Appendix 1

Theorems from Functional Analisys Used in This Book

A1.1. Convergence in Hilbert Space

Let H be real separable Hilbert space with the scalar product (u, v) and the norm $\|u\| = (u, u)^{1/2}$.

Definition 1. *The sequence u_n strongly converges to u ($u_n \xrightarrow{S} u$) if* $\lim\limits_{n\to\infty} \|u_n - u\| = 0$.

Definition 2. *The sequence u_n weakly converges to u($u_n \xrightarrow{W} u$) if $\forall\ v \in H$* $\lim\limits_{n\to\infty} (v, u_n) = (v, u)$.

Definition 3. *The sequence u_n weakly converges to u on the set $K \subset H$ ($u_n \xrightarrow{K} u$) if* $\forall\, v \in K,\ \lim\limits_{n\to\infty} (v, u_n) = (v, u)$.

Theorem 1. If $u_n \xrightarrow{S} u$ then $u_n \xrightarrow{W} u$.

Theorem 2. If $u_n \xrightarrow{W} u$ then $\|u_n\| \le C,\ C = \text{const}$.

Theorem 3. If the set K is dense in H, $u_n \xrightarrow{K} u$, $\|\dot{u}_n\| \le$ const then $u_n \xrightarrow{W} u$.

Theorem 4. If $u_n \xrightarrow{W} u$, $\|u_n\| \to \|u\|$ then $u_n \xrightarrow{S} u$.

Theorem 5. If $u_n \xrightarrow{W} u$, $\|u_n\| \le \|u\|$ then $u_n \xrightarrow{S} u$.

Theorem 6. If $u_n \xrightarrow{W} u$, $\lim\limits_{n\to\infty} \|u_n\| \le \|u\|$ then $u_n \xrightarrow{S} u$.

Theorem 7. Bounded set in H is weak compact.

A1.2. Theorems on Linear Operators

Theorem 8 (Banach). Let X and Y be Banach spaces and $A : X \to Y$ be linear bounded one-to-one operator. Then $A^{-1} : Y \to X$ is bounded.

Theorem 9 (Banach, Steinhaus). Let X and Y be Banach spaces and $A_n : X \to Y$ be the sequence of linear bounded operators. If for every $u \in X$ the sequence $A_n u$ is bounded in Y-norm (or $\forall u \in X\ A_n u \xrightarrow{S} Au$) then the sequence A_n is uniformly bounded in $X \to Y$-operator norm.

Theorem 10 (Fredholm). Let H_1 and H_2 be Hilbert spaces and $A : H_1 \to H_2$ be linear bounded operator with the closed range, and $A^* : H_2 \to H_1$ be the adjoint operator. Then $R(A) = N(A^*)^{\perp}$, where R and N mean range and null-space respectively.

A1.3. Sobolev Spaces in Domain

Let $\Omega \subset R^n$ be some domain with the cone condition. It means that it is possible to touch every point of its boundary by the cone with fixed angle and size inside of domain and outside of it.

Sobolev space $W_p^m(\Omega)$ consists of the functions which are integrable at the power $p \geq 1$ simultanuously with their m-th partial derivatives. The norm in the space $W_p^m(\Omega)$ is defined as follows:

$$\|u\|_{W_p^m(\Omega)} = \left(\sum_{0\leq|\alpha|\leq m} \int_{\Omega} (D^\alpha u)^p d\Omega \right)^{1/p}.$$

If $p = 2$ when $W_2^m(\Omega)$ becomes the Hilbert space with the scalar product

$$(u, v)_{W_2^m(\Omega)} = \sum_{0\leq|\alpha|\leq m} \int_{\Omega} D^\alpha u \cdot D^\alpha v d\Omega.$$

Theorem 11. Let Ω be the domain with the cone condition. Then the linear bounded prolongation operator $P : W_2^m(\Omega) \to W_2^m(R^n)$ does exist,

$$\|Pu\|_{W_2^m(R^n)} \leq \|P\| \cdot \|u\|_{W_2^m(\Omega)}.$$

Theorem 12. If $m > n/2$ then the space $W_2^m(\Omega)$ is continuously embedded into space $C(\bar{\Omega})$ of continuous functions with uniform norm,

$$\forall\, u \in W_2^m(\Omega) \quad \|u\|_{C(\Omega)} \leq C\|u\|_{W_2^m}, \quad C = \text{const}.$$

Moreover, this embedding operator is compact one.

Theorem 13. If $m > s$ then the space $W_2^m(\Omega)$ is continuously embedded into $W_2^s(\Omega)$. Moreover, this embedding operator is compact one.

Theorem 14. The space $W_2^m(\Omega)$ is continuously embedded into the space $W_p^k(\Omega)$ when $k \le p \le \infty$, $\quad k - n/p \le m - n/2$ (except $k = m - n/2$ & $p = \infty$).

Remark. It is possible to introduce the Hilbert spaces $W_2^\alpha(R^n)$ with real (may be negative) index α with the help of well-known property of Fourier transform

$$F(D^\alpha u) = (i\xi)^\alpha F(u), \quad D^\alpha u = F^{-1}((i\xi)^\alpha F(U)).$$

For the case of bounded domain Ω the spaces $W_2^\alpha(\Omega)$ can be constructed with the prolongation-restriction theorems. Theorems 12, 13, 14 are valid also for the real indices.

Appendix 2

On Software Investigations in Splines

During the last 20 years the various software in splines was produced in the Computing Center USSR Academy of Science at Novosibirsk in the Laboratory of Applied Numerical Analysis headed by prof. V.A.Vasilenko. The first variant of software library LIDA (Library on Data Approximation) appeared at the beginning of 80-th (main language was ALGOL-60), the second variant LIDA-2 was created for large computers in 1982 (main language was FORTRAN-IV). More than 50 scientific and industrial organizations have used this software in the USSR and outside (France, Czechoslovakia, Bulgaria). The main fields of applications were geology, geophysics, geodesy, engineering, ecology, medicine, etc. The most powerful variant of our library is LIDA-3. This software was realized in 1987 for large and personal IBM-compatible computers (main languages are FORTRAN and C). The full title of this library is "Software library on data approximation and digital signal and image processing" but the 2-nd part concerning with the special decompositions of digital filters is not presented here. The main authors of LIDA-3 are prof. V.A.Vasilenko, dr. A.J.Rozhenko, dr. A.Yu.Bezhaev, dr. A.V.Kovalkov.

The software library LIDA-3 (division Approximation) consists of two parts: one-dimensional approximation and multi-dimensional approximation, particulary at the scattered meshes. This library includes the independent (for user) subpackages for the solution of approximation problems of different types. As a rule, some variational principle is used for the construction of the suitable spline. Here we discribe briefly the functional possibilities of subpackages

A2.1. One-Dimensional Case

Subpackage ODD. Interpolation and smoothing by polynomial splines of odd degrees by point evaluations, general Anselone-Laurent algorithm is used. Choice of the optimal smoothing parameter from the residual principle. Simultaneous solution of several problems at the same mesh. Generation of informative mesh for noised data using splines on convex sets and fast algorithm for active constrains.

Subpackage EVEN. Interpolation and smoothing by polynomial splines of even degrees by local integrals. Choice of smoothing parameter from the residual principle. Simultaneous solution of several problems at the same mesh.

Subpackage HERM. Hermitian spline interpolation by point evaluations of function and its derivatives. Various interpolation conditions at every mesh point. Simultaneous solution of several problems.

Subpackage LSPL. Interpolation by quasipolynomial L-splines, using special B-splines. Various types of boundary conditions. Approximation of fast oscilations, boundary and interior layers.

Subpackage RATIO. Interpolation by ratio of two polynomial splines. An efficient approximation of boundary and interior layers, separation of pole singularities.

Subpackage APPREX. Analysis of the mesh function to find the latent law of quasipolynomial type, based on the discrete convolution operator which annihilates (or almost annihilates) given data.

A2.2. Multi-Dimensional Case

Subpackage MULODD. Interpolation and smoothing of n-variable function at the n-dimensional grid by the polynomial splines of n variables and different odd degrees with respect to variables. An efficient storage location for the huge problems, in particular with external storage.

Subpackage GREEN. Interpolation and smoothing of n-variable function at the n-dimensional scattered mesh where point evaluations of function or some linear differential expessions are given. Based on the representation of D^m-spline with the help of Green function for polyharmonic operator. Linear algebraic system with dense matrix arises, algorithm is efficient when the number of interpolation points is about 300 per 1 Mbyte main memory. Choice of smoothing parameter.

Subpackage FINEL. Intended for the same problem as subpackage GREEN, but algorithm is based on finite element technique for D^m-splines. B-splines are used as finite elements. Sparse matries, fast iterative method for the solution of algebraic systems. Effective storage distribution. Number of interpolation points is about 6000 per 1 Mbyte for 2-dimensional problems and about 3000 per 1 Mbyte for 3-dimensional case. Choice of smoothing parameter.

Subpackage RAPAS. Approximation of n-variable function given at the scattered mesh by the generalized ratio whose numerator and denominator are linear combinations of given functions (for example polynomials or splines). The best uniform approximation is provided in the mesh points only. RAPAS is intended mainly for data compession to obtain the generalized ratio with a small number of coefficient instead a huge number of original measurements at the scattered points.

Subpackage SFERA. Intended for interpolation of the function given at the scattered mesh points on the unit 3-dimensional sphere. Based on the traces of D^m-splines on the smooth manifold, in this case on the sphere. This approximation is invariant with respect to any parametrization of the sphere. May be used for interpolation of the surfaces of bodies which are one-to-one images of the sphere points.

Subpackage BREAK. Interpolation of 2-dimensional mesh function given at the scattered mesh points by finite element analog of discontinuous D^m-splines. The discontinuaty lines can be automatically determined by the density parameter of mesh and "jump" level, or these lines can be a priori given. As finite elements the special discontinuous B-splines are used. Please note that only discontinuaties of the 1-st type are possible here.

Subpackage RATIO2. Interpolation of 2-dimensional mesh function given at the scattered mesh points by the ratio of two finite element analogs of D^m-splines. It is possible to approximate the functions with the pole singularities or with the boundary and interior layers. The classical Gibbs oscillation effect practically disappears in this interpolation. The discontunuaty lines are also separated automatically in this algorithm.

Subpackage SIGPI. Intended for $\Sigma\Pi$-approximation (and data compression) of the mesh function of two variables which is given at the huge uniform rectangular grid (like digital image). As the functions of one variable in $\Sigma\Pi$-decomposition the discrete analogs of the polynomial splines are used, suitable B-splines are obtained by the discrete convolutions. Subpackage is effective for data compression with the given error level in various Hilbert norms, especially for the image compression, including color images. After the decomposition of color image into black-white component and red-green-blue components they can be compressed separately.