of $\bar{\theta}_0 = 0$, one can use a similar numerical algorithm. Here, formula (17) and its difference analog are used instead of (16).

A similar approach can be used to numerically solve the inverse 1FB and 2FB problems.

8. Inverse 3FB problem (with information for the family of angles $\boldsymbol{\theta_0}$ and unknown functions $\boldsymbol{v(z)}$, $\boldsymbol{\rho(z)}$, and $\boldsymbol{\varphi_0(\xi, \theta_0)}$)

Inverse 3FB problem. Let Θ_0 be a sequence defined in the inverse 2HS and 2FB problems with a limiting point $\bar{\theta}_0$. Let, at $0 \le \xi < \infty$, the values $u(H, \xi, \theta_0)$ of the solution $u(z, \xi, \theta_0)$ to direct 2FB problem with conditions (3)–(8) for any $\theta_0 \in \Theta_0$ be given. Moreover, the numbers v_0 and μ_0 are specified. The number H and the functions $v(z)$ and $\mu(z)$ at $0 \le z \le H$ and the function $\varphi_0(\xi, \theta_0)$ at $0 \le \xi < \infty$, $\theta_0 \in \Theta_0$ from conditions (3)–(8) are unknown. It is necessary to determine the number H and the functions $v(z)$ and $\mu(z)$, at $0 \le z \le H$, $\varphi_0(\xi, \theta_0)$ at $-\infty < \xi < \infty$ and $\theta_0 \in \Theta_0$.

Let $\tilde{\varphi}_0(k, \theta_0)$ denote the Fourier transform of the function $\varphi_0(\xi, \theta_0)$ over the variable ξ: $\varphi_0(\xi, \theta_0) = \int_{-\infty}^{\infty} \tilde{\varphi}(k, \theta_0)\, \mathrm{e}^{ik\xi}\, dk$.

Let γ be a contour on the complex plane $\theta_0 = \theta_{01} + i\theta_{02}$ consisting of the ray $\theta_{01} = 0$, $\theta_{02} \ge 0$, the section $0 \le \theta_{01} \le \pi/2$, $\theta_{02} = 0$, and the ray $\theta_{01} = \pi/2$, $\theta_{02} \ge 0$. The inverse 3FB problem is formulated in [8]; this paper also contains the proof of

Theorem 3. *Let conditions* (A) *of Theorem 1, its condition* (B) *for any* $\theta_0 \in \Theta_0$, *and the condition*

(D′) $\dfrac{dv}{dz} = \dfrac{d\mu}{dz} = 0$

be satisfied.

Also, let the following condition be satisfied:

(E) *two real numbers k_1 and k_2 are given. They are nonzero and have the following properties*: *the functions $\tilde{\varphi}_0(k_1, \theta_0)$ and $\tilde{\varphi}_0(k_2, \theta_0)$ are analytical in some domain D_γ, which contains the contour γ; the quantities $\tilde{\varphi}_0(k_1, \theta_0)$ and $\tilde{\varphi}_0(k_2, \theta_0)$ are nonzero on the contour γ and take either real or pure imaginary values on the contour γ* (*not necessarily simultaneously; the values of $\tilde{\varphi}_0(k_1, \theta_0)$ and $\tilde{\varphi}_0(k_2, \theta_0)$ are unknown*)[1].

[1] In the case of $\varphi_0 = \varphi_0(\xi)$ (the shape of the incident wave does not depend on θ_0), condition (E) is formulated as follows: two real numbers, k_1 and k_2, are given; they are nonzero and such that the numbers $\tilde{\varphi}(k_1)$ and $\tilde{\varphi}(k_2)$ are either real or pure imaginary (not necessarily simultaneously; the numbers $\tilde{\varphi}_0(k_1)$ and $\tilde{\varphi}(k_2)$ are unknown).

Then the inverse 3FB problem has the unique solution $\{H, v(z), \mu(z), \varphi_0(\xi, \theta_0)\}$.

Let us briefly present the proof of Theorem 3 and the solution to inverse 3FB problem. They are based on the theorem presented below that was proved in [8] (see also [23, 25]).

Theorem 4. *Let conditions* (A) *and* (B) *of Theorem 1 and condition* (D′) *of Theorem 3 in the direct FB problem with conditions* (3)–(8) *be satisfied. Then for any* θ_0 *satisfying condition* (H) *the solution* $u(z, \xi, \theta_0)$ *to the direct FB problem can be represented in the form*

$$u(z, \xi, \theta_0) = \int_{-\infty}^{\infty} U(z, k, \theta_0)\, \mathrm{e}^{ik\xi}\, dk, \qquad 0 \le z \le H, \quad -\infty < \xi < \infty, \tag{35}$$

$$\begin{gathered} U(z, k, \theta_0) = 2\cos\theta_0 \sigma_0 \mu^{-1/2}(0) \mu^{-1/2}(z) ik\tilde{\varphi}_0(k, \theta_0) \times \\ \frac{\Phi(z, k, \theta_0)\Psi'(H, k, \theta_0) - \Psi(z, k, \theta_0)\Phi'(H, k, \theta_0)}{\Phi'(H, k, \theta_0) + ik\, \dfrac{\cos\theta_0 \sigma^0}{\mu(0)}\, \Psi'(H, k, \theta_0)}, \\ 0 \le z \le H, \qquad -\infty < k < \infty. \end{gathered} \tag{36}$$

Here $\sigma^0 = \mu_0/v_0$, *and the functions* $\Phi(z, k, \theta_0)$ *and* $\Psi(k, z, \theta_0)$ *are solutions to the equation*

$$\frac{d^2U}{dz^2} + \Big\{p(z) + \frac{k^2}{\bar{v}^2(z, \theta_0)}\Big\}U = 0, \tag{37}$$

where

$$p(z) = -\frac{1}{2}\, a^2(z) - \frac{1}{2}\, a'(z) = -\frac{1}{2}\, \frac{\mu''}{\mu} + \frac{1}{4}\Big(\frac{\mu'}{\mu}\Big)^2, \quad a(z) = \frac{\mu'}{\mu}, \tag{38}$$

with the initial conditions, respectively:

$$\Phi(0, k, \theta_0) = 1, \qquad \Phi'(0, k, \theta_0) = \tilde{\alpha} = \frac{1}{2}\, a(0) = \frac{1}{2}\, \frac{\mu'(0)}{\mu(0)},$$

$$\Psi(0, k, \theta_0) = 0, \qquad \Psi'(0, k, \theta_0) = 1.$$

Here and below the prime denotes differentiation with respect to z. *The integral in formula* (35) *is a Fourier integral* (*that is, the formula of its inversion is valid*).

The following lemma was proved in [8]:

Lemma 2. *The functions* $\Phi(z, k, \theta_0)$ *and* $\Psi(z, k, \theta_0)$ *for each pair of fixed values* $z \in [0, H]$ *and* $k \in (-\infty, \infty)$ *are integer analytical functions* θ_0 *as well as the complex variable*

$$s = -k^2 \sin^2\theta_0 / v_0^2. \tag{39}$$

Taking into account formulas (4) and (38), we can write down equation (37) in the following form:

$$-\frac{d^2U}{dz^2} + q(k,z)U = sU, \qquad 0 \le z \le H,$$

where the function

$$q(k,z) = -\Big\{p(z) + \frac{k^2}{v^2(z)}\Big\} \tag{40}$$

is continuous by virtue of condition (A) at any $k \in (-\infty, \infty)$. Let us consider for the given $k = k_1$ the following two regular Sturm–Liouville problems with the same condition at the point $z = H$ and different conditions at the point $z = 0$:

$$\begin{aligned} &-y'' + q(k_1,z)y = sy, \qquad 0 \le z \le H,\\ &\cos\varphi y(0) + \sin\varphi y'(0) = 0, \qquad y'(H) = 0 \qquad (\operatorname{ctg}\varphi = -\tilde\alpha); \end{aligned} \tag{41}$$

$$\begin{aligned} &-y'' + q(k_1,z)y = sy, \qquad 0 \le z \le H,\\ &y(0) = 0, \qquad y'(H) = 0, \end{aligned} \tag{42}$$

where the numbers H and $\tilde\alpha$ and the functions $v(z)$ and $\mu(z)$, in terms of which the function $q(k_1,z)$ is expressed, according to (38) and (40), are taken from the direct FB problem.

With the use of the given numbers k_1 and v_0 and the sequence Θ_0 using (39), we construct a sequence of numbers $s_1, s_2, \dots, s_n, \dots$ $(s_n = s(k_1, \theta_0^{(n)}))$. Let S_{k_1} denote this sequence. It is evident that S_{k_1} converges to the finite limiting point $\bar s = s(k_1, \bar\theta_0)$. By virtue of formulas (35) and (36) and the identity $\Phi\Psi' - \Phi'\Psi \equiv 1$, we obtain

$$u(H,\xi,\theta_0) = \int_{-\infty}^{\infty} \frac{2\cos\theta_0\sigma^0\mu^{-1/2}(0)\mu^{-1/2}(H)ik\tilde\varphi_0(k,\theta_0)}{\Phi'(H,k,\theta_0) + ik\,\frac{\cos\theta_0\sigma^0}{\mu(0)}\,\Psi'(H,k,\theta_0)}\,\mathrm{e}^{ik\xi}\,dk.$$

Here, inverting the integral at $k = k_1$ and for any $\theta_0 \in \Theta_0$, we determine values of the function

$$\begin{gathered} A(k_1,\theta_0) = c\Big\{\Phi'(H,k_1,\theta_0) + ik\,\frac{\cos\theta_0\sigma^0}{\mu(0)}\,\Psi'(H,k_1,\theta_0)\Big\},\\ c = \sigma^0\mu^{1/2}(0)\mu^{1/2}(H)/\tilde\varphi_0(k_1,\theta_0), \end{gathered} \tag{43}$$

for any $\theta_0 \in \Theta_0$. Separating the real and the imaginary parts of the function $A(k_1,\theta_0)$, we determine values of the functions $\{c\Phi'(H,k_1,\theta_0)\}$ and $\{c\beta\Psi'(H,k_1,\theta_0)\}$ for any $\theta_0 \in \Theta_0$ (where $\beta = \sigma^0/\mu(0)$) (the functions Φ and Ψ are real for real k and θ_0).

Let us consider Φ and Ψ as functions of the variable $s = -k_1^2 \sin^2\theta_0 / v_0^2$: $\Phi(z, k_1, \theta_0) = \Phi(z, k_1, s)$, $\Psi(z, k_1, \theta_0) = \Psi(z, k_1, s)$. By virtue of Lemma 2 and condition (E), the functions $\tilde{\Phi}(s) = \{c\Phi'(H, k_1, s)\}$ and $\tilde{\Psi}(s) = \{c\beta\Psi'(H, k_1, s)\}$ are analytical functions of s in some domain D_0 of the plane s. The domain D_0 contains a real axis into which the contour γ when mapping $s = s(k, \theta_0)$ of form (39) at any $k \in (-\infty, \infty)$ is transformed. We found their values for any $s \in S_{k_1}$. Accordance to the uniqueness theorem for analytical functions, the functions $\tilde{\Phi}(s)$ and $\tilde{\Psi}(s)$ are determined uniquely in the domain D_0. We can analytically continue $\tilde{\Phi}(s)$ and $\tilde{\Psi}(s)$ to the entire straight line $-\infty < s < \infty$. Hence, all real zeros of the functions $\Phi'(H, k_1, s)$ and $\Psi'(H, k_1, s)$ are determined uniquely.

It should be noted that the set of all real zeros of the functions $\Phi'(H, k_1, s)$ and $\Psi'(H, k_1, s)$ forms a spectrum of boundary value problems (41) and a spectrum of boundary value problems (42), respectively. This follows from the general theory of the classical Sturm–Liouville problem [18, 34]. Accordance to the results obtained by G. Borg, L.A. Chudov, and V.A. Marchenko [18, 34], these spectra uniquely determine the numbers $\tilde{\alpha}$ and H and the function $q(k_1, z)$ at $0 \le z \le H$. To solve this problem, let us use the methods by V.A. Marchenko, M.G. Krein, M.G. Gasymov, and B.M. Levitan [18, 34] to determine the regular Sturm–Liouville operator on the basis of two spectra.

Performing the same operations for $k = k_2$, we find the function $q(k_2, z)$. Then we determine the functions $v(z)$ and $\rho(z)$ by the following formulas:

$$v(z) = \left\{ \frac{k_2^2 - k_1^2}{q(k_2, z) - q(k_1, z)} \right\}^{1/2}, \qquad p(z) = -q(k_1, z) + \frac{k_1^2}{v^2(z)}$$

at $0 \le z \le H$. With the help of the found number $\tilde{\alpha} = a(0)$ and formula (38), we find the function $a(z)$. Hitherto the number μ_0 was not used. If μ_0 is specified, we find the number $\mu(0) = \sigma_0/\beta$ (the number β is preliminarily determined from (43)). Finally, from the equation $\mu' - a(z)\mu = 0$ we find $\mu(z)$. Then, using the reconstructed values H, $v(z)$, and $\mu(z)$, we construct the function $B(k, \theta_0) = \Phi'(H, k, \theta_0) + ik \frac{\cos\theta_0 \sigma^0}{\mu(0)} \Psi'(H, k, \theta_0)$ at $\theta_0 \in \Theta_0$, $-\infty < k < \infty$. Inverting the Fourier integral for the function $u(H, \xi, \theta_0)$, we find the function $U(H, k, \theta_0)$. Then we find the function $\tilde{\varphi}_0(k, \theta_0) = \frac{\mu^{1/2}(0)\mu^{1/2}(H)}{2\sigma^0 \cos\theta_0 ik} B(k, \theta_0) U(H, k, \theta_0)$ at $\theta_0 \in \Theta_0$, $-\infty < k < \infty$. And, finally, we find the function $\varphi_0(\xi, \theta_0)$ at $\theta_0 \in \Theta_0$, $-\infty < \xi < \infty$. Theorem 3 is proved.

Remark 5. The statement of the inverse 3FB problem, in which the values $u(0, z)$ and $u_z(0, \xi)$ are given instead of $u(H, \xi, \theta_0)$ (see [8, 23, 25]), is investigated in a similar way.

Remark 6. Inverse problems similar to inverse problems 1, 2, and 3 were formulated and investigated in [5, 7, 23, 25] also for the elliptic case $v(z) > v_0/\sin\theta_0$. In this case, theorems similar to 1–4 are valid. Here, the corresponding problems from [5, 7] (these are inverse problems 3.2, 5.1, and 5.2 in [23, 25]) "play the role" of the inverse 1HS and 1FB problems, and formula (28) "plays the role" of formula (16). Statements with the fixed boundary $z = H$ can be considered in a similar way [6–12].

9. Possible application areas

The inverse problems of the plane wave scattering by inhomogeneous layers considered in [3–12, 19, 21] and in this paper have an important practical interpretation. First, seismic problems on determining the upper surface layer of Earth's cross-section using a horizontal component of the SH type of the seismic vector of displacements are reduced to these problems in the case of a layer with a free boundary. This component is measured at the free boundary (the daily surface) excited by a wave coming from depth of the half-space on which the inhomogeneous layer lies. There exist three-component devices that can measure this component. Second, these are problems of determining the characteristics of the inhomogeneous layer which give a reflected wave or a field at a free boundary with given required properties, for instance, the least possible in magnitude. The inverse problems under consideration also belong to the problems determining the structure of the sea or the lake precipitation with the use of remote sensing data from the water reservoir surface considered in [13, 14, 17]. Possible application areas of these inverse problems include problems of location, tomography, and non-destructive diagnostics.

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