Bull. Nov. Comp. Center, Num. Anal., 5 (1994), 1–25

Direct and inverse problems assosiated with inclined passing of SH-waves through 1d inhomogeneous medium*

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Wave process in a one-dimensional vertically-inhomogeneous medium induced by a sounding impulse moving from the depth is considered. Mathematical background of the algorithm for the reconstruction of the medium's mechanical parameters is given when the form of the initial wave and the surface seismogram are known. Theoretical results are illustrated by the numerical examples.

Observations of wave fields in internal points of the medium, i.e., boreholes, are used in seismoprospecting to obtain more comprehensive information on the geologic medium under study. Over the recent years, a special method of vertical seismic profiling (VSP) has been developed using observations in a number of points along the borehole. One of the difficulties of this method is the absence of direct information on the form of the initial wave generating interference vibrations in the layer and at the earth's surface. It is hard to solve the inverse dynamic problem of seismoprospecting (VSP in particular) without this information.

The given paper deals with direct and inverse problems of dynamics of horizontally polarized waves in a vertically inhomogeneous elastic medium. It is supposed that measurements of wave displacements were carried out both at the free surface of the earth and inside of it at one or several depths. Theorems of the existence and uniqueness of solutions not only for the inverse, but also for the direct problem are presented, since we didn't manage to find the consideration of these issues in terms of classical mathematical physics in spite of the fact that they were sufficiently well studied theoretically. Moreover, consideration of these questions in the rigorous, but adapted form can be of interest in geophysics.

*This work was supported in part by Russian Fund of Fundamental Research under Grant 93-05-8638.

1. Statement of the problem

Let the Cartesian coordinates in $\mathbf{R}^3$ be x, y, z. We suppose that the half-space $z \geq 0$ is filled with an elastic isotropic medium having the Lame parameters λ, μ and density ρ. The process of wave propagation in this medium is described by the general system of dynamic elasticity theory equations

$$(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,U + \mu\,\Delta U + (\mathrm{grad}\,\lambda)\,\mathrm{div}\,U + (\mathrm{grad}\,\mu)\,U' + U'\,\mathrm{grad}\,\mu = \rho\,U_{tt}.$$

Here $U = (X, Y, Z)$ is the displacement vector of the medium points, U' is the Jacobi matrix of the mapping $(x, y, z) \to U(x, y, z)$ and t is time. Absence of external influences on the free surface $z = 0$ is interpreted as fulfilment of the boundary conditions

$$\lambda\,\mathrm{div}\,U + 2\mu\,Z_z = 0, \quad X_z + Z_x = 0, \quad Y_z + Z_y = 0.$$

In the subsequent discussion it will be supposed that the functions λ, μ, ρ are twice continuously differentiable and depend only on z, and the displacement vector is parallel to the y-axis, i.e., $X = Z = 0$. Then the initial system will be substantially simplified

$$\begin{aligned}
&(\lambda+\mu)\,(Y_y)_z + \lambda_z Y_y = 0,\\
&(\lambda+\mu)\,(Y_y)_x = 0,\\
&(\lambda+\mu)\,Y_{yy} + \mu\,\Delta Y + \mu_z Y_z = \rho\,Y_{tt},
\end{aligned}$$

and the boundary conditions will have the form

$$Y_y\Big|_{z=0} = 0, \quad Y_z\Big|_{z=0} = 0.$$

The first equation of this system together with the first boundary condition form the homogeneous Cauchy problem over the derivative Y_y, due to which $Y_y = 0$. Thus, $Y = Y(x, z, t)$, and

$$\mu\,(Y_{xx} + Y_{zz}) + \mu_z Y_z = \rho\,Y_{tt}, \quad z > 0; \tag{1}$$

$$Y_z\Big|_{z=0} = 0. \tag{2}$$

Further we shall restrict our consideration to the case, when the functions μ, ρ are constant for $z > H > 0$ and equal to some known values μ_0, ρ_0. Under these conditions the equation (1) in the domain $\{\, z > H,\ -\infty < x < \infty \,\}$ has particular solutions of the plane waves

$$Y(x,z,t)=\varphi\Big(t+\frac{z\cos\alpha-x\sin\alpha}{v_0}\Big), \tag{3}$$

where α is the angle formed by the direction of the wave movement and the negative direction of z-axis; $v_0=\sqrt{\mu_0/\rho_0}$ is the velocity of wave propagation; φ is an arbitrary twice differentiable function. As a rule, it will be supposed that the function φ vanishes outside of some interval (a,∞). This implies that the wave has in the plane (x,z) a linear leading edge moving with speed v_0 in the direction of the vector $(\sin\alpha,-\cos\alpha)$.

Let us suppose that the wave of the type (3) moving from the domain $z>H$ in the direction of the boundary $z=0$ was generated by means of external influences. Scattering from the inhomogeneities of the medium in the band $0<z<H$, this wave will generate a wave process $Y(x,z,t)$ in the half-plane $z>0$. Due to symmetry (displacement along the x-axis is equivalent to time lag) this process should have the form

$$Y(x,z,t)=w(z,\tau),\qquad \tau=t-\frac{x\sin\alpha}{v_0}.$$

Passing in (1)–(2) to the new function $w(z,\tau)$ and assuming

$$r(z)=\rho(z)-\mu(z)\frac{\sin^2\alpha}{v_0^2},$$

we obtain

$$(\mu w_z)_z=r\,w_{\tau\tau},\quad z>0. \tag{4}$$

$$w_z(0,\tau)=0. \tag{5}$$

The function $w(z,\tau)$ coincides with the initial wave (3) until its leading edge reaches the domain $z<H$. This condition can be formulated in the following way. There exists such finite τ_0, that

$$w(z,\tau)=\varphi\Big(\tau+\frac{z\cos\alpha}{v_0}\Big),\quad 0\le z<\infty,\quad \tau<\tau_0. \tag{6}$$

The relation (6) characterizes the function w at $\tau\to-\infty$ and serves as an analogue of initial data.

Depending on the sign of the coefficient $r(z)$ the equation (4) can be hyperbolic, elliptic or of a mixed type. In our further discussion $r(z)$ will be considered positive at all $z\ge 0$, and this guarantees hyperbolicity of the equation (4). This condition can always be fulfilled by choosing the angle α small enough.

Now we can formulate two basic problems that are associated with the system (4)–(6) and will be considered in the subsequent sections of the given paper.

Direct problem. *Let the medium characteristics $\mu(z)$, $\rho(z)$, the angle α and the function $\varphi(\tau)$ be given. It is necessary to find the function $w(z,\tau)$ satisfying the equation* (4)*, the boundry condition* (5) *and the "initial" condition* (6).

Inverse problem. *Let the characteristics of the medium $\mu(z)$ and $\rho(z)$ on the interval $(0,H)$ be unknown. Only values of these functions, their first derivatives in the point $z=0$, and constants ρ_0, μ_0 and α are given. Moreover, the form $\varphi(\tau)$ of the initial wave and the regime of vibrations $w(0,\tau)$ of boundary points caused by this wave are known. It is necessary to find the relationship between the unknown functions μ and ρ on the interval $(0,H)$. To be more precise, we shall present the algorithm allowing the reconstruction of the function $\sigma(z) = \sqrt{\mu(z)r(z)}$, determining the required relationship between μ and ρ.*

It should be noted that a similar problem is studied in [1]. There is also considered the wave process satisfying the equation (4), but the plane wave generating it moves from the half-space $z<0$ filled with an elastic homogeneous medium with the known parameters. In this case the corresponding direct problem reduces to the search for a solution to the equation (4) that is equal to zero for $\tau \le 0$ and satisfies the inhomogeneous condition of the form

$$w_z(0,\tau) - k\, w_\tau(0,\tau) = \varphi(\tau)$$

at $z=0$. Here the inverse problem consists in the determination of the function $\sigma(z)$ in the half-line $0 < z < \infty$ using the known values $\varphi(\tau)$ and $w(0,\tau)$. Substantial difference of these statements from the statements presented in the given paper will be clear from the further consideration.

2. Direct problem

Let us consider generalized (in the Sobolev sense) solutions to the problem (4)–(6) determined in the half-plane $\{\, z \ge 0,\ -\infty < \tau < \infty \,\}$ and belonging to the space W_2^2 in any region of the form $\{\, z \ge 0,\ \tau_1 \le \tau \le \tau_2 \,\}$. For any solution of this type the energy conservation law (see, for example, [2])

$$\frac{d}{d\tau}\int_0^\infty \left[\mu(z)w_z^2(z,\tau) + r(z)w_\tau^2(z,\tau)\right] dz = 0$$

is valid and from it directly follows uniqueness of the solution to the direct problem. This law also implies that the solution does not depend on τ_0

that is present in the condition (6): solutions w' and w'' corresponding to different values of τ_0' and τ_0'' coincide.

The evolution system (4)–(5) is naturally connected with the spectral problem

$$-(\mu u_z)_z = \lambda r(z) u, \quad 0 < z < \infty; \tag{7}$$

$$u_z(0) = 0. \tag{8}$$

The corresponding differential operator $u \to -(\mu u_z)_z$ acts in the space $L_2(0,\infty)$ and is defined for all functions $u \in W_2^2(0,\infty)$, satisfying the boundary condition (8). Let u be the eigenfunction of the problem (7)–(8) corresponding to the eigenvalue λ. Multiplying the equality (7) by $\bar{u}$ and integrating over z, we obtain

$$\int_0^\infty \mu(z)|u_z(z)|^2\,dz = \lambda \int_0^\infty r(z)|u(z)|^2\,dz.$$

As μ and r are positive, λ is necessarily real and positive.

In order to study the solvability of the problem (4)–(6), we represent it in a more convenient form. Following [3], we set

$$v(z) = \sqrt{\mu(z)/r(z)}, \quad \xi = \int_0^z \frac{ds}{v(s)}, \quad h = \int_0^H \frac{ds}{v(s)}.$$

Physically $v(z)$ means propagation velocity of perturbations at the depth z, and ξ means time of wave path from the free surface to the depth z. Passing from (z,τ) to new independent variables (ξ,τ), we obtain

$$w_{\xi\xi} + [\ln\sigma(\xi)]_\xi\, w_\xi = w_{\tau\tau},$$

where

$$\sigma(\xi) = \sqrt{\mu(z(\xi))\, r(z(\xi))}.$$

In the point $\xi = 0$ we have $w_\xi = 0$ as before.

In order to correct the new form of the condition (6) we choose τ_0 such that the support of the function $\varphi(\tau + z\cos\alpha/v_0)$ will lie in the domain $z > H$ at $\tau < \tau_0$, where the velocity $v(z)$ is constant and equal to $v_0/\cos\alpha$. Then $(\xi - h)v_0 = (z - H)\cos\alpha$, and the condition (6) will have the form

$$w(\xi,\tau) = \varphi(\tau + \xi - h + H\cos\alpha/v_0), \quad 0 < \xi < \infty, \quad \tau < \tau_0.$$

And, finally, we assume

$$u(\xi,\tau) = \sqrt{\sigma(\xi)}\, w(\xi,\tau).$$

After trivial transformations we come to the problem

$$u_{\xi\xi} - q(\xi)\, u = u_{\tau\tau}, \qquad \xi > 0, \qquad -\infty < \tau < \infty; \tag{9}$$

$$u_\xi - k\, u = 0, \qquad \xi = 0, \qquad -\infty < \tau < \infty; \tag{10}$$

$$u(\xi,\tau) = f(\xi+\tau), \quad 0 < \xi < \infty, \qquad \tau < \tau_0. \tag{11}$$

Here

$$\begin{aligned} q(\xi) &= (\ln\sqrt{\sigma})_{\xi\xi} + \left[(\ln\sqrt{\sigma})_\xi\right]^2, \quad 0 \le \xi < \infty; \quad k = \frac{\sigma'(0)}{2\sigma(0)}; \\ f(\tau) &= \sqrt{\sigma_0}\cdot\varphi(\tau - h + H\cos\alpha/v_0), \quad -\infty < \tau < \infty, \\ \sigma_0 &= \left[\mu_0(\rho_0 - \mu_0 \sin^2\alpha/v_0^2)\right]^{1/2}. \end{aligned}$$

Recall that μ, ρ and, consequently, σ are twice continuously differentiable, and $\sigma(\xi) \equiv \sigma_0$ for $\xi \ge h$. So, the function $q(\xi)$ is continuous at $\xi \ge 0$ and vanishes in the half-line $\xi \ge h$. The function f together with φ are equal to zero outside of some interval $[a,\infty)$.

Together with (9)–(11), let us consider the spectral problem

$$U_{\xi\xi} - q(\xi)\, U = -\omega^2 U, \quad \xi > 0; \tag{12}$$

$$U_\xi - k\, U = 0, \quad \xi = 0, \tag{13}$$

obtained as a result of the formal application of the Fourier transform

$$U(\xi,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\tau\omega}\, u(\xi,\tau)\, d\tau$$

to the equations (9)–(10). As the coefficient $q(z)$ vanishes in the half-line $\xi \ge h$, the function $\exp(i\omega\xi)$ will be the solution to the equation (12) at $\xi \ge h$ and any complex ω. Continuing the solution for the whole interval $0 \le \xi < h$ we obtain the function $e(\xi,\omega)$ that is determined and satisfies the equation (12) at all complex ω and any $\xi \ge 0$; it is holomorphic over the variable ω in the whole complex plane at any fixed $\xi \ge 0$ and coincides with $\exp(i\omega\xi)$ at $\xi \ge h$ and any ω.

Let us enumerate the properties of $e(\xi,\omega)$ that will be necessary further. It is easy to verify (see [4–5]), that if $\omega \ne 0$, then $e(\xi,\omega)$ satisfies the integral equation

$$e(\xi,\omega) = e^{i\omega\xi} + \int_{\xi}^{\infty} \frac{\sin\omega(\xi-\eta)}{\omega} q(\eta)e(\eta,\omega)\,d\eta.$$

Hence, using the method of successive approximations, we can easily derive

Lemma 1. *At $\omega \neq 0$ the function $e(\xi,\omega)$ and its derivatives e', e'' with respect to the variable ξ can be represented in the form*

$$e^{(k)}(\xi,\omega) = e^{i\omega\xi}\left[(i\omega)^k + \psi_k(\xi,\omega)\right], \tag{14}$$

where ψ_k are functions that are holomorphic over ω, continuous over the combination of arguments (together with the derivatives with respect to ξ of the order $2-k$), equal to zero for $\xi \geq h$ and such that

$$|\psi_k(\xi,\omega)| \leq C_k(|\omega|+1)^{k-1}\exp[\alpha(|\operatorname{Im}\omega| - \operatorname{Im}\omega)],$$
$$0 \leq \xi < h, \quad C_k > 0, \quad \alpha > 0. \tag{15}$$

Details of the proof are standard and we do not give it here.

For each ω the functions $e(\xi,\omega)$ and $e(\xi,-\omega)$ are solutions of one and the same equation (12). Their Wronskian for $\xi \geq h$ can be easily computed in the explicit form

$$e'(\xi,\omega)e(\xi,-\omega) - e(\xi,\omega)e'(\xi,-\omega) = 2i\omega. \tag{16}$$

As the equation (12) does not contain U_ξ, the Wronskian (see, for example, [4]) does not depend on ξ. Consequently, the formula (16) is valid also at $\xi < h$. So, if $\omega \neq 0$, then $e(\xi,\omega)$ and $e(\xi,-\omega)$ form the fundamental system of solutions to the equation (12). Note that for real ω the functions $e(\xi,\omega)$ and $e(\xi,-\omega)$ are complex conjugate, i.e.,

$$e(\xi,-\omega) = \overline{e(\xi,\omega)}, \quad \xi \geq 0, \quad \operatorname{Im}\omega = 0. \tag{17}$$

Substituting $e(\xi,\omega)$ into the boundary condition (13), we obtain the entire analytical function

$$s(\omega) = e'(0,\omega) - k_e e(0,\omega),$$

that will be important in the further constructions.

Lemma 2. *The function $s(\omega)$ does not vanish at $\operatorname{Im}\omega \geq 0$, $\omega \neq 0$, but it has a zero of the first order at $\omega = 0$.*

Proof. If $\operatorname{Im}\omega > 0$ and $s(\omega) = 0$, then $e(\xi,\omega)$ is non trivial solution to the problem (12) (13) belonging to the space $W_2^2(0,\infty)$. Turning back from ξ to the initial independent variable z and assuming that

$$u(z) = e(\xi(z), \omega) \cdot [\sigma(\xi(z))]^{-1/2},$$

we obtain the eigenfunction of the spectral problem (7)–(8), corresponding to the non-positive eigenvalue $\lambda = -\omega^2$. This contradicts the properties of eigenvalues of the problem (7)–(8).

Let $s(\omega)$ vanish at some real $\omega \neq 0$. Then, on the basis of (17), not only $e(\xi, \omega)$, but also $e(\xi, -\omega)$ will satisfy the boundary condition (13). As $e(\xi, \omega)$ and $e(\xi, -\omega)$ form the fundamental system of solutions, then all the solutions to the equation (12) will satisfy the condition (13), and this is impossible.

It remains to consider the exceptional value $\omega = 0$. In this case the function $e(\xi, 0)$ together with its derivative $e_\omega(\xi, 0)$ over ω satisfy one and the same equation $U'' - q(\xi)U = 0$. By the transformation $U = \sqrt{\sigma(\xi)}V$ this equation reduces to the form $V'' + [\ln \sigma(\xi)]'V' = 0$, and after that its general solution

$$U(\xi) = \sqrt{\sigma(\xi)} \left[C_1 \int_0^\xi \frac{d\zeta}{\sigma(\zeta)} + C_2 \right].$$

can be easily found. The correlations

$$e(\xi, 0) = 1, \quad e_\omega(\xi, 0) = i\xi$$

should be valid for $\xi \geq h$, and therefore,

$$e(\xi, 0) = \sqrt{\sigma(\xi)/\sigma_0},$$

$$e_\omega(\xi, 0) = i[\sigma_0 \sigma(\xi)]^{1/2} \left[\int_0^\xi \frac{d\zeta}{\sigma(\zeta)} - \int_0^h \frac{d\zeta}{\sigma(\zeta)} \right] + ihe(\xi, 0).$$

Hence, it directly follows that

$$s(0) = 0, \quad s_\omega(0) = i\sqrt{\sigma_0/\sigma(0)} \neq 0. \qquad \square$$

Now we can solve the spectral problem (12)–(13), applying the standard reasoning from the scattering theory (see [2, 5]). Any solution $U(\xi, \omega)$ of the equation (12) decomposes into the linear combination

$$U(\xi, \omega) = a(\omega)e(\xi, \omega) + b(\omega)e(\xi, -\omega).$$

At $\xi \geq h$ this decomposition is equivalent to

$$U(\xi,\omega) = a(\omega)e^{i\xi\omega} + b(\omega)e^{-i\xi\omega}.$$

The term $a(\omega)\exp(i\xi\omega)$ corresponds to the "incoming wave" and determines the asymptotics of the solution $u(\xi,\tau)$ for the initial problem (9)–(11) at $\tau \to -\infty$. Owing to the condition (11) the coefficient $a(\omega)$ should coincide with the Fourier transform $F(\omega)$ of the data $f(\tau)$ for the direct problem. The coefficient $b(\omega)$ corresponds to the "outgoing wave" and determines the asymptotics of the solution at $\tau \to \infty$. It will be found from the boundary condition (13):

$$U_\xi(0,\omega) - k\,U(0,\omega) = F(\omega)\,s(\omega) + b(\omega)\,s(-\omega) = 0.$$

According to Lemma 2 this equation is solvable relative to $b(\omega)$ at all real $\omega \neq 0$ and even at all ω from the half-plane $\operatorname{Im}\omega < 0$. So,

$$U(\xi,\omega) = F(\omega)\,e(\xi,\omega) - \frac{s(\omega)}{s(-\omega)}F(\omega)\,e(\xi,-\omega). \tag{18}$$

The function $S(\omega) = s(\omega)/s(-\omega)$ is one of the main objects of investigation in the scattering theory. It determines the so-called scattering operator, relating the coefficients $a(\omega)$ and $b(\omega)$. Based on Lemma 2, $S(\omega)$ is analytical for $\operatorname{Im}\omega \leq 0$, $\omega \neq 0$ and in the point $\omega = 0$ it has a removable singularity. Owing to (17) $s(-\omega) = \bar{s}(\omega)$ on the real axis, and therefore, $|S(\omega)| = 1$ at $\operatorname{Im}\omega = 0$. If $\operatorname{Im}\omega < 0$, then the value $|S(\omega)|$ can be easily estimated, based on Lemma 1. And, finally, we conclude

$$|S(\omega)| = \begin{cases} 1, & \operatorname{Im}\omega = 0; \\ O\left(\exp(2\alpha|\operatorname{Im}\omega|)\right), & \operatorname{Im}\omega < 0. \end{cases} \tag{19}$$

The Heuristic considerations used in the derivation of (18) can be put on an absolutely rigorous basis, if we show that the function

$$u(\xi,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega\tau}\,U(\xi,\omega)\,d\omega, \tag{20}$$

where $U(\xi,\omega)$ is found according to the formula (18), is really the solution to the initial problem (9)–(11). For this we shall need a description of some functional spaces.

Let us denote by V the set of functions $u(\xi,\tau)$, continuous in the half-plane $\{\xi \geq 0,\ -\infty < \tau < \infty\}$ and having the following properties:

(i) mappings $\xi \to u(\xi,\cdot)$ and $\tau \to u(\cdot,\tau)$ take values in the spaces $L_2(-\infty,\infty)$ and $L_2(0,\infty)$ and are continuous over ξ and τ, respectively;

(ii) the norm

$$\|u\|^2 = \sup_{\xi} \int_{-\infty}^{\infty} |u(\xi,\tau)|^2\, d\tau + \sup_{\tau} \int_{0}^{\infty} |u(\xi,\tau)|^2\, d\xi$$

is finite.

Supplement of V according to this norm will be denoted by V_2. It is natural that traces on any lines parallel to coordinate axes are defined for elements from V_2, and the property (i) is fulfilled for these traces.

The set of functions $u(\xi,\tau)$, all the derivatives $D_\xi^k D_\tau^m u$ of which of the order $0 \le k+m \le l$ belong to V, will be denoted by the symbol V^l. Supplement of V^l over the norm

$$\|u\|_l^2 = \sum_{k+m=0}^{l} \| D_\xi^k D_\tau^m u\|^2$$

will be called V_2^l-space. It can be easily verified that the elements from V_2^l have generalized derivatives of the order l, and for each derivative $D_\xi^k D_\tau^m u$ continuous mappings $\xi \to D_\xi^k D_\tau^m u(\xi,\cdot)$ and $\tau \to D_\xi^k D_\tau^m u(\cdot,\tau)$ are defined, with the values in the spaces $W_2^{l-k-m}(-\infty,\infty)$ and $W_2^{l-k-m}(0,\infty)$, respectively. The space V_2^0 is identified with V_2, and the norm $\|\cdot\|_0$ is identified with the norm $\|\cdot\|$. The norm in the classical space W_2^l will be denoted by $|\cdot|_l$.

Let the function $\Phi(\omega)$ belong to $L_2(-\infty,\infty)$, and $v(\xi,\omega)$ be continuous at $\xi \ge 0$, $-\infty < \omega < \infty$, and such that

$$|v(\xi,\omega)| \le \begin{cases} C_0(|\omega|+1)^{-1}, & \xi \le h; \\ 0, & \xi > h. \end{cases}$$

Consider the improper integral

$$u(\xi,\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\Phi(\omega)v(\xi,\omega)\, d\omega, \tag{21}$$

that is understood as the limit of proper integrals

$$u_n(\xi,\tau) = \int_{B_n} e^{i\omega\tau}\Phi(\omega)v(\xi,\omega)\, d\omega.$$

where $\{P_n\}$ is the arbitrary sequence of finite expanding intervals, the union of which is equal to $(-\infty, \infty)$.

Lemma 3. *The functions $u_n(\xi, \tau)$ belong to V_2 and form a fundamental sequence in this space. The corresponding limit $u(\xi, \tau)$ is identified with the value of the improper integral (21) and satisfies the inequality*

$$\|u\|_0 \leq C \cdot |\Phi|_0,$$

where C does not depend on Φ.

The proof directly follows from the Hölder inequality and the Parseval equality, and therefore, we do not give it here. It should only be noted, that the conclusion of the Lemma will remain valid for the functions

$$u(\xi, \tau) = \int_{-\infty}^{\infty} e^{i\omega(\tau \pm \xi)} \Phi(\omega) \psi(\xi, \omega)\, d\omega,$$

where $\Phi(\omega)$ belongs to $L_2(-\infty, \infty)$. Under the improper integral we understand its principal value.

Now we can formulate and prove the basic statement of the given section.

Theorem 1. *Let the function $f(\tau)$, determining the form of the initial wave, belong to the space W_2^2 and vanish outside of some interval $[a, \infty)$. Then the problem (9)–(11) has a unique solution in the space V_2^2, and the estimate*

$$\|u\|_2 \leq C \cdot |f|_2 \tag{22}$$

is valid, where C does not depend on f.

Proof. Uniqueness was proved earlier, and therefore, we shall take up the question of existence. We shall show that the required solution $u(\xi, \tau)$ is given by the formula (20), in which the function $U(\xi, \omega)$ is determined from the equality (18). For this we consider the auxiliary integral

$$u_n(\xi, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{i\omega\tau} U(\xi, \omega)\, d\omega. \tag{23}$$

Obviously, the function u_n is twice continuously differentiable, and the operator of differentiation can be put under the integral over the variable It follows from this and from (12)–(13) that u_n satisfies the equations

(9)–(10). If it will be proved that at $n\to\infty$ every derivative $D_\xi^k D_\tau^m u_n$ of the order $k+m\le 2$ tends to some limit in the space V_2, the function $u(\xi,\tau)$ will be the element of V_2^2 and will also satisfy the equations (9)–(10).

Derivatives of u_n can be easily computed from (18) and (23) using (14)

$$D_\xi^k D_\tau^m u_n = J_{1n} + J_{2n} + J_{3n}, \tag{24}$$

where

$$J_{1n} = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{i\omega(\tau+\xi)}(i\omega)^{k+m}F(\omega)\,d\omega$$

$$J_{2n} = -\frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{i\omega(\tau-\xi)}(i\omega)^{k+m}F(\omega)S(\omega)\,d\omega$$

$$J_{3n} = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{i\omega\tau}(i\omega)^{m}F(\omega)\left[e^{i\omega\xi}\psi_k(\xi,\omega) - e^{-i\omega\xi}S(\omega)\psi_k(\xi,-\omega)\right]d\omega.$$

The expression $(i\omega)^l F(\omega)$ is the Fourier transform of the derivative $f^{(l)}(\tau)$. For $l\le 2$ all these derivatives together with the functions $(i\omega)^l F(\omega)$ belong to the space L_2. But then, on the basis of (19) and the remark to Lemma 3 the integrals J_{1n}, J_{2n} have limits in the space V_2, and norms of limiting functions are majorized by the value

$$C\,|(i\omega)^{k+m}F(\omega)|_0 = C\,|D_\tau^{k+m} f(\tau)|_0 \le C\,|f|_2.$$

Existence of the limit at J_{3n} together with the analogous estimate of the norm of the limiting function also follows from Lemma 3, but the formulas (15) and (19) should be used.

Thus, the function $u(\xi,\tau)$ really satisfies the system (9)–(10), and the inequality (22) is fulfilled for it. It remains to verify the condition (11). This can be done very easily, when f is finite and infinitely differentiable. Really, by the Paley-Wiener theorem [6] the Fourier transform $F(\omega)$ of this function is analytical in the whole complex plane and at any $p\ge 0$ it allows the estimate

$$|F(\omega)| \le \frac{A_p}{(|\omega|+1)^p}\exp(a\,\mathrm{Im}\,\omega), \tag{25}$$

where a is the left boundary of supp f. Let us assume that $k=m=0$ in (24). Owing to the inversion formula of the Fourier transform the integral J_{1n} will tend to $f(\xi+\tau)$. For the computation of the limit of J_{2n} we shall use analyticity of the integrand and pass on to the integration over the semicircle $\Gamma_n = \{\,|\omega| = n,\ \mathrm{Im}\,\omega\le 0\,\}$. For $\omega\in\Gamma_n$ on the basis of (19) and (25) we have

$$|e^{-i\omega\xi}F(\omega)S(\omega)| = O\left((|\omega|+1)^{-p}\exp[(2\alpha-a)|\operatorname{Im}\omega|]\right).$$

But then, according to the well-known Jordan lemma, the integral J_{2n}' will tend to zero at all $\tau < a-2\alpha$. Similarly, using (15), (19), (25) and the Jordan lemma it is determined that J_{3n} tends to zero at $\tau < a-2\alpha-h$.

In the general case there exists the sequence $f_n(\tau)$ of infinitely differentiable finite functions, converging to f in the space W_2^2. This sequence should always be chosen so that the supports of functions f_n will lie in the interval $[a-\varepsilon,\infty)$, where $\varepsilon > 0$. Let us denote by $v_n(\xi,\tau)$ the solution of the problem (9)–(11) with "initial data" $f_n(\tau)$. As it was just shown, $v_n(\xi,\tau) = f_n(\xi+\tau)$ at $\tau < a-\varepsilon-2\alpha-h$. On the other hand, the inequality (22) is valid for the difference $u - v_n$. So,

$$\|u - v_n\|_2 \le C\,|f - f_n|_2 \to 0 \quad (n \to \infty).$$

It directly follows from this that at all $\tau < a-\varepsilon-2\alpha-h$

$$u(\xi,\tau) = \lim_{n\to\infty} v_n(\xi,\tau) = \lim_{n\to\infty} f_n(\xi+\tau) = f(\xi+\tau). \qquad \square$$

3. Inverse problem

After we have passed from the old variables (z,w) to new variables (ξ,u) the inverse problem stated in Section 1 can be formulated in the following way: the form $f(\tau)$ of the initial wave and the value $g(\tau) = u(0,\tau) = \sqrt{\sigma(0)}\,w(0,\tau)$ of the corresponding solution to the direct problem (9)–(11) are known. It is necessary to find the coefficient $q(\xi)$ of the equation (9). Further the inverse problem will be solved in this formulation and, following [3], we shall try to reduce it to the well-studied inverse spectral problem for the Sturm-Liouville equation.

First of all let us recall the necessary information from the spectral theory of differential operators. We denote the operator

$$u(\xi) \to -u''(\xi) + q(\xi)\,u(\xi),$$

acting in $L_2(0,\infty)$ by A. Its domain of definition consists of all the functions belonging to $W_2^2(0,\infty)$ and satisfying the boundary condition

$$u'(0) - k\,u(0) = 0.$$

This operator is, evidently, self-adjoint and, therefore, its spectrum lies on the real axis $\operatorname{Im}\lambda = 0$ of the complex plane. Moreover, the finiteness of $q(\xi)$ guarantees (see [4]) that the whole half-line $\lambda > 0$ consists of the points of continuous spectrum, but the spectrum in the half-line $\lambda < 0$ can be only discrete. In our case operator A cannot have any eigenvalues also in the half-line $\lambda \leq 0$. Otherwise, if we return to the initial variable z, we shall find non-positive eigenvalues of the spectral problem (7)–(8), but this is impossible.

Consider the solution of Cauchy problem

$$-\theta''(\xi,\lambda) + q(\xi)\theta(\xi,\lambda) = \lambda\theta(\xi,\lambda),$$
$$\theta(0,\lambda) = 1, \quad \theta'(0,\lambda) = k,$$

determined at all $\xi \geq 0$ and any complex λ. Function $\theta(\xi,\lambda)$ generates the integral transform

$$v(\lambda) = \int_0^\infty u(\xi)\theta(\xi,\lambda)\,d\xi. \tag{26}$$

It is known [4] that this transform isometrically maps $L_2(0,\infty)$ onto some weight space $L_{2,\rho}(-\infty,\infty)$ consisting of all ρ-measurable functions $v(\lambda)$ such that

$$\int_{-\infty}^\infty |v(\lambda)|^2\,d\rho(\lambda) < \infty.$$

Initial operator A is transformed into multiplication by λ by means of (26), and the mapping inverse to (26) has the form

$$u(\xi) = \int_{-\infty}^\infty v(\lambda)\theta(\xi,\lambda)\,d\rho(\lambda). \tag{27}$$

Here $\rho(\lambda)$ is non-decreasing continuous on the right real-valued function that is called spectral distribution function of operator A. Point set of the function growth coincides with the spectrum A, and the continuous function $\rho(\lambda)$ corresponds to the operator with continuous spectrum. The properties of operator A justified above make it possible to correct the inversion formula (27) and fulfil the integration over the interval $(0,\infty)$.

The classical inverse Sturm-Liouville problem consists in the determination of the coefficient $q(\xi)$ using the given spectral function $\rho(\lambda)$. There are several effective methods for the solution of this problem (see [3–5]), owing to which the inverse problem that we formulated can be considered to be solved, if any way of construction of $\rho(\lambda)$ using the given $f(\tau)$ and $g(\tau)$ will be found. Below we shall try to find such constructions.

Let us suppose that the function $f(\tau)$ satisfies the conditions of the Theorem 1, and turn our attention to the equality (18) determining the formal Fourier transform $U(\xi,\omega)$ from the solution of the direct problem. By Theorem 1 the solution $u(\xi,\tau)$ belongs to the space V_2^2. It is possible to apply the usual Fourier transform over the variable τ to the elements of this space. Consequently, $U(\xi,\omega)$ is not only formal, but it is the actual Fourier transform of $u(\xi,\tau)$. In particular, $U(0,\omega)$ coincides with the Fourier transform of the function $g(\tau) = u(0,\tau)$ that will be denoted by $G(\omega)$. Taking into account (16) and (18) we have

$$\begin{aligned} G(\omega) &= F(\omega)\left[e(0,\omega) - S(\omega)\,e(0,-\omega)\right] \\ &= \frac{F(\omega)}{s(-\omega)}\left[e(0,\omega)\,e'(0,-\omega) - e'(0,\omega)\,e(0,-\omega)\right] = -F(\omega)\frac{2i\omega}{s(-\omega)}. \end{aligned}$$

It follows from this equation and from the absence of zeros of the function $s(\omega)$ at all real $\omega \neq 0$, that $F(\omega)$ and $G(\omega)$ can vanish only simultaneously. Moreover,

$$s(\omega) = \frac{2i\omega\,F(-\omega)}{G(-\omega)} \tag{28}$$

in every point ω, where $F(\omega) \neq 0$. If the set of these points is everywhere dense (for example, in the case of finite f), then the relation (28) continuously extends to all values $\omega \neq 0$.

So, in principle, the data f and g of the inverse problem make it possible to compute the function $s(\omega)$. It turns out that the required spectral function $\rho(\lambda)$ is connected with $s(\omega)$ by means of a simple formula that was, probably, first proved in [5] for the boundary condition $u(0,\tau) = 0$. The same statement is obtained in [7] for the case of general boundary conditions, but it used one result of M.G. Kreĭn [8], the proof of which was not published. We present the new derivation of this formula, based on another idea.

Theorem 2. *Spectral function $\rho(\lambda)$ of operator A is infinitely differentiable for $\lambda > 0$, and the equality*

$$\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi|s(\sqrt{\lambda})|^2}. \tag{29}$$

is valid.

Proof. Note that the function $\theta(\xi,\lambda)$ satisfies the equation (12) at $\omega = \sqrt{\lambda}$, and that is why it decomposes into the linear combination of $e(\xi,\sqrt{\lambda})$ and $e(\xi,-\sqrt{\lambda})$. Coefficients of this decomposition are easily computed from the initial conditions $\theta(0,\lambda) = 1$ and $\theta'(0,\lambda) = h$ using formula (16)

$$\theta(\xi,\lambda) = -\frac{s(-\sqrt{\lambda})}{2i\sqrt{\lambda}}\, e(\xi,\sqrt{\lambda}) + \frac{s(\sqrt{\lambda})}{2i\sqrt{\lambda}}\, e(\xi,-\sqrt{\lambda}). \tag{30}$$

As $f(\tau)$ let us choose some finite infinitely differentiable function and consider the corresponding solution $u(\xi,\tau)$ of the direct problem (9)–(11). According to Theorem 1, the function u belongs to the space V_2^2 and, consequently, at any fixed τ it is in the domain of definition of operator A. This makes it possible to apply the transform (26) to both parts of the equation (9). Assuming that

$$v(\lambda,\tau) = \int_0^\infty u(\xi,\tau)\theta(\xi,\lambda)\, d\xi, \quad \lambda > 0, \tag{31}$$

and taking into account that operator A will transform into multiplication by λ, we obtain the elementary equation $v_{\tau\tau} = -\lambda v$, general solution to which has the form

$$v(\lambda,\tau) = \alpha(\lambda)\exp(i\tau\sqrt{\lambda}) + \beta(\lambda)\exp(-i\tau\sqrt{\lambda}).$$

Coefficients α and β will be determined from the condition (11). Let $\tau \le \tau_0$, then $u(\xi,\tau) = f(\xi+\tau)$, and such $\tau_1 \le \tau_0$ will be found that if $\tau \le \tau_1$, then the support of the function $f(\xi+\tau)$ lies in the domain $\xi > h$. In the same domain $e(\xi, \pm\sqrt{\lambda})$ coincides with $\exp(\pm i\xi\sqrt{\lambda})$. Consequently, substituting (30) into (31), we shall have for $\tau \le \tau_1$

$$\begin{aligned} v(\lambda,\tau) &= \frac{1}{2i\sqrt{\lambda}} \int_0^\infty f(\xi+\tau)\left[s(\sqrt{\lambda})e^{-i\xi\sqrt{\lambda}} - s(-\sqrt{\lambda})e^{i\xi\sqrt{\lambda}}\right] d\xi \\ &= \frac{\sqrt{2\pi}\, s(\sqrt{\lambda})}{2i\sqrt{\lambda}} F(\sqrt{\lambda})\, e^{i\tau\sqrt{\lambda}} - \frac{\sqrt{2\pi}\, s(-\sqrt{\lambda})}{2i\sqrt{\lambda}} F(-\sqrt{\lambda})\, e^{-i\tau\sqrt{\lambda}}. \end{aligned}$$

Due to analyticity of $v(\lambda,\tau)$ over the variable τ this relation will be valid also at $\tau > \tau_1$.

Now let us substitute the found decomposition of the function $v(\lambda,\tau)$ into the inversion formula (27) and assume that $\xi = 0$ in it. Taking into account that $\theta(0,\lambda) = 1$ we obtain

$$g(\tau) = u(0,\tau) = \int_0^\infty v(\lambda,\tau)\theta(0,\lambda)\, d\rho(\lambda) =$$

$$\sqrt{2\pi}\int_0^\infty \frac{s(\sqrt{\lambda})}{2i\sqrt{\lambda}} F(\sqrt{\lambda})\, e^{i\tau\sqrt{\lambda}}\, d\rho(\lambda) - \sqrt{2\pi}\int_0^\infty \frac{s(-\sqrt{\lambda})}{2i\sqrt{\lambda}} F(-\sqrt{\lambda})\, e^{-i\tau\sqrt{\lambda}}\, d\rho(\lambda).$$

In the first of the integrals we pass to the new variable $\omega = \sqrt{\lambda}$, and in the second one – to the variable $\omega = -\sqrt{\lambda}$. Finally, $g(\tau)$ will have the form

$$g(\tau) = \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{s(\omega)}{2i|\omega|} F(\omega)\, e^{i\tau\omega}\, d\rho(\omega^2). \tag{32}$$

On the other hand, the Fourier image $G(\omega)$ of the function $g(\tau)$ is related with $F(\omega)$ by means of the equality (28), due to which

$$g(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega)\, e^{i\tau\omega}\, d\omega = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2i\omega}{s(-\omega)} F(\omega)\, e^{i\tau\omega}\, d\omega. \tag{33}$$

The obtained different representations of the function $g(\tau)$ allow the computation of ρ. For this we take an arbitrary infinitely differentiable function $\varphi(\omega)$ that is equal to zero outside of the interval $[0, \sqrt{\lambda}]$, multiply the equality (32) by its Fourier image $\Phi(\tau)$ and integrate over τ. As the improper integral (i.e., its principal value) in (32) converges to $g(\tau)$ in the metric of the space $L_2(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} \Phi(\tau) g(\tau)\, d\tau = \lim_{n\to\infty} \int_{-\infty}^{\infty} \Phi(\tau) \left[\int_{-n}^{n} \frac{s(\omega)}{2i|\omega|} F(\omega)\, e^{i\tau\omega}\, d\rho(\omega^2) \right] d\tau.$$

After the rearrangement of integrations that is possible on the basis of the Fubini theorem, we find

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(\tau) g(\tau)\, d\tau &= \lim_{n\to\infty} 2\pi \int_{-n}^{n} \frac{s(\omega)}{2i|\omega|} F(\omega) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\tau) e^{i\tau\omega} d\tau \right] d\rho(\omega^2) \\ &= 2\pi \int_{0}^{\sqrt{\lambda}} \frac{s(\omega)}{2i\omega} F(\omega) \varphi(\omega)\, d\rho(\omega^2). \end{aligned}$$

Similarly, it follows from (33) that

$$\int_{-\infty}^{\infty} \Phi(\tau) g(\tau)\, d\tau = -\int_{0}^{\sqrt{\lambda}} \frac{2i\omega}{s(-\omega)} F(\omega) \varphi(\omega)\, d\omega.$$

Comparing the obtained formulas and using the arbitrariness of $\varphi(\omega)$, we conclude that for any $\lambda > 0$

$$\pi \int_0^{\sqrt{\lambda}} \frac{s(\omega)}{\omega} F(\omega)\, d\rho(\omega^2) = \int_0^{\sqrt{\lambda}} \frac{2\omega}{s(-\omega)} F(\omega)\, d\omega.$$

Hence, applying integration by parts, we obtain

$$\pi \frac{s(\sqrt{\lambda})}{\sqrt{\lambda}} F(\sqrt{\lambda})\, \rho(\lambda) = \pi \int_0^{\sqrt{\lambda}} \rho(\omega^2) \left[\frac{s(\omega)}{\omega} F(\omega)\right]' d\omega + \int_0^{\sqrt{\lambda}} \frac{2\omega}{s(-\omega)} F(\omega)\, d\omega.$$

Diferentiating the both parts of this equality, it is easy to verify that ρ has a derivative in every point $\lambda > 0$, for which $F(\sqrt{\lambda}) \neq 0$, and the formula (29) is valid. It only remains for us to note that F is the Fourier transform of the finite function, and therefore it is analytical and can vanish only on the denumerable set of isolated points. But then, due to the continuity of the right hand side of (29) at $\lambda > 0$, the equality (29) is valid also in these exceptional points. □

Note, finally, that for any summable function $f(\tau)$ and any real l, modules of the Fourier images $f(\tau)$ and $f(\tau + l)$ are identical. So, the spectral function $\rho(\lambda)$ determined by the formulas (28) and (29) and, hence, the required σ and q do not change under shifts of arguments of the data f or g of the inverse problem. Critical values H (or h) of arguments z (or ξ), determining boundaries of the areas, where the medium characteristics are not known, influence only the argument's shift of the function $f(\tau)$. Therefore it is not necessary to give the value H at all for the solution of the inverse problem. Moreover, it can be found after the determination of coefficients q or σ.

4. Numerical experiments

As it was shown in the previous section, the formulas (28) and (29) permit the spectral function $\rho(\lambda)$ to be found in the explicit form for the initial differential operator, if the data f and g of the corresponding inverse problem are known. Further reconstruction of the coefficients $q(\xi)$ and k can be realized in many ways. One of the most effective methods has been developed by M.G. Kreĭn [9] in 1954. Let us recall the main idea of this algorithm.

Let $\rho(\lambda)$ be a non-decreasing function, vanishing for $\lambda < 0$. It is necessary to find out whether there is a differential operator with the spectral function $\rho(\lambda)$. Its coefficients $q(\xi)$ and k should also be found. Assume that $\nu(\lambda) = \rho(\lambda) - 2\sqrt{\lambda}/\pi$ and consider the improper integral

$$a(x) = \int_1^\infty \frac{\cos\sqrt{\lambda}x}{\lambda}\, d\nu(\lambda).$$

For existance of the required operator it is sufficient (see, for example, [4]) that the function $a(x)$ have continuous derivatives up to the 4-th order at all $x \leq 0$. This restriction concerns only the behaviour of $\nu(\lambda)$ when $\lambda \to \infty$. In the practical calculations, the function ν is usually considered to be equal to zero for the large values of λ. Consequently, the given requirement is fulfilled automatically.

If this condition is fulfilled, the reconstruction of the coefficients $q(\xi)$ and k reduces to the search for a solution to the following integral equation of Kreǐn

$$y(x, 2\xi) + \int_0^{2\xi} J(x - \zeta) y(\zeta, 2\xi)\, d\zeta = 1, \quad 0 \leq x \leq 2\xi, \tag{34}$$

where

$$J(x) = \frac{1}{2} \int_0^\infty \cos(\sqrt{\lambda}x)\, d\nu(\lambda).$$

It can be shown [3–5, 9] that the operator in the left-hand side of the equality (34) is positive. Therefore, this equation has the unique solution $y(x, 2\xi)$. The required values $q(\xi)$, k and $\sigma(\xi)$ are expressed by means of y using the formulas

$$q(\xi) = \frac{1}{y(2\xi, 2\xi)} \frac{d^2}{d\xi^2} y(2\xi, 2\xi), \qquad k = \frac{d}{d\xi} \ln y(2\xi, 2\xi) \Big|_{\xi=0},$$

$$\sigma(\xi) = \sigma(0) \cdot y^2(2\xi, 2\xi).$$

The described method can be realized very effectively if the algorithm of the fast Fourier transform will be used to find the spectral function by the formulas (28)–(29), and if the Levinson recursion (see [10]) will be used to solve the equation (34). On this basis we have made up a program for the computation of the solution to the inverse problem. This program has demonstrated a sufficiently high efficiency and accuracy of calculations for the test examples. The numerical experiments were conducted according to the following scheme. At first the solution to the direct problem, i.e., the synthetic surface seismogram $g(\tau)$ was computed for the known medium model $\sigma(\xi)$ and the given form of the incident wave $f(\tau)$. Then the function f and some segments of the synthetic seismogram g of different length were used as the initial data for the calculation of the solution to the inverse problem. As a result, accuracy of the medium reconstruction depending on the used segment of the synthetic seismogram was estimated. The angle of incidence of the initial wave was considered to be equal to zero.

The medium model was based on the real measurements presented by the State Geological Corporation 'Eniseigeofizica'. They contained information about density and velocity at 228 points in the borehole 'Chambinskaya-114' up to the depth of 3362 metres. Then the function $\sigma(\xi)$ computed for these real data was averaged and smoothed. Further calculations were carried out for the smooth $\sigma(\xi)$. The diagrams of the original and smoothed functions are shown in Figure 1.

The initial wave $f(\tau)$ was analytically described by the smooth finite function. It has been tested on several wave types. However, no dependence between the wave form and the accuracy of the medium reconstruction was found in our experiments. The incident wave of the type

$$f(\tau) = \begin{cases} \pi^2(1-10\tau)^2\{1-\exp[-(25\pi\tau)^2]\}\sin 50\pi\tau, & 0 < \tau < 0.1 \text{ (sec)}, \\ 0, & \text{otherwise}, \end{cases}$$

and corresponding synthetic seismogram are presented in Figure 2.

And, finally, Figure 3 shows the results of the medium reconstruction using the initial segments of this synthetic seismogram, corresponding to the time intervals of 1.6 sec (dotted line), 2.4 sec (dashed line) and 3.2 sec (thin line). The quality of reconstruction improves with the increase of the time interval. The graphs of the initial and reconstructed functions are practically identical when the length of the time interval is more than 4 sec.

The numerical experiments were performed at the work station SUN SPARC and on IBM PC 386/387. The run time for the inverse problem did not exceed 2–30 sec, depending on the required accuracy and the length of the used seismogram segment.

5. Some conclusions and generalizations

The statement of inverse problem that we considered is typical for the scattering theory, tomography, flaw detection and other problems where it is natural to suppose that the form $f(\tau)$ of the initial wave is known. As a rule, direct information on the form of the initial wave is absent in the problems of seismoprospecting. Here values of the full wave field (i.e., sum of incoming and scattered waves) in one or several points deep inside the medium and on its surface are natural observed characteristics of the wave

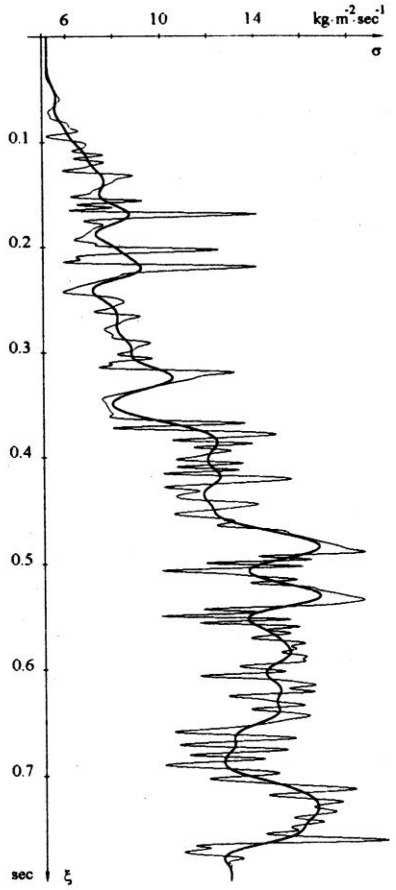

Figure 1. Original medium

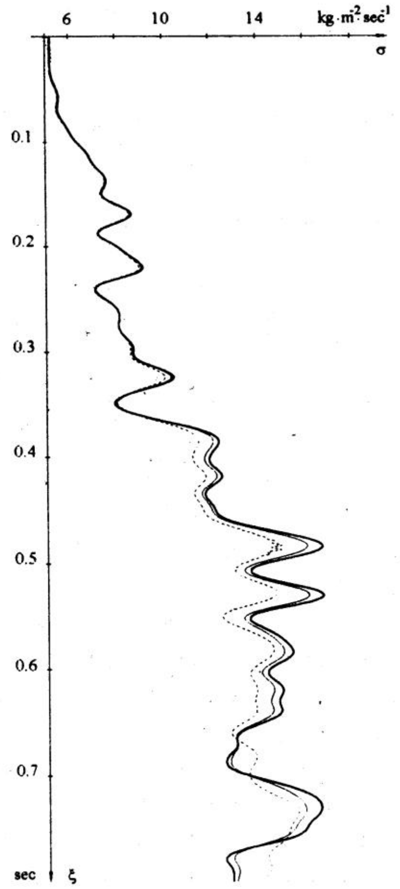

Figure 3. Restored medium

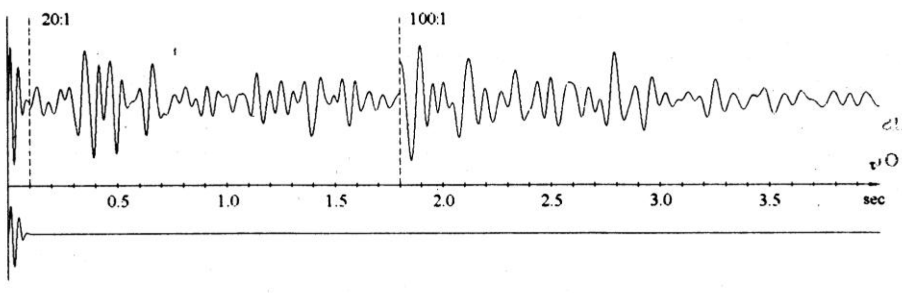

Figure 2. Synthetic seismogram and incident wave

process. This fact hinders direct application of the algorithms described above to the problems of seismoprospecting. In the given section these algorithms will be adapted to the needs of geophysics.

First of all let us find out, what information on the medium structure and character of the wave process is sufficient for the reconstruction of the unknown form of the initial wave. For this we again turn our attention to the equation (18) relating the Fourier images of functions $u(\xi,\tau)$ and $f(\tau)$. Differentiating this equation over the variable ξ we come to the system

$$\begin{cases} U(\xi,\omega) = F(\omega)\, e(\xi,\omega) - S(\omega)F(\omega)\, e(\xi,-\omega), \\ U_\xi(\xi,\omega) \;= F(\omega)\, e'(\xi,\omega) - S(\omega)F(\omega)\, e'(\xi,-\omega). \end{cases} \tag{35}$$

Hence, taking into account (16) we get

$$F(\omega) = \frac{1}{2i\omega}[U_\xi(\xi,\omega)\, e(\xi,-\omega) - U(\xi,\omega)\, e'(\xi,-\omega)]. \tag{36}$$

Consequently, the function $F(\omega)$ can be found if the values $e(\xi,\omega)$, $e'(\xi,\omega)$ and $U(\xi,\omega)$, $U_\xi(\xi,\omega)$ in some point ξ are known. Note that $e(\xi,\omega)$ is always known in the domain $\xi \geq h$ where it coincides with $\exp(i\xi\omega)$. Therefore, it is sufficient to know the full wave field $u(\xi,\tau)$ and its derivative with respect to ξ in some point $\xi \geq h$ for the reconstruction of $f(\tau)$ or its Fourier image $F(\omega)$.

This brings up the question: to what extent are the medium parameters determined by the values $u(x,\tau)$ and $u_\xi(x,\tau)$ at some $x < h$? Naturally, the function $g(\tau) = u(0,\tau)$ is considered to be known as before. We shall show that these data uniquely determine the coefficient $q(\xi)$ in equation (9) only in the interval $0 \leq \xi \leq x$, while at $\xi > x$ the values $q(\xi)$ can be, generally speaking, arbitrary.

Let $u(\xi,\tau)$ be the solution of the problem (9)–(11) with some $f(\tau)$ satisfying the conditions of Theorem 1. We choose some value x from the interval $(0,h)$ and arbitrarily change the function $\sigma(\xi)$ outside of the interval $[0,x]$ so that it will remain positive, twice continuously differentiable and constant at all sufficiently large ξ. Together with $\sigma(\xi)$, $q(\xi)$ will also change correspondingly. These modified functions σ and q will be denoted by $\tilde{\sigma}$ and $\tilde{q}$. On the interval $[0,x]$ the evident identities $\tilde{\sigma}(\xi) \equiv \sigma(\xi)$, $\tilde{q}(\xi) \equiv q(\xi)$ are valid.

Theorem 3. *There exists the function $\tilde{f}(\tau)$ satisfying the conditions of Theorem 1 and such that the solution $\tilde{u}(\xi,\tau)$ of the problem (9)–(11) with the coefficient $\tilde{q}(\xi)$ and "initial data" $\tilde{f}(\tau)$ coincides with $u(\xi,\tau)$ at all $\xi \leq x$, $-\infty < \tau < \infty$.*

Proof. On the basis of Theorem 1 we pass from evolutional to the corresponding spectral problems. The symbol $\tilde{e}(\xi,\omega)$ will be used to denote the function playing the same part in the equation

$$-\tilde{U}_{\xi\xi}+\tilde{q}(\xi)\tilde{U}=\omega^2\tilde{U}, \tag{37}$$

that $e(\xi,\omega)$ in the equation (12). It is sufficient to find solution of the equation (37) coinciding with $U(\xi,\omega)$ on the interval $0\le\xi\le x$ and decomposing into the linear combination

$$\tilde{U}(\xi,\omega)=\tilde{F}(\omega)\,\tilde{e}(\xi,\omega)+\tilde{G}(\omega)\,\tilde{e}(\xi,-\omega), \tag{38}$$

where $\tilde{F}(\omega)$ is the Fourier image of some function $\tilde{f}(\tau)$ from the class W_2^2 with the support that is bounded from the left. It will be shown that the solution of the Cauchy problem for the equation (37) with the initial conditions

$$\tilde{U}(x,\omega)=U(x,\omega),\quad \tilde{U}_\xi(x,\omega)=U_\xi(x,\omega) \tag{39}$$

satisfies these requirements.

As the coefficients $q(\xi)$ and $\tilde{q}(\xi)$ on the interval $[0,x]$ coincide, the functions $U(\xi,\omega)$ and $\tilde{U}(\xi,\omega)$ are also equal to each other at $\xi\le x$ and all ω due to the uniqueness of the solution to the Cauchy problem. In particular, $\tilde{U}(\xi,\omega)$ together with $U(\xi,\omega)$ satisfy the boundary condition (13).

Let us find the coefficients of decomposition (38). For this we substitute the expression (38) into the boundary conditions (39), right hand sides of which will be determined from the relations (35). As a result we obtain the system of linear equations for $\tilde{F}$ and $\tilde{G}$, and solving it we get

$$\begin{aligned}\tilde{F}(\omega)&=\frac{F(\omega)}{2i\omega}\big[E'(x,\omega)\tilde{e}(x,-\omega)-E(x,\omega)\tilde{e}'(x,-\omega)\big],\\ \tilde{G}(\omega)&=\frac{F(\omega)}{2i\omega}\big[E(x,\omega)\tilde{e}'(x,\omega)-E'(x,\omega)\tilde{e}(x,\omega)\big],\end{aligned}$$

where

$$E(x,\omega)=e(x,\omega)-S(\omega)\,e(x,-\omega),\quad E'(x,\omega)=e'(x,\omega)-S(\omega)\,e'(x,-\omega).$$

Note that the functions $\tilde{F}$ and $\tilde{G}$ have a removable singularity at $\omega=0$. In fact, on the basis of Lemma 2 we have $s(0)=0$, $s_\omega(0)\ne 0$, and therefore, $S(0)=-1$, $E(x,0)=2e(x,0)$, $E'(x,0)=2e'(x,0)$. From this, using the representations

$$e(\xi,0)=[\sigma(\xi)/\sigma_0]^{1/2},\quad \tilde{e}(\xi,0)=[\tilde{\sigma}(\xi)/\tilde{\sigma}_0]^{1/2},$$

found in Lemma 2, we obtain

$$E'(x,\omega)\tilde{e}(x,-\omega) - E(x,\omega)\tilde{e}'(x,-\omega)\Big|_{\omega=0} =$$
$$[\sigma_0\tilde{\sigma}_0]^{-1/2}[\sigma'(x)(\tilde{\sigma}(x)/\sigma(x))^{1/2} - \tilde{\sigma}'(x)(\sigma(x)/\tilde{\sigma}(x))^{1/2}].$$

This expression is equal to zero, because the functions σ and $\tilde{\sigma}$ coincide (together with their derivatives) at the point x. Similarly,

$$E(x,\omega)\tilde{e}'(x,\omega) - E'(x,\omega)\tilde{e}(x,\omega)\Big|_{\omega=0} = 0.$$

So, $\tilde{F}$ and $\tilde{G}$ are analytical in the half-plane $\operatorname{Im}\omega \leq 0$, and due to (14) and (15) their modules are majorized by the value $C\exp(-\gamma \operatorname{Im}\omega)$, where C and γ are positive constants. All the necessary properties of the function $\tilde{F}$ are the result of this.

Firstly, $f(\tau)$ belongs to W_2^2. So, the product $F(\omega)\ (1+\omega^2)$ belongs to $L_2(-\infty,\infty)$. But then $\tilde{F}(\omega)\ (1+\omega^2)$ is also in $L_2(-\infty,\infty)$ and, consequently, $\tilde{F}(\omega)$ is the Fourier transform of some function $\tilde{f}(\tau)$ from the class W_2^2.

Secondly, support of the function $f(\tau)$ lies in the domain $a \leq \tau$. By the Paley-Wiener theorem (see [6]) this is equivalent to the analyticity of $F(\omega)$ in the half-plane $\operatorname{Im}\omega < 0$ and the existence of the uniform estimate

$$\int_{-\infty}^{\infty} |F(\lambda + i\mu)|^2\, d\lambda \leq \mathrm{const}\cdot e^{2a\mu}, \quad \mu < 0.$$

After the replacement of a by $a-\gamma$, the function $\tilde{F}$ will also satisfy the same estimate and therefore $\operatorname{supp}\tilde{f}$ will lie in the domain $a-\gamma \leq \tau$. □

So, if the form of the incoming wave is unknown, nothing definite can be said about the medium structure outside of the observation interval. Whatever the acceptable medium parameters in the domain $\xi > x$, the form of the initial wave can always be chosen so that the response caused by it will coincide with the observed wave field on the whole interval $0 \leq \xi \leq x$ and in all time moments. Nevertheless, the result of Theorem 3 is still positive, as it allows the reconstruction of the coefficients $\sigma(\xi)$ and $q(\xi)$ on the interval $0 \leq \xi \leq x$ using the given values

$$u(x,\tau) = \varphi(\tau), \quad u_\xi(x,\tau) = \psi(\tau), \quad u(0,\tau) = g(\tau).$$

Really, let the values of the function σ and its first and second derivatives in the point $\xi = x$ be known additionally. We construct the new function $\tilde{\sigma}(\xi)$ so that it will coincide on the interval $[0,x]$ with the function

$\sigma(\xi)$ that is already available but still unknown. For $\xi > x$ we determine this function in any known way, only the requirements of Theorem 3 should be fulfilled. In accordance with the conclusion of Theorem 3 there exists such initial wave $\tilde{f}(\tau)$ that the solution $\tilde{u}(\xi,\tau)$ corresponding to this wave will satisfy the conditions

$$\tilde{u}(x,\tau) = \varphi(\tau), \quad \tilde{u}_\xi(x,\tau) = \psi(\tau), \quad \tilde{u}(0,\tau) = g(\tau). \tag{40}$$

Moreover, at $\xi \geq x$ the function $\tilde{\sigma}(\xi)$ and the coefficient $\tilde{q}(\xi)$ generated by it are known. So, values of the special solution $\tilde{e}(\xi,\omega)$ to the equation (12) and its derivative $\tilde{e}'(\xi,\omega)$ in the domain $\xi \geq x$ and, in particular, in the point x itself can be found. But then the Fourier image $\tilde{F}(\omega)$ of hypothetical initial wave $\tilde{f}(\tau)$ is easily determined from the conditions (40) using the formula (36). Thus, we again obtain the above-considered inverse problem on the reconstruction of the function $\tilde{\sigma}(\xi)$ using the given $\tilde{f}(\tau)$ and $\tilde{u}(0,\tau)$. Solution of this problem gives full information on the values of the initial function $\sigma(\xi)$ on the interval $[0,x]$.

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Translated by O.A. Yudina