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Stability of numerical methods for solving stochastic differential equations

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This paper deals with the problem of mean-square stability of numerical methods for solving SDE's. We introduce the notion of the stiff in a mean-square sense system of SDE's, the practical verification of which is not difficult. In the capacity of the investigated family of numerical methods the generalization of two-stage Rosenbrock's methods is considered.

1. Introduction

The use of numerical methods for solving stochastic differential equations (SDE) in the statistical modeling of the dynamics of various stochastic objects often runs into the problem of the instability of the numerical solution. This fact stimulates the further evolution of the theory of stability of numerical methods for solving SDE's. In the construction of such a theory one can use many ideas from the excellent paper by Dahlquist who introduced the notion of A-stability of numerical methods for solving stiff systems of ODE's [1]. Analysis of stability of numerical methods for solving SDE's with additive noises was carried out in [2], and here we do not consider such SDE. During studying the stability of solution of general SDE's we can use notions of asymptotic stability in probability and asymptotic p-stability [3].

Analysis of asymptotic stability in probability of numerical methods for solving SDE's was made in [4]. Also the notion of stiff in probability system of SDE's was introduced in this paper. We must note that at present the practical verification of the stiffness in probability is hindered because there do not exist effective numerical algorithmes for calculating the spectrum of characteristic Lyapunov's exponents of the system of SDE's.

In this paper the basic attention will be given to questions of the mean-square stability of numerical methods for solving SDE's. Also we introduce here the notion of the stiff in a mean-square sense system of SDE's, the practical verification of which is not difficult. The difficulties of the numerical solving of stiff SDE are also discussed. In the capacity of the investigated family of numerical methods the generalization of two-stage Rosenbrock's methods is considered [5]. The generalized implicit Runge-Kutta methods [6] are not studied because of the complex program's realisation and large computational costs of these methods for solving non-linear systems of SDE's. For the numerical methods for solving SDE's the sufficient condition of mean-square stability has been obtained when these methods are applied to the asymptotically mean-square stable linear systems of SDE's with multiplicative noises. We carry out the investigation of the mean-square stability of known numerical methods for solving SDE's. Also we present results of numerical experiments obtained with the help of the dialogue system `Dynamics and Control`.

2. The mean-square stability of zero solution of SDE

We give the general non-linear autonomous system of SDE's in the Ito-sense in the following differential form

$$dy(t) = f(y(t))dt + \sum_{k=1}^{M} \sigma^{(k)}(y(t))dw_k(t), \tag{2.1}$$

$$y(0) = y_0, \tag{2.2}$$

where f, $\sigma^{(k)}$, $k = 1, \ldots, M$ are N-dimensional vector functions. We always suppose that $f(0) = 0$, $\sigma^{(k)}(0) = 0$, $k = 1, \ldots, M$.

Definition 1 [3]. The solution $y(t) \equiv 0$ of the system of SDE's (2.1) is said to be an asymptotically stable in a mean-square sense if for any $\varepsilon > 0$ there is such $\delta > 0$ that for $t \geq 0$ and $|y_0| < \delta$ expectation $\langle |y(t; y_0)|^2 \rangle < \varepsilon$, moreover $\lim_{t\to\infty} \langle |y(t; y_0)|^2 \rangle = 0$.

Here $y(t; y_0)$ is the solution of the system of SDE's (2.1) with initial condition (2.2).

The linear system of SDE's of *the first approximation*:

$$dy(t) = Ay(t)dt + \sum_{k=1}^{M} S^{(k)}y(t)dw_k(t), \tag{2.3}$$

where $A = \frac{\partial f}{\partial y}(0)$, $S^{(k)} = \frac{\partial \sigma^{(k)}}{\partial y}(0)$, $k = 1, \ldots, M$, correspond to the system of SDE's. In [7] it is proved that zero-solution of the system (2.1) is stable in probability providing zero solution of the system of SDE's (2.3) is asymptotically mean-square stable.

For SDE's (2.3) we can write down the equation on the matrix of second moments of the solution $\Gamma(t) = \langle y(t)y^T(t)\rangle$:

$$\frac{d\Gamma}{dt} = A\Gamma + \Gamma A^T + \sum_{k=1}^{M} S^{(k)}\Gamma(S^{(k)})^T. \tag{2.4}$$

If we write the matrix equation (2.4) in the form of the system of ODE's

$$\frac{d\widetilde{\Gamma}}{dt} = \widetilde{A}\widetilde{\Gamma}, \tag{2.5}$$

where

$$\widetilde{\Gamma} = (\gamma_{11}, \ldots, \gamma_{1N}, \gamma_{22}, \ldots, \gamma_{2N}, \ldots, \gamma_{N-1,N-1}, \gamma_{N-1,N}, \gamma_{N,N}),$$

then necessary and sufficient condition of the asymptotic mean-square stability of zero solution of the system (2.3) is the following requirement

$$Re\lambda_j(\widetilde{A}) < 0, \quad j = 1, \ldots, N(N+1)/2. \tag{2.6}$$

In [8] it is proposed that the more simple sufficient condition of asymptotic mean-square stability is the following: $\lambda_j(\hat{A}) < 0$, $j = 1, \ldots, N$, where

$$\hat{A} = A + A^T + \sum_{k=1}^{M} S^{(k)}(S^{(k)})^T.$$

Also there is a proof of

Theorem 1. *The solution of SDE* (2.3) *satisfies the following estimates:*

$$|y_0|^2 \exp(\lambda_{min}t) \;\le\; \langle |y(t)|^2\rangle \;\le\; |y_0|^2 \exp(\lambda_{max}t), \tag{2.7}$$

where $\lambda_{min} = \min \lambda_j(\hat{A})$, $\lambda_{max} = \max \lambda_j(\hat{A})$.

According to this theorem we can introduce

Definition 2. The system of SDE's (2.1) is said to be a stiff in a mean-square sense if for its first approximation (2.3) the following conditions are fulfilled:

$$\lambda_{min} \ll \lambda_{max} < 0. \tag{2.8}$$

Moments of solutions of stiff in a mean-square sense systems of SDE's contain quick and slow components.

The solution of the scalar SDE in the Ito-sense

$$dy(t) = \alpha y(t)dt + sy(t)dw(t) \tag{2.9}$$

with constant real coefficients can be written in the form of

$$y(t) = y_0 \exp(\hat{\alpha}t + sw(t)),$$

where $\hat{\alpha} = \alpha - s^2/2$. We also have an equation for $\gamma(t) = \langle y^2(t) \rangle$:

$$\frac{d\gamma}{dt} = (2\alpha + s^2)\gamma. \tag{2.10}$$

At once from (2.10) we see that zero solution of SDE (2.9) is asymptotically mean-square stable providing $\alpha < -s^2/2$. Note, the requirement of asymptotic stability in probability is considerably less stiff: $\alpha < s^2/2$.

3. The asymptotic mean-square stability of numerical methods for solving SDE

We write down the family of numerical methods for solving SDE (2.1) in the form

$$\begin{aligned}
\widetilde{y}_{n+1} &= y_n + \widetilde{p}_1 k_1 + \sqrt{h}\widetilde{q}_1 \sum_{k=1}^{M} \sigma^{(k)}(y_n)\zeta_n^{(k)}, \\
y_{n+1} &= y_n + p_1 k_1 + p_2 k_2 + \sqrt{h} \sum_{k=1}^{M} \left(q_1 \sigma^{(k)}(y_n) + q_2 \sigma^{(k)}(\widetilde{y}_{n+1})\right)\zeta_n^{(k)}, \\
k_1 &= \left[I - ha\frac{\partial f}{\partial y}(y_n)\right]^{-1} \left[hf(y_n) + \beta_{10}\sqrt{h} \sum_{k=1}^{M} \sigma^{(k)}(y_n)\zeta_n^{(k)}\right], \\
k_2 &= \left[I - ha\frac{\partial f}{\partial y}(y_n)\right]^{-1} \left[hf(\widetilde{y}_{n+1}) + \beta_{21}\sqrt{h} \sum_{k=1}^{M} \sigma^{(k)}(\widetilde{y}_{n+1})\zeta_n^{(k)}\right],
\end{aligned} \tag{3.1}$$

where y_n, $n = 0, 1, \ldots, K$ are the values of the approximate solution of the system of SDE's (2.1) at the mesh nodes with respect to time $\{t_n\}$; h is the step size of the integration at the node t_n; $\widetilde{p}_1$, p_1, p_2, $\widetilde{q}_1$, q_1, q_2, β_{10}, β_{21}, a are real parameters of the method; $\{\zeta_n^{(k)}\}$, $n = 0, 1, \ldots, K-1$, $k = 1, \ldots, M$ is a sequence of independent in totality normal random values:

$$\sqrt{h}\zeta_n^{(k)} = w^{(k)}(t_{n+1}) - w^{(k)}(t_n).$$

Under the condition $\sigma^{(k)}(y) \equiv 0$, $k = 1, \ldots, M$ methods (3.1) are transformed into two-stage Rosenbrock's methods [5]. While $a = 0$, methods (3.1) are the generalization of explicit two-stage Runge-Kutta's methods. Parameters $\widetilde{p}_1$, p_1, p_2, $\widetilde{q}_1$, q_1, q_2, β_{10}, β_{21} are determined from the requirement of the provision of the given order of the convergence of a method in a certain sense, and with the help of the parameter a it is possible to improve properties of the stability of the numerical method.

Definition 3. The numerical method is said to be asymptotically stable in a mean-square sense with step size $h > 0$ (with respect to the given SDE) if under its application with this step size to the given asymptotically stable in a mean-square sense system of SDE's the following condition is fulfilled

$$\lim_{n \to \infty} \langle |y_n|^2 \rangle = 0. \tag{3.2}$$

Definition 4. The numerical method is said to be A-stable in a mean-square sense if the condition (3.2) is fulfilled by integrating with any step size $h > 0$ the scalar asymptotically stable in a mean-square sense SDE (2.9).

Under the application of the numerical method from the family (3.1) to the system of SDE's (2.3), we receive the following recursion formula

$$y_{n+1} = R_n(h) y_n, \quad n = 0, 1, 2, \ldots, \tag{3.3}$$

where $R_n(h)$ is a sequence of joint-independent and with y_n and having the same distribution random $N \times N$-dimensional *matrices of the transition.* If we denote $R^2 = \langle R_n \times R_n \rangle$, defining

$$R \times R = \begin{bmatrix} r_{11}R & \cdots & r_{1N}R \\ \cdots & \cdots & \cdots \\ r_{N1}R & \cdots & r_{NN}R \end{bmatrix}, \tag{3.4}$$

then $\lim_{n \to \infty} \langle |y_n|^2 \rangle = 0$ providing all eigenvalues of $N^2 \times N^2$-dimensional matrix R^2 are situated inside of the unit circle [3]. Even when $N = 2$, matrix (3.4) has 4×4-dimension and the receiving of analytical expressions of all eigenvalues matrix R^2 as the function of h can be complicated enough. It's more easy to check another condition, received from the theorem defining the conditions of the stability of the product of independent matrices [3]:

Theorem 2. *It is sufficiently for the asymptotic stability in a mean-square sense with $h > 0$ of the numerical method with respect to the asymptotically* *stable in a mean-square sense system of SDE's that the solution X of the equation*

$$\langle R_n^T(h) X R_n(h)\rangle - X = -Q \tag{3.5}$$

for any positive definite matrix Q is represented by a positive definite matrix.

In the application of the numerical method of family (3.1) to the SDE (2.9) we receive in formula (3.3) that the function $R_n(h)$ is scalar and the numerical method is asymptotically stable in a mean-square sense with step h providing

$$\langle R_n^2(h)\rangle \; < \; 1. \tag{3.6}$$

It is not difficult to verify that for the numerical methods of family (3.1)

$$y_{n+1} = y_n + \left[I - ah\frac{\partial f}{\partial y}(y_n)\right]^{-1}\left[hf(y_n) + \sqrt{h}\sum_{k=1}^{M}\sigma^{(k)}(y_n)\zeta_n^{(k)}\right], \tag{3.7}$$

for which

$$R_n(h) = (I - ahA)^{-1}\Big(I + (1-a)hA + \sqrt{h}\sum_{k=1}^{M} S^{(k)}\zeta_n^{(k)}\Big),$$

the condition (3.6) is fulfilled for any $h > 0$ in case $a \geq 0.5$. Thus, for methods like (3.7), the condition $a \geq 0.5$ is the requirement of the A-stability in a mean-square sense and it coincides with the requirement of the A-stability of one-stage Rosenbrock's methods for solving ODE's.

For Euler's method

$$y_{n+1} = y_n + hf(y_n) + \sqrt{h}\sum_{k=1}^{M}\sigma^{(k)}(y_n)\zeta_n^{(k)} \tag{3.8}$$

we have

$$R_n(h) = I + Ah + \sqrt{h}\sum_{k=1}^{M} S^{(k)}\zeta_n^{(k)}$$

and an unequality (3.6) is fulfilled when $0 < h < -(2\alpha + s^2)/\alpha^2$. Limitation on h becomes much more stiff when we increase $-\alpha$ or approximate s^2 to -2α.

Mil'stein's methods [6]

$$y_{n+1} = y_n + hf(y_n) + \sqrt{h}\sigma^{(1)}(y_n)\zeta_n^{(1)} + \frac{h}{2}\frac{\partial\sigma^{(1)}}{\partial y}(y_n)\sigma^{(1)}(y_n)((\zeta_n^{(1)})^2 - 1) \tag{3.9}$$

and Platen's method [4]

$$\begin{aligned} y_{n+1} &= y_n + hf(y_n) + \sqrt{h}\sigma^{(1)}(y_n)\zeta_n^{(1)} \\ &+ \tfrac{\sqrt{h}}{2}\left[\sigma^{(1)}(y_n + \sqrt{h}\sigma^{(1)}(y_n)) - \sigma^{(1)}(y_n)\right]((\zeta_n^{(1)})^2 - 1) \end{aligned} \tag{3.10}$$

for SDE in the Ito-sense with a single noise has the same matrix of transition:

$$R_n(h) = I + Ah + \sqrt{h}S^{(1)}\zeta_n^{(1)} + \tfrac{h}{2}S^{(1)}(S^{(1)})^T((\zeta_n^{(1)})^2 - 1)$$

and for them unequality (3.6) is satisfied when

$$0 < h < -(2\alpha + s^2)/(\alpha^2 + s^4/2).$$

The latter unequality is even more limitative than the similar in Euler's method especially when value $s^2 \approx -2\alpha$ is large.

By the using of numerical methods (3.8)–(3.10) for statistical simulating of a stiff in a mean-square sense system of SDE's the step size of the integration is defined by the value $|\lambda_{min}|$.

If we substitute the transfer matrix of A-stable in a mean-square sense method (3.7) with $a = 0.5$ in the left side of the equation (3.5), then we receive in view of independence $\zeta_n^{(k_1)}$ and $\zeta_n^{(k_2)}$ providing $k_1 \neq k_2$ that

$$\begin{aligned} &\Big\langle \Big(I + \tfrac{h}{2}A^T + \sqrt{h}\sum_{k=1}^{M}(S^{(k)})^T\zeta_n^{(k)}\Big)(I - \tfrac{h}{2}A^T)^{-1}X \\ &\times(I - \tfrac{h}{2}A)^{-1}\Big(I + \tfrac{h}{2}A + \sqrt{h}\sum_{k=1}^{M} S^{(k)}\zeta_n^{(k)}\Big)\Big\rangle - X \\ &= (I + \tfrac{h}{2}A^T)Y(I + \tfrac{h}{2}A) + h\sum_{k=1}^{M}(S^{(k)})^TYS^{(k)} \\ &\quad - (I - \tfrac{h}{2}A^T)Y(I - \tfrac{h}{2}A) \\ &= h\Big(A^TY + YA + \sum_{k=1}^{M}(S^{(k)})^TYS^{(k)}\Big), \end{aligned} \tag{3.11}$$

where $Y = (I - \frac{h}{2}A^T)^{-1}X(I - \frac{h}{2}A)^{-1}$. According to Arnold's theorem [9], the system of SDE's (2.3) is an asymptotically stable in a mean-square sense if there exist a positive-definite matrix Y which satisfies matrix equation

$$A^TY + YA + \sum_{k=1}^{M}(S^{(k)})^TYS^{(k)} = -Q_y \tag{3.12}$$

for some symmetric positive-definite matrix Q_y. Let $\widetilde{Y}$ be positive-definite matrix which satisfies equation (3.12). Then, in view of the negative definity of A, the matrix $\widetilde{X} = (I - \frac{h}{2}A^T)\widetilde{Y}(I - \frac{h}{2}A)$ is positivly definite and satisfies the equation

$$\langle R_n^T(h)\widetilde{X}R_n(h)\rangle - \widetilde{X} = -hQ_y.$$

It means that method (3.7) with $a = 0.5$ is asymptotically stable in a mean-square sense with any step size $h > 0$ concerning asymptotically stable in a mean-square sense system of SDE's (2.3).

4. Numerical tests

Making numerical tests, the dialogue system **Dynamics and Control**, which was developed in Novosibirsk Computing Center of Sibirian Division of Russian Academy of sciences, was used. This dialogue system presents essentially new way of the making of the numerical test in a dialogue of user and computer. The fundamental principles of this dialogue system contain creation of maximal eases for users, possibility of quick transition from a statement of a problem to the obtaining of the result of calculations in graphical or numerical forms, the automatic fulfillment of routine work. The dialogue system includes various modern numerical algorithmes of linear algebra, solutions of systems of SDE's and ODE's, the minimization of a function. In the bank of numerical algorithms for solving SDE we included the most used numerical methods, algorithms of a variable step size [10], and algorithms of checking the asymptotic stability in a mean-square sense of linear systems of SDE's (2.3), and also an algorithm of the estimating of moment's functions (2.4) of solutions of linear systems of SDE's (2.3).

Example 1.

$$dy = -5ydt + 3ydw(t), \quad y(0) = 1. \tag{4.1}$$

For SDE in the Ito-sense (4.1) we have $\langle y(t)\rangle = \exp(-5t)$ and $\langle y^2(t)\rangle = \exp(-t)$. Concerning to the SDE (4.1) Euler's method is an asymptotically stable in a mean-square sense when $0 < h < 0.04$, and methods (3.9) and (3.10) are asymptotically stable in the same sense when $0 < h < 0.015$. By means of methods (3.7)–(3.10) we simulated 1000 paths per method for (4.1) with step size $h = 0.2$ on the interval $[0; 5]$. With such step size we have for method (3.8) $\gamma_{n+1} = 1.8\gamma_n$, and for methods (3.9) and (3.10) $\gamma_{n+1} = 3.42\gamma_n$.

In Figure 1 graphs of the second moment $\gamma(t) = \exp(-t)$ and the estimate of $\hat{\gamma}(t)$, received by method (3.7) are presented. Figure 2 depicts graphs of estimates of $ln\hat{\gamma}(t)$, received by methods (3.8)–(3.10).

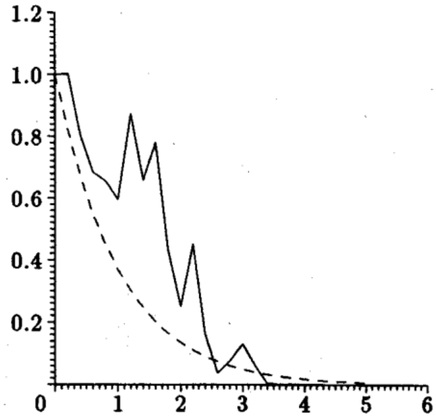

Figure 1. The second moment $\gamma(t)$ (dashed line) and its estimate by method (3.7) (solid line)

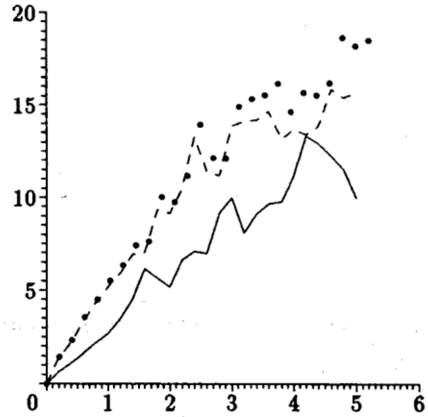

Figure 2. The logarithm of the estimate of the second moment by Euler's method (solid line), by Mil'stein's method (dashed line) and by Platen's method (point line)

Example 2.

$$\begin{aligned} dy_1 &= (-10y_1 + y_2)dt + (y_1 + y_2)dw_1(t), \\ dy_2 &= (-y_1 - y_2)dt + (-0.25y_1 + 0.25y_2)dw_1(t). \end{aligned} \tag{4.2}$$

For SDE in the Ito-sense (4.2) matrix $\widetilde{A}$ in (2.5) has the form

$$\begin{bmatrix} -19 & 4 & 1 \\ -1.25 & -11 & 1.25 \\ 0.0625 & -2.125 & -1.9375 \end{bmatrix}.$$

As eigenvalues of matrix $\widetilde{A}$ are negative: $\lambda_1 = -2.21$, $\lambda_2 = -11.42$, $\lambda_3 = -18.31$, the system of SDE's (4.2) is an asymptotically stable in a mean-square sense. Notice that the system of SDE's (4.2) is a weak stiff in the sense of introduced Definition 2. By means of methods (3.7) and (3.8) with step size $h = 0.2$ we simulated 1000 paths of the solution of SDE on the interval [0;5].

In Figure 3 graphs of the exact second moment $\gamma_{11}(t)$ and statistical estimate which was received by method (3.7) is given. In Figure 4 we show graph of the estimate received by method (3.8).

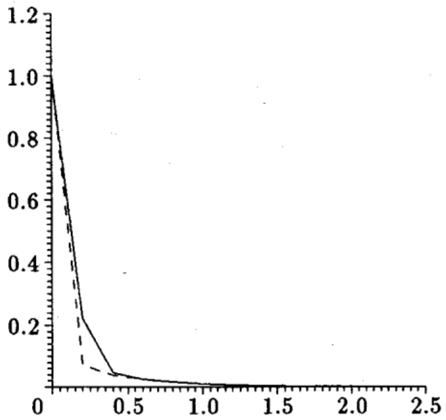

Figure 3. The exact second moment (dashed line) and its estimate by method (3.7) (solid line)

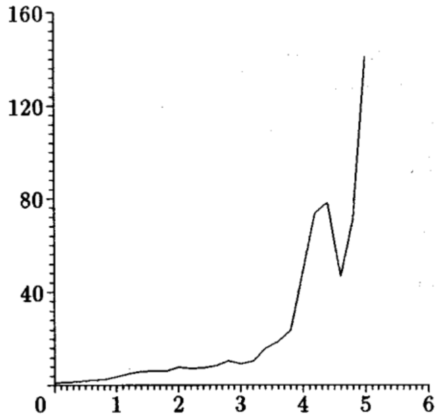

Figure 4. The estimate of the second moment by Euler's method

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