Bull. Nov. Comp. Center, Num. Anal., 1 (1993)

Certain aspects of application of numerical methods for solving SDE systems

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The problems of determining the structure of numerical method, the choice of its parameters, analysis of meansquare or weak convergence of the numerical solution to the true one are much more complicated for systems of SDE, than for those of ODE. Nevertheless, many theoretical and practical ideas of the numerical methods of ODE solution can be transferred or extended onto the numerical methods of SDE solution. In particular, the notion of absolute stability has induced searching for the methods conserving the one-dimensional distribution of the model SDE solution. Elementary transfer of the absolute stability notion onto the numerical methods of SDE solution is not constructive, since it is possible that the numerical method be absolutely stable for the given integration step-size, while the critical situations occur in the process of trajectory simulation. A cause of this phenomena is an unstable growth of the variance of the numerical solution. Thus it is necessary to define for a numerical method of SDE solution not only the stability region in the sense of ODE, but also the conservation conditions of the variance of the stationary solution of the model equations.

For statistical simulation of the solution trajectories of homogeneous in time SDE systems

$$y(t) = y_0 + \int_0^t f(y(s))ds + \int_0^t \sigma(y(s))dw(s), \quad 0 \le t \le t_{\text{fin}} \tag{1}$$

we will apply methods from the family

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + \tilde{p}_1 k_1 + \sqrt{h}\tilde{q}_1\sigma(y_n)\zeta_n, \\ y_{n+1} &= y_n + p_1 k_1 + p_2 k_2 + \sqrt{h}(q_1\sigma(y_n) + q_2\sigma(\tilde{y}_{n+1}))\zeta_n, \\ k_1 &= [I - ha\tfrac{\partial f}{\partial y}(y_n)]^{-1}[hf(y_n) + \beta_{10}\sqrt{h}\sigma(y_n)\zeta_n], \\ k_2 &= [I - ha\tfrac{\partial f}{\partial y}(y_n)]^{-1}[hf(\tilde{y}_{n+1}) + \beta_{21}\sqrt{h}\sigma(\tilde{y}_{n+1})\zeta_n], \end{aligned} \tag{2}$$

where $y_n, n = 0, 1, \dots, K$ are the values of the approximate solution of SDE system (1) at the mesh nodes with respect to time $\{t_n\}$, h is the integration stepsize at the node t_n; $\tilde{p}_1, p_1, p_2, \tilde{q}_1, q_1, q_2$, $\beta_{10}, \beta_{21}, a$ are real parameters of the method, $\{\zeta_n\}$, $n = 0, 1, \ldots, K-1$ is a sequence of mutually independent normal random vectors with independent in totality components ζ_{nj}, $j = 1, \ldots, N$ having zero mean and variance one. Moreover, mutual independence of the vectors ζ_n and y_n is assumed for each $n = 0, \ldots, K-1$. These properties will take place if we set

$$\sqrt{h}\zeta_n = w(t_{n+1}) - w(t_n),$$

what is true in the sequel.

Definition 1. *The method is said to be asymptotically unbiased with the stepsize h, if while applying it with this stepsise to the scalar linear SDE*

$$y(t) = y_0 - \alpha \int_0^t y(s)ds + \sigma w(t), \tag{3}$$

where $\alpha > 0, \sigma \neq 0$ are real coefficients, the distribution of the numerical solution y_n converges as $n \to \infty$ to the normal distribution with zero mean and variance $\sigma^2/(2\alpha)$.

Definition 2. *The interval $(x_0, 0)$ is said to be the interval of asymptotic unbiasedness of the method, if the latter is asymptotically unbiased with any stepsize $h > 0$, for which $-\alpha h \in (x_0, 0)$.*

The property of asymptotic unbiasedness is concordant with the asymptotic behaviour of the distribution of the true solution of SDE (3), in particular, with the behaviour of mean and variance of the true solution as $t \to \infty$:

$$< y(t) >= e^{-\alpha t} < y_0 > \to 0,$$

$$D_y(t) = e^{-2\alpha t} D_0 + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \to \frac{\sigma^2}{2\alpha},$$

where D_0 is the variance of the random variable y_0.

A numerical solution y_n of the model SDE (3), obtained by a method from family (2), is a normal random variable for any $n = 0, \ldots, K$, if only y_0 is a normal random variable. Due to the fact that the normal distribution is completely determined by its mean and variance, for the construction of the asymptotic unbiasedness interval of the method from family (2) it is sufficient to show for which αh conditions

$$\lim_{n \to \infty} < y_n >= 0, \tag{4}$$

$$\lim_{n \to \infty} Dy_n = \tfrac{\sigma^2}{2\alpha}, \tag{5}$$

where Dy_n is the variance of the random variable y_n, are satisfied.

Applying a method from family (2) to SDE (3), obtain

$$y_{n+1} = R(x)y_n + Q(x)\sigma\sqrt{h}\zeta_n, \tag{6}$$

where $x = -\alpha h$ and

$$R(x) = \frac{1 + (1-2a)x + (1/2 - 2a + a^2)x^2}{(1-ax)^2}, \tag{7}$$

$$Q(x) = \frac{1 - [a(1+q_1+q_2) - p_2(\tilde{p}_1\beta_{10} + p\tilde{q}_1)]x + a[a(q_1+q_2) - p_2\tilde{q}_1]x^2}{(1-ax)^2}. \tag{8}$$

With regard to independence of the random variables y_n and ζ_n from (6), we find:

$$< y_{n+1} >= R(x) < y_n >, \tag{9}$$

$$Dy_{n+1} = R^2(x)Dy_n + Q^2(x)\sigma^2 h. \tag{10}$$

For $|R(x)| < 1$ from (9), (10) derive

$$\lim_{n\to\infty} < y_n >= 0,$$

$$\lim_{n\to\infty} Dy_n = \frac{\sigma^2}{2\alpha} \cdot \frac{(-2x)Q^2(x)}{1 - R^2(x)}.$$

It means that conditions (4), (5) will be satisfied for

$$Q^2(x) = \frac{1 - R^2(x)}{-2x} \tag{11}$$

in the interval $(x_0, 0)$, for which $|R(x)| < 1$. For instance, for the method

$$y_{n+1} = y_n + [I - \frac{h}{2}\frac{\partial f}{\partial y}(y_n)]^{-1}[hf(y_n) + \sqrt{h}\sigma(y_n)\zeta_n] \tag{12}$$

with the functions $R(x) = \frac{1+x/2}{1-x/2}$ and $Q(x) = \frac{1}{1-x/2}$, equality (11) and inequality $|R(x)| < 1$ hold for all $x < 0$. It means that method (12) is asymptotically unbiased with the asymptotic unbiasedness interval $(-\infty, 0)$.

For a linear SDE system of the form

$$y(t) = y_0 + A\int_0^t y(s)ds + \sigma w(t), \tag{13}$$

where A and σ are constant matrices of size $N \times N$, the mean of the solution $y(\cdot)$ is defined by the expression

$$< y(t) >= e^{At} < y_0 >,$$

and the covariance matrix of the vector $y(t)$ as a function of t satisfies equation

$$\begin{aligned} \frac{dD_y(t)}{dt} &= AD_y(t) + D_y(t)A^T + \sigma\sigma^T, \\ D_y(0) &= D_0, \end{aligned}$$

where D_0 is the covariance matrix of the random vector y_0. If the matrix A is negative definite, then there exists a stationary normal distribution of the solution $y(t)$ as $t \to \infty$, and thereto

$$< y(t) > \longrightarrow 0,$$

$$D_y(t) \longrightarrow D,$$

where the matrix D is a solution of the continuous Lyapunov equation

$$AD + DA^T = -\sigma\sigma^T.$$

Moreover, if y_0 is a normal random vector with $< y_0 >= 0$ and $D_0 = D$, then the solution is a Gaussian stationary stochastic process with the correlation function

$$R_y(\tau) = De^{A^T|\tau|}.$$

Applying method (12) to a linear SDE system (13), we obtain recursive formula

$$y_{n+1} = \mathbf{R}(Ah)y_n + \sqrt{h}\mathbf{Q}(Ah)\sigma\zeta_n,$$

where

$$\mathbf{R}(Ah) = (I - \frac{h}{2}A)^{-1}(I + \frac{h}{2}A),$$

$$\mathbf{Q}(Ah) = (I - \frac{h}{2}A)^{-1}$$

Then, with regard to independence of the vectors y_n and ζ_n, we have

$$\begin{aligned} < y_{n+1} > &= \mathbf{R}(Ah) < y_n >, \\ Dy_{n+1} &= \mathbf{R}(Ah)Dy_n\mathbf{R}^T(Ah) + h\mathbf{Q}(Ah)\sigma\sigma^T\mathbf{Q}^T(Ah). \end{aligned} \tag{14}$$

For the stationary numerical solution of SDE (13) relation (14) takes the form

$$D = \mathbf{R}(Ah)D\mathbf{R}^T(Ah) + h\mathbf{Q}(Ah)\sigma\sigma^T\mathbf{Q}^T(Ah). \tag{15}$$

Multiplying (15) from the left by $(I - \frac{h}{2}A)$ and from the right by $(I - \frac{h}{2}A)^T$, we come to the continuous Lyapunov equation. It means that method (12) conserves the stationary distribution of the true solution of SDE system (13) during the calculation with any integration stepsize.

Substituting (7) and (8) into (11), we easily make sure that there is no other asymptotically unbiased method in family (2). With this regard, let us slightly weaken the requirements to the method.

Definition 3. *The method is said to be ε - asymptotically biased in the interval $(x_0, 0)$, if in application to the model SDE (3) with the fixed integration stepsize h, such that $-\alpha h \in (x_0, 0)$, the distribution of the numerical solution y_n converges as $n \to \infty$ to the normal distribution with zero mean and variance $\frac{\sigma^2}{2\alpha}\hat{d}(x)$, $x = -\alpha h$, thereto*

$$|\hat{d}(x) - 1| \leq \varepsilon$$

for all $x \in (x_0, 0)$.

For the methods from family (2) we have

$$\hat{d}(x) = \frac{(-2x)Q^2(x)}{1 - R^2(x)}.$$

So, for instance, applying the method

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + k_1, \\ y_{n+1} &= y_n + \tfrac{3}{4}k_1 + \tfrac{1}{4}k_2 + \tfrac{\sqrt{h}}{4}(\sigma(\tilde{y}_{n+1}) - \sigma(y_n))\zeta_n, \\ k_1 &= [I - \tfrac{h}{4}\tfrac{\partial f}{\partial y}(y_n)]^{-1}[hf(y_n) + \sqrt{h}\sigma(y_n)\zeta_n], \\ k_2 &= [I - \tfrac{h}{4}\tfrac{\partial f}{\partial y}(y_n)]^{-1}[hf(\tilde{y}_{n+1}) + \sqrt{h}\sigma(\tilde{y}_{n+1})\zeta_n], \end{aligned} \tag{16}$$

to SDE (3), we obtain

$$y_{n+1} = \left[\frac{1 + x/4}{1 - x/4}\right]^2 y_n + \frac{1}{(1 - x/4)^2}\sqrt{h}\sigma\zeta_n,$$

whence $\hat{d}(x) = 1/(1 + x^2/16)$. It is easy to calculate, that method (16) is 0.05 -asymptotically biased in the interval $(-0.91, 0)$.

For the well-known method

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + hf(y_n) + \sqrt{h}\sigma(y_n)\zeta_n, \\ y_{n+1} &= y_n + \tfrac{h}{2}[f(y_n) + f(\tilde{y}_{n+1})] + \tfrac{\sqrt{h}}{2}[\sigma(y_n) + \sigma(\tilde{y}_{n+1})]\zeta_n \end{aligned} \tag{17}$$

we know that in application to SDE (3)

$$y_{n+1} = (1 + x + x^2/2)y_n + (1 + x/2)\sqrt{h}\sigma\zeta_n$$

and

$$\hat{d}(x) = \frac{1 + x + x^2/4}{1 + x + x^2/2 + x^3/8}.$$

Method (17) is 0.05 -asymptotically biased in the interval $(-0.4, 0)$. The generalized Euler method

$$y_{n+1} = y_n + hf(y_n) + \sqrt{h}\sigma(y_n)\zeta_n, \tag{18}$$

for which $\hat{d}(x) = 1/(1 + x/2)$, is 0.05 -asymptotically biased in the interval $(-0.095, 0)$.

Definition 4 [1]. *A SDE system is said to be asymptotically p-stable, $p > 0$, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for $|y_0| < \delta$ the inequality $< |y(t)|^p >< \varepsilon$ takes place for all $t \geq 0$, thereto $\lim_{t\to\infty} < |y(t)|^p >= 0$.*

For $p = 1$ SDE system is said to be asymptotically stable in the mean, for $p = 2$ it is said to be asymptotically stable in the meansquare.

Note that the asymptotical p-stability for larger p implies the asymptotical p-stability for smaller p.

Definition 5. *The method is said to be asymptotically p -stable with stepsize h, if while applying it with that stepsize to the asymptotically p -stable SDE, the equality $\lim_{n\to\infty} < |y_n|^p >= 0$ is satisfied.*

If $p = 1$, the method is said to be asymptotically stable in the mean, if $p = 2$, it is asymptotically stable in the meansquare.

Let us consider now the model scalar SDE in the sense of Itô of the form

$$y(t) = y_0 - \alpha \int_0^t y(s)ds + \sigma \int_0^t y(s)dw(s), \tag{19}$$

where $\alpha > 0$, $\sigma \neq 0$ are real coefficients. For the determined initial value $y_0 \neq 0$ the density of the one-dimensional distribution of the solution of (19) for any fixed t is lognormal:

$$p(t, y) = \frac{1}{y\sigma\sqrt{2\pi t}} \exp\left\{-\frac{((\alpha + \sigma^2/2)t + \ln(y/y_0))^2}{2\sigma^2 t}\right\}, \quad y > 0. \tag{20}$$

The mean of the solution of (19) is specified by expression

$$< y(t) >= y_0 \exp(-\alpha t) \to \begin{cases} 0, & \text{if } \alpha > 0, \\ \infty, & \text{if } \alpha < 0, \end{cases} \qquad \text{for } t \to \infty.$$

Similarly, the variance of the solution is specified by expression

$$Dy(t) = y_0^2(e^{\sigma^2 t} - 1)e^{-2\alpha t} \to \begin{cases} 0, & \text{if } \alpha > \sigma^2/2, \\ \infty, & \text{if } \alpha < \sigma^2/2, \end{cases} \qquad \text{for } t \to \infty.$$

Let now SDE (19) be asymptotically stable in the meansquare. Since the mean and the second moment of the solution $y(\cdot)$ satisfy equations

$$\frac{dm}{dt} = -\alpha m,$$

$$\frac{d\gamma}{dt} = (-2\alpha + \sigma^2)\gamma,$$

then the requirement of asymptotic stability in the mean is satisfied for $\alpha > 0$, and in the meansquare - for $\alpha > \frac{\sigma^2}{2}$.

Applying the Euler method (18) to (19), obtain

$$y_{n+1} = (1 - h\alpha + \sqrt{h}\sigma\zeta_n)y_n,$$

whence

$$< y_{n+1} >= (1 - h\alpha) < y_n > \to 0$$

as $n \to \infty$, if $0 < h\alpha < 2$ and

$$< y_{n+1}^2 >= ((1 - h\alpha)^2 + h\sigma^2) < y_n^2 > \to 0$$

as $n \to \infty$, if

$$0 < h\alpha < \frac{(2\alpha - \sigma^2)}{\alpha}.$$

Applying the method (12) to SDE (19), one comes to

$$y_{n+1} = \frac{1 - \frac{h}{2}\alpha + \sqrt{h}\sigma\zeta_n}{1 + \frac{h}{2}\alpha} y_n,$$

whence

$$< y_{n+1} >= \frac{1 - \frac{h}{2}\alpha}{1 + \frac{h}{2}\alpha} < y_n > \to 0$$

as $n \to \infty$ for any $h > 0$, if only $\alpha > 0$, and

$$< y_{n+1}^2 >= \frac{(1 - \frac{h}{2}\alpha)^2 + h\sigma^2}{(1 + \frac{h}{2}\alpha)^2} < y_n^2 > \to 0$$

as $n \to \infty$ for any $h > 0$, if only $\alpha > \sigma^2/2$.

As we see, method (12) is asymptotically stable in the meansquare with any stepsize $h > 0$ in application to meansquare asymptotically stable SDE (19).

Let us consider now a linear SDE system in the sense of Itô of the form

$$y(t) = y_0 + A \int_0^t y(s)ds + \sum_{j=1}^{l} \sigma^{(j)} \int_0^t y(s)dw_j(s), \tag{21}$$

where A and $\sigma^{(j)}$, $j = 1, \ldots, l$ are constant matrices of size $N \times N$, $w_j(\cdot)$, $j = 1, \ldots, l$ are mutually independent one-dimensional standard Wiener processes, y_0 is a random vector, independent of all $w_j(t)$ for $t \geq 0$.

SDE system (21) is meansquare asymptotically stable, if the linear ODE system for the matrix $\gamma_y(t)$ of the second moments of the solution vector $y(t)$

$$\frac{d\gamma_y}{dt} = A\gamma_y + \gamma_y A^T + \sum_{j=1}^{l} \sigma^{(j)} \gamma_y \sigma^{(j)T}$$

is exponentially stable. Unfortunately, none has succeeded yet in expressing conditions for asymptotic meansquare stability for SDE system (21) in terms of the eigenvalues of arbitrary matrices A and $\sigma^{(j)}$, $j = 1, \ldots, l$.

In practical application of numerical methods for simulation of the SDE system solution, there arises a problem of choosing the integration stepsize and accuracy of simulation of the values of solution at the mesh nodes, corresponding to this stepsize. It is natural to try to construct a procedure of automatic choice of the integration stepsize based on the requirement to provide for a certain given condition on the calculation accuracy.

Let us construct a variable step algorithm with the calculation accuracy control on each step of the process of simulating the values of each separate trajectory of the solution sought for, based on the methods from family (2).

Let us set

$$\tilde{y}_{n+1} = y_n + k_1 + \sqrt{h}(1 - \beta_{10})\sigma(y_n)\zeta_n \tag{22}$$

in method (2). Then the Taylor expansion of the numerical solution $\tilde{y}_{n+1}(h)$ in the vicinity of the point $h = 0$ is

$$\tilde{y}_{n+1} = y_n + \sqrt{h}\sigma(y_n)\zeta_n + hf(y_n) + \beta_{10} a h^{3/2} \sum_{j=1}^{N} \Lambda_j f(y_n)\zeta_{nj} + \ldots \tag{23}$$

In case of general nonlinear SDE system, the local error value of method

(2), (22) at the mesh node t_{n+1} under consistency conditions is

$$\begin{aligned}\delta_{n+1} &= \sum_{j_1,j_2=1}^{N} \Lambda_{j_2}\sigma_{\cdot j_1}(y_n)[\int_0^h \Delta w_{j_2}^{t_n}(s)dw_{j_1}^{t_n}(s) \\ &\quad - \tfrac{1}{2}\Delta w_{j_2}^{t_n}(h)\Delta w_{j_1}^{t_n}(h)] + O(h^{3/2}).\end{aligned} \tag{24}$$

On the other hand,

$$y_{n+1} - \tilde{y}_{n+1} = \frac{h}{2}\sum_{j_1,j_2=1}^{N} \Lambda_{j_2}\sigma_{\cdot j_1}(y_n)\zeta_{nj_2}\zeta_{nj_1} + O(h^{3/2}). \tag{25}$$

Comparing (24) and (25), we see that the vector

$$\hat{\delta}_{n+1} = y_{n+1} - \tilde{y}_{n+1}$$

has the same structure and the same order of magnitude $O(h)$, as δ_{n+1}, thus it can be used as an estimate of the local error value δ_{n+1}. This means of estimation the simulation error can be used, for instance, for methods (16) with $\beta_{10} = 1$ and (17) with $\beta_{10} = 0$.

Let us construct a procedure of integration stepsize variation, keeping in mind that the absolute value of the calculation error estimate vector on each step should not exceed the given value $\varepsilon > 0$.

A recurrent step from the node t_n to t_{n+1} is considered to be a success, if

$$|\hat{\delta}_{n+1}| < \varepsilon. \tag{26}$$

In this case the stepsize for calculation started at the node t_{n+1} may be increased $(1.1)^k$ times, where k is the maximum positive integer, for which inequality

$$(1.21)^k|\hat{\delta}_{n+1}|D < \varepsilon, \tag{27}$$

where $D = \max\{1, |\zeta_{n+1}|\}$ holds. Including the value D into inequality (27) will not enable the stepsize to grow overwhelmingly, if large sample values of components of the random vector ζ_{n+1} are simulated.

If the calculation accuracy requirement (26) is not fulfilled, the integration stepsize is reduced $(1.1)^k$ times, where k is a minimum positive integer, for which the inequality

$$\frac{|\hat{\delta}_{n+1}|}{(1.1)^k} < \varepsilon$$

holds. Afterwards the value of the simulated trajectory at the node t_{n+1} is recalculated with the reduced stepsize, though the functions $f(y)$, $\sigma(y)$,

$\frac{\partial f}{\partial y}(y)$ at the point y_n., and also the random vector ζ_n are not subjects for recalculation.

When constructing variable step algorithms for SDE system solution, one may generalize well-known variable step algorithms for ODE system solution, based on the Runge-Kutta or Rosenbrock type methods.

Let us consider a family of numerical methods for SDE solution, which is slightly more general, than (2):

$$\begin{aligned}
y_{n+1}^{(0)} &= y_n, \\
y_{n+1}^{(i)} &= y_n + \sum_{j=1}^{i} \beta_{i+1,j} k_j + \sqrt{h} \sum_{j=1}^{i} \alpha_{i+1,j} \sigma(y_{n+1}^{(j-1)}) \zeta_n, \\
& \qquad i = 1, \dots, m, \\
k_j &= [I - ha \tfrac{\partial f}{\partial y}(y_n)]^{-1} [hf(y_{n+1}^{(j-1)}) + \sqrt{h} \sigma(y_{n+1}^{(j-1)}) \zeta_n], \\
& \qquad j = 1, \dots, m, \\
y_{n+1} &= y_{n+1}^{(m)}.
\end{aligned} \tag{28}$$

If now we take a numerical method of ODE solution as a basis, i.e., if we specify the parameters a, m, β_{ij} in advance, then the parameters α_{ij} can be chosen so that finally method (28) attains the meansquare convergence order 1. For SDE in the sense of Itô to this end it suffices to require conditions

$$\sum_{j=1}^{m} \alpha_{m+1,j} = 0,$$

$$\sum_{j=2}^{m} (\alpha_{m+1,j} + \beta_{m+1,j}) \sum_{l=1}^{j-1} (\alpha_{j,l} + \beta_{j,l}) = 0$$

to be fulfilled. For instance, the well-known 6-stage Runge-Kutta method of order 5 from the algorithm RKF45 [2], can be generalized in the following way:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{16}{135} k_1 + \frac{6656}{12825} k_3 + \frac{28561}{56430} k_4 - \frac{9}{50} k_5 + \frac{2}{55} k_6 \\
&+ \sqrt{h} (\frac{119}{135} \sigma(y_n) - \frac{6656}{12825} \sigma(y_{n+1}^{(2)}) - \frac{28561}{56430} \sigma(y_{n+1}^{(3)}) \\
&+ \frac{9}{50} \sigma(y_{n+1}^{(4)}) - \frac{2}{55} \sigma(y_{n+1}^{(5)})) \zeta_n,
\end{aligned} \tag{29}$$

where

$$y_{n+1}^{(0)} = y_n,$$

$$
\begin{aligned}
y_{n+1}^{(1)} &= y_n + \frac{1}{4}k_1, \\
y_{n+1}^{(2)} &= y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2, \\
y_{n+1}^{(3)} &= y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3, \\
y_{n+1}^{(4)} &= y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4, \\
y_{n+1}^{(5)} &= y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5, \\
k_j &= hf(y_{n+1}^{(j-1)}) + \sqrt{h}\sigma(y_{n+1}^{(j-1)})\zeta_n, \qquad j = 1, \dots, 6.
\end{aligned}
$$

Difference

$$\hat{\delta}_{n+1} = y^*_{n+1} - \tilde{y}^*_{n+1},$$

where

$$y^*_{n+1} = y_n + \frac{16}{135}k^*_1 + \frac{6656}{12825}k^*_3 + \frac{28561}{56430}k^*_4 - \frac{9}{50}k^*_5 + \frac{2}{55}k^*_6,$$

$$\tilde{y}^*_{n+1} = y_n + \frac{25}{216}k^*_1 + \frac{1408}{2565}k^*_3 + \frac{2197}{4104}k^*_4 - \frac{1}{5}k^*_5,$$

$$k^*_j = hf(y_{n+1}^{(j-1)}), \quad j = 1, \dots, 6$$

can be used as an estimate of the error $\hat{\delta}_{n+1}$ of method (29). Indeed, the estimate $\hat{\delta}_{n+1}$ is close to the difference between the numerical solutions of the ODE system $\frac{dy}{dt} = f(y)$, obtained by the Runge-Kutta methods of order 5 and 4 by one step.

The numerical solution accuracy control is carried out via testing inequalities

$$|\hat{\delta}_{n+1,i}| < \frac{|y_{ni}| + |y_{n+1,i}|}{2}\varepsilon_{\text{rel}} + \varepsilon_{\text{abs}} \equiv \varepsilon_i, \qquad i = 1, \dots, N,$$

where $\varepsilon_{\text{rel}}, \varepsilon_{\text{abs}}$ are respectively relative and absolute admissible calculation errors. Integration from the node t_n to t_{n+1} is considered to be a failure, if only

$$|\hat{\delta}_{n+1,i}| \geq 10\varepsilon_i$$

for some $i \in \{1, \dots, N\}$. The integration stepsize varies in accordance with the formula

$$h_{\text{new}} = h_{\text{old}} \cdot k_h,$$

where $k_h = \min\{5, \varepsilon_T/D\}$ with increasing stepsize and $k_h = \max\{0.1, \varepsilon_T\}$ with reducing stepsize. Here

$$\varepsilon_T = \frac{0.9}{(\max\limits_{1\le i\le N} |\hat{\delta}_{n+1,i}|/\varepsilon_i)^{1/5}}, \quad D = \max\{1, |\zeta_{n+1}|\}.$$

The modified procedure of the integration stepsize variation from the algorithm RKF45 is designed to make the integration stepsize more stable and to minimize the number of "recursions" in the process of SDE solution. The presented method of estimating the numerical solution error and the above procedure of integration stepsize variation enables one to modify easily the software realization of the algorithm RKF45, introduced in [2] for the solution of ODE systems to the systems of SDE. Therefore, the algorithm constructed can be used most efficiently for the simultaneous numerical solution of ODE and SDE systems. Such situations occur frequently in the problems of analysis and synthesis of the optimal control by stochastic systems.

The demand for statistical simulation of the SDE system solutions arises in many problems of the modern theory of automated control. It is connected with the complexification of the problems, solved by controlled dynamic systems, with the advanced requirements to them, related to the stochastic character of useful signals and perturbations. Stochasticity of processes occurring in real automated systems brings about additional essential difficulties concerning the theoretical solution of the problem of automated control synthesis, which increases in its turn the significance of statistical simulation. In the process of study one frequently encounters the necessity to solve the Riccati differential equations, to simulate statistically the filtration algorithms, and the dynamics of the controlled system.

Numerical procedures of solving the Cauchy problems, occurring in the automated control theory, are usually characterized by the following features:

- necessity of joint solution of ODE and SDE systems,
- components of the systems solved obtaining segments of rapid transition,
- necessity to solve unstable systems,
- large dimensionality of the systems solved.

Let the signal model be described by a nonlinear SDE system (1), where $f(y)$ is a differentiable vector function of size N, and $\sigma(y)$ is a matrix of size $N \times N$.

Let the observation model have the form

$$z(t) = H(y(t)) + G\delta(t), \tag{30}$$

where $z(t)$ is an observation vector of size $r, H(y)$ is a differentiable vector function of size r, G is a constant non-singular matrix of size $r \times r$, $\delta(\cdot)$ is the "standard observation noise", the Gaussian stochastic process with zero mean and correlation function $R_\delta(\tau) = I\delta_D(\tau)$, where $\delta_D(\tau)$ is the Dirak delta-function. The processes $\delta(\cdot)$ and $y(\cdot)$ are assumed to be independent.

For estimation of the state vector $y(t)$ by the results of observation $z(t)$, the approximate nonlinear filtration algorithms are applied. For instance, the first order filter has the form [3]:

$$\begin{aligned} \frac{d\hat{y}(t)}{dt} &= f(\hat{y}(t)) + P(t)\frac{\partial H^T(\hat{y}(t))}{\partial y}(GG^T)^{-1}[z(t) - H(\hat{y}(t))], \\ \hat{y}(0) &= \langle y_0 \rangle, \end{aligned} \tag{31}$$

where the symmetric covariance matrix $P(t)$ of the estimation error vector is determined from the matrix equation

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial f(\hat{y}(t))}{\partial y}P + P\frac{\partial f^T(\hat{y}(t))}{\partial y} \\ &\quad - P\frac{\partial H^T(\hat{y}(t))}{\partial y}(GG^T)^{-1}\frac{\partial H(\hat{y}(t))}{\partial y}P + \sigma(\hat{y}(t))\sigma^T(\hat{y}(t)), \\ P(0) &= D_0. \end{aligned} \tag{32}$$

Here D_0 is the covariance matrix of the random vector y_0.

Let the linear mathematical model of the controlled object have the form

$$y(t) = y_0 + \int_0^t (Ay(s) + Bu(s))ds + \sigma w(t), \quad 0 \le t \le t_{\text{fin}}, \tag{33}$$

where $u(t)$ is the synthesized d -dimensional vector function of the controlling effects, B is a constant matrix of size $N \times d$, and the observation model is

$$z(t) = Hy(t) + G\delta(t), \tag{34}$$

where H is a constant matrix of size $r \times N$. Then the dynamics of the controlled object for the minimized functional

$$J(u) = \frac{1}{2} \langle \int_0^{t_{\text{fin}}} (y^T(t)Qy(t) + u^T(t)Ru(t))dt + y^T(t_{\text{fin}})P_{\text{fin}}y(t_{\text{fin}}) \rangle, \tag{35}$$

where Q, R, P_{fin} are constant symmetric matrices, thereto the matrix R is positive definite, according to the Bellman optimality priniciple and the separation principle, with regard to the Kalman-Bucy filter, is determined by equations [3]:

$$dy(t) = [Ay(t) - BR^{-1}B^TP(t)\hat{y}(t)]dt + \sigma dw(t), y(0) = y_0, \quad (36)$$

$$\frac{dP(t)}{dt} + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + Q = 0, P(t_{\text{fin}}) = P_{\text{fin}}, \quad (37)$$

$$\frac{d\hat{y}(t)}{dt} = (A - BR^{-1}B^TP(t))\hat{y}(t) + \tilde{P}(t)H^T(GG^T)^{-1}[z(t) - H\hat{y}(t)], \quad \hat{y}(0) = < y_0 >, \quad (38)$$

$$\frac{d\tilde{P}(t)}{dt} = A\tilde{P}(t) + \tilde{P}(t)A^T - \tilde{P}(t)H^T(GG^T)^{-1}H\tilde{P}(t) + \sigma\sigma^T, \tilde{P}(0) = D_0. \quad (39)$$

Here D_0 is the covariance matrix of the random vector $y_0, P(t)$ and $\tilde{P}(t)$ are symmetric matrix-valued $N \times N$ functions, the solutions of the Riccati equations.

Before solving numerically systems (36), (38), one has to solve the Riccati equations (37), (39) first. The matrix differential Riccati equations before the numerical solution are reduced to ODE systems for the elements of the matrices $P(t)$ and $\tilde{P}(t)$. Meanwhile, one has to take into account that the appearing ODE systems might be stiff. Solving the Riccati equation (37) numerically in forward time scale one has also to take account of possible instability of the resulting ODE system, which may bring about large error values. Thus it is recommended to save the values of $P(t_i)$, calculated in backward time scale and the values $\tilde{P}(t_j)$, calculated in forward time scale, and then apply those for the solution of system (36), (37), possibly using interpolation. With regard to these features of joint ODE and SDE systems, it is recommended to apply the generalized variable step algorithm for their numerical integration.

Example 1. Let a linear stochastic controlled system have the form

$$\begin{aligned} dy_1 &= (y_2 + u_1)dt + dw_1(t) + 0.1dw_2(t), \\ dy_2 &= (y_3 + u_2)dt + dw_2(t), \\ dy_3 &= (-6y_2 - 5y_3 + u_3)dt + 0.1dw_3(t), \\ dy_4 &= (y_1 + 40y_2 - 10y_4 + 2u_1 + u_2)dt + 0.2dw_4(t), \\ dy_5 &= (y_1 + 40y_2 - 2y_5 + 3u_1 + 2u_2 + u_3)dt + 2dw_5(t), \end{aligned} \quad (40)$$

and $y(0) = y_0$ be a normal random vector with independent components, $< y_{0i} >= 10i, Dy_{0i} = 0.01, i = 1, \ldots, 5$. Note that the uncontrolled system

(40) is conditionally stable, since one of the eigenvalues of the matrix A of SDE system equals zero. Only two components of the solution are measured:

$$\begin{aligned} z_1 &= y_4 + \delta_1, \\ z_2 &= y_5 + \sqrt{2}\delta_2. \end{aligned} \tag{41}$$

It is easy to verify, that system (40), (41) is observable and controllable [3]. The functional to be minimized will be defined in the form

$$J(u) = \frac{1}{2} < \int_0^{t_{\text{fin}}} \left(\sum_{i=1}^{5} y_i^2(t) + \sum_{i=1}^{3} u_i^2(t)\right)dt + \sum_{i,j=1}^{5} y_i(t_{\text{fin}})y_j(t_{\text{fin}}) > . \tag{42}$$

The variable step algorithm RKF45 with $\varepsilon_{\text{abs}} = \varepsilon_{\text{rel}} = 10^{-3}$ will be used for the solution of the Riccati equations (37), (39), corresponding to problem (40)-(42). Each Riccati equation in this case is equivalent to an ODE system of size 15. Figure 1 illustrates the behaviour of the component $\tilde{p}_{22}(t)$ of the solution of the Riccati equation (39). Here we observe both rapid and steady-state segments of the solution.

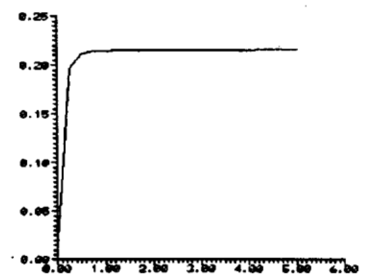

Figure 1. The component $\tilde{p}_{22}(t)$

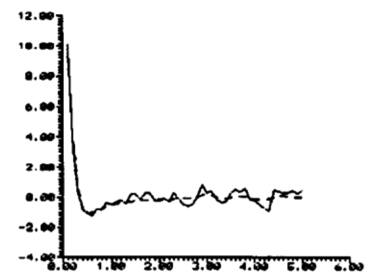

Figure 2. A trajectory of the stochastic process $y_1(t)$ and of its estimate $\hat{y}_1(t)$

Systems (36) and (38) of size 10, corresponding to equations (40), was solved by the variable step algorithm (29) with

$$\varepsilon_{\text{abs}} = \varepsilon_{\text{rel}} = 10^{-3}.$$

Figure 2 presents the charts of a trajectory of the stochastic process $y_1(t)$ (the solid line) and of its estimate $\hat{y}_1(t)$ (the dotted line), obtained by means of the Kalman-Bucy filter.

A complexity and an awkwardness of mathematical models for solving problems of analysis and synthesis of an automatic control of dynamical systems demand the large preliminary work when setting input data, when choosing a suitable numerical algorithm, when organizing a digital and a graphical output information. The realization of this job may be considerably facilitated by using the achivements of modern programming. In particular, special compilator can be used for representating functions of diffusion and drift; the system of analytical calculations can be used for writing the derivation of the function; the work with input and output information and a choice of numerical algorithm and its parameters may be produced in limits of a modern users interface with using "menu" and "windows". During the calculation of the problem we must have an information about arising extremal situations; statistical and reference information; possibility to interrupt the work of numerical algorithm in any time and to renew it after some correction. All the enumerated possibilities are planned to realize in the dialogue system "Dynamics and Control". In the dialogue system we will realize algorithms of linear and non-linear filtrations, an estimate of parameters of ordinary and stochastic differential systems, algorithms of modelling dynamics of controlled systems. The dialogue system is based on modern numerical methods of linear algebra, algorithms for solving ODE and SDE, algorithms of minimization of the function.

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