Bull. Nov. Comp. Center, Num. Anal., 6 (1994), 1–7

A-m.s.-stable numerical methods for solving stochastic differential equations in the Ito-sense*

S.S. Artemiev and T.L. Stepanova

In this paper we investigate the asymptotic mean-square stability (m.s.-stability) of the family of numerical methods for solving SDE's in the Ito-sense generalizing Rosenbrock's type methods. The connection between the asymptotic m.s.-stability of the numerical method for solving SDE and the absolute stability of the corresponding Rosenbrock's type method are shown. Examples of A-m.s.-stable numerical methods are given.

1. Introduction

One of the most important characteristics of the numerical method for solving stochastic differential equations (SDE) is its stability. The stability in a mean-square sense (m.s.-stability) of numerical methods is usually connected with the generalization of the implicit Runge-Kutta type methods [1–4]. Problems arising here mainly refer to the complex program's realization and large computational costs of these methods. The more suitable family from this point of view is the special family of explicit methods generalizing Rosenbrock's type methods (RTM) [5]. In this paper we consider such family of methods for solving SDE in the Ito-sense and prove that methods from this family are asymptotically m.s.-stable in integrating with any step size in any asymptotically mean-square stable linear system of SDE's with multiplicative noises, if original RTM are A-stable.

2. The family of numerical methods

The non-linear autonomous system of SDE's is given in the following differential form

*Supported by the Russian Foundation of Fundamental Research under Grant 94-01-00074.

$$dy(t) = f(y(t))dt + \sum_{k=1}^{M} \sigma^{(k)}(y(t))dw_k(t), \tag{1.1}$$

$$y(0) = y_0, \tag{1.2}$$

where f, $\sigma^{(k)}$, $k = 1,\dots,M$ are N-dimensional vector-functions, $w_k(\cdot)$, $k = 1,\dots,M$ are independent in the totality standard Wiener processes, y_0 is N-dimensional random vector. We always assume that $f(0) = 0$, $\sigma^{(k)}(0) = 0$, $k = 1,\dots,M$.

The first approximation linear system of SDE's

$$dy(t) = Ay(t)dt + \sum_{k=1}^{M} S^{(k)}y(t)dw_k(t), \tag{1.3}$$

corresponds to the system (1.1), where

$$A = \frac{\partial f}{\partial y}(0), \quad S^{(k)} = \frac{\partial \sigma^{(k)}}{\partial y}(0), \quad k = 1,\dots,M.$$

Definition 1 [6]. The numerical method is said to be an asymptotically m.s.-stable with the step size $h > 0$ (with respect to the given system of SDE's), if under its application with this step size to the asymptotically stable in a mean-square sense system of SDE's the following condition:

$$\lim_{n\to\infty} \langle |\, y_n \,|^2 \rangle = 0. \tag{1.4}$$

is fulfilled.

Here $\{y_n\}$ are the values of the numerical solution of the system of SDE's at the nodes of the mesh $\{t_n\}$, $\langle\cdot\rangle$ is the operation of taking the mean value.

Definition 2. The numerical method is called A-m.s.-stable, if the condition (1.4) is fulfilled by integrating with any step size $h > 0$ any asymptotically stable in a mean-square sense linear system of SDE's (1.3).

Definition 3. The numerical method is called A_0-m.s.-stable, if the condition (1.4) is fulfilled by integrating with any step size $h > 0$ the scalar assymptotically stable in a mean-square sense SDE

$$dy(t) = \alpha y(t)dt + sy(t)d\omega(t). \tag{1.5}$$

Here α, s are constant real coefficients such that for SDE in the Ito-sense the unequality $2\alpha < -s^2$ is fulfilled.

For statistical simulating the trajectories of the solutions of the systems of SDE's (1.1) we will use methods in the form

$$y_{n+1} = y_n + \sum_{i=1}^{m} p_i k_i + \sqrt{h}\Big[I - ah\frac{\partial f}{\partial y}(y_n)\Big]^{-m} \sum_{k=1}^{M} \sigma^{(k)}(y_n)\zeta_n^{(k)},$$
$$k_i = h\Big[I - ah\frac{\partial f}{\partial y}(y_n)\Big]^{-1} f\big(y_n + \sum_{j=1}^{i-1} \beta_{ij}k_j\big), \quad i = 1,\ldots,m. \tag{1.6}$$

Here $\{y_n\}$ are the values of the approximate solutions of the system of SDE's (1.1) at the mesh nodes with respect to time $\{t_n\}$; h is the step size of the integration at the node t_n; p_i, β_{ij}, a are real parameters of the method, with $a > 0$; $\{\zeta_n^{(k)}\}$, $k = 1,\ldots,M$ is a sequence of independent in totality normal random vectors

$$\sqrt{h}\zeta_n^{(k)} = w_k(t_{n+1}) - w_k(t_n),$$

m is the number of stages of the method. Methods (1.6) are the generalization of m-stage RTM for solving ordinary differential equations (ODE's). Further we always assume that the mesh $\{t_n\}$ is uniform and the original m-stage method has the m-th order of the consistency.

Under the application of the numerical method of the family (1.6) to the system of SDE's (1.3) we receive the following recursion formula:

$$y_{n+1} = \tilde{\boldsymbol{R}}_n(h)y_n, \quad n = 0,1,2,\ldots, \tag{1.7}$$

where $\{\tilde{\boldsymbol{R}}_n(h)\}$ is the sequence of mutually independent and independent with y_n, equally distributed $N \times N$ – dimensional random matrices of the transition. Under its application to the SDE (1.5) $\{\tilde{\boldsymbol{R}}_n(h)\}$ is the sequence of independent normal random values.

The matrix of the transition $\tilde{\boldsymbol{R}}_n(h)$ of the m-stage method from the family (1.6) can be written in the form of [5]:

$$\begin{aligned}\tilde{\boldsymbol{R}}_n(h) &= (I - ahA)^{-m}\Big[I + \sum_{j=1}^{m}\Big\{\frac{1}{j!} + \sum_{l=1}^{j}\frac{(-a)^l C_m^l}{(j-l)!}\Big\}(hA)^j + \sqrt{h}\sum_{k=1}^{M} S^{(k)}\zeta_n^{(k)}\Big] \\ &= (I - ahA)^{-m}\Big[I + (1 - ma)hA + \pi(hA) + \sqrt{h}\sum_{k=1}^{M} S^{(k)}\zeta_n^{(k)}\Big],\end{aligned} \tag{1.8}$$

where $\pi(hA)$ is m-th degree matrix polinomial in the form of

$$\pi(x) = c_2x^2 + \ldots + c_mx^m.$$

3. The theorem of the asymptotic m.s.-stability of numerical methods

Theorem 1. *In order the method of the family* (1.6) *be A-m.s.-stable, the sufficient condition is that the corresponding RTM be A-stable.*

Proof. To prove this theorem we need:

Theorem (Lyapunov [7]). *If the matrix* $\boldsymbol{R}$ *is converging (i.e., its spectrum belongs to the interior of the unit circle), we have for every symmetric positive-definite matrix* B *the only solution* X *of the equation*

$$\boldsymbol{R}^T X \boldsymbol{R} - X = B \tag{2.1}$$

is the symmetric negative-definite matrix.

Theorem (Hasminski [8]). *For the asymptotic stability in a mean-square sense of the system of SDE's* (1.3) *a necessary (resp. sufficient) condition is that, for every (resp. for some) symmetric positive-definite matrix* Q, *the matrix equation*

$$YA + A^TY + \sum_{k=1}^{M}(S^{(k)})^T Y S^{(k)} = Q \tag{2.2}$$

has a symmetric negative-definite matrix.

Theorem (Hasminski [8]). *For the asymptotic stability in a mean-square sense of the process* (1.7) *with the step size* $h > 0$ *it is necessary for every, and sufficient for some, symmetric positive-definite matrix* $C(h)$ *the solution* X *of the equation*

$$\langle \tilde{\boldsymbol{R}}_n^T(h) X \tilde{\boldsymbol{R}}_n(h)\rangle - X = C(h) \tag{2.3}$$

be the symmetric negative-definite matrix.

Let $R(z)$ be the stability function of the A-stable RTM, i.e., $|R(z)| < 1$ for every z, situated at the left side of the complex plane. The matrix function $\boldsymbol{R}(hA)$ exists if the complex function $R(z)$ is defined at the spectrum of the matrix hA. If $h\lambda_1(A), \ldots, h\lambda_N(A)$ are eigenvalues of the matrix hA, then $R(h\lambda_1(A)), \ldots, R(h\lambda_N(A))$ are eigenvalues of matrix $\boldsymbol{R}(hA)$ [9]. Because for every stable matrix A unequalities

$$|R(h\lambda_i(A))| < 1, \quad i = 1, \ldots, N$$

are fulfilled for every step size $h > 0$, matrix $\boldsymbol{R}(hA)$ is converging for every step size $h > 0$.

Let Y be some symmetric negative-definite solution of (2.2) with the symmetric positive-definite matrix Q and let $X = [I - ahA^T]^m Y [I - ahA]^m$; it follows that the matrix X is symmetric negative-definite matrix for every stable matrix A and for every step size $h > 0$. Substituting X and $\tilde{\boldsymbol{R}}_n(h)$ from (1.8) at the left-hand side of (2.3), we receive in view of the equality

$$(I - ahA)^m = I - mahA + \gamma(hA), \quad \gamma(x) \equiv g_2 x^2 + \ldots + g_m x^m$$

and the properties of random values $\{\zeta_n^{(k)}\}$, $k = 1, \ldots, M$, that

$$\begin{aligned}
&\langle \tilde{\boldsymbol{R}}_n(h)^T X \tilde{\boldsymbol{R}}_n(h) \rangle - X \\
&\quad = \Big\langle \big[I + (1 - ma)hA^T + \pi(hA^T) + \sqrt{h} \sum_{k=1}^{M} \zeta_n^{(k)} (S^{(k)})^T\big][I - ahA^T]^{-m} \times \\
&\qquad X[I - ahA]^{-m}\big[I + (1 - ma)hA + \pi(hA) + \sqrt{h} \sum_{k=1}^{M} S^{(k)} \zeta_n^{(k)}\big]\Big\rangle - X.
\end{aligned}$$

Making substitution of the variable X by the variable Y and calculating the expectation we receive that

$$\begin{aligned}
&\langle \tilde{\boldsymbol{R}}_n(h)^T X \tilde{\boldsymbol{R}}_n(h) \rangle - X \\
&\quad = \big[I + (1 - ma)hA^T + \pi(hA^T)\big]Y\big[I + (1 - ma)hA + \pi(hA)\big] + \\
&\qquad h \sum_{k=1}^{M} (S^{(k)})^T Y S^{(k)} - \big[I - mahA^T + \gamma(hA^T)\big]Y\big[I - mahA + \gamma(hA)\big] \\
&\quad = h\Big(YA + A^T Y + \sum_{k=1}^{M} (S^{(k)})^T Y S^{(k)}\Big) + \delta(hA, Y) \\
&\quad = hQ + \delta(hA, Y) \equiv C(h),
\end{aligned}$$

where $\delta(hA, Y)$ is the matrix polynomial depending on matrices hA and Y, but not on matrices $S^{(k)}$, $k = 1, \ldots, M$, with the minimal degree with respect to h in δ is equal to 2, and the meaning of this polynomial is the symmetric matrix. Let us show that so defined matrix $C(h)$ is positive-definite for every $h > 0$.

If we substitute the symmetric negative-definite matrix X in (2.1), where $\boldsymbol{R}$ is the converging, for every $h > 0$, matrix of the transition of the correspondng RTM, then we receive with the help of the Lyapunov theorem that the symmetric matrix $B \equiv h(YA+A^TY)+\delta(hA,Y)$ is positive-definite for every $h>0$. But the symmetric matrix $YA+A^TY$ as well as the matrix Q is positive-definite.

So for every $h>0$ the matrix $C(h)$ is symmetric positive definite and the equation (2.3) has the symmetric negative-definite solution X. But it means that the sufficient condition of the second Hasminski theorem is fulfilled and our theorem is proved. □

Corollary 1. *If RTM is A-stable, then the corresponding method from the family* (1.6) *is A_0-m.s.-stable.*

Let us present examples A-m.s.-stable 1-, 2-, 3-stage methods of the family (1.6). In every case methods have the first order of the convergence in a mean-square sense:

$$\max_n \langle |y_n - y(t_n)|^2\rangle = O(h).$$

$$m=1: \quad a=0.5, \quad p_1=1,$$

$$m=2: \quad a=0.25, \quad p_1=0.75, \quad p_2=0.25, \quad \beta_{21}=1,$$

$$m=3: \quad a=\frac{1}{3}, \quad p_1=\frac{17}{12}, \quad p_2=-\frac{5}{4}, \quad p_3=\frac{5}{6},$$

$$\beta_{21}=\frac{2}{5}, \quad \beta_{31}=\frac{29}{30}, \quad \beta_{32}=-\frac{1}{6}.$$

Let us note that A-m.s.-stable methods with the second order of convergence in a mean-square sense even for the simplest SDE (1.5) have not still be constructed.

References

[1] G.N. Mil'stein, *The Numerical Integration of Stochactic Differential Equations*, Sverdlovsk University, Sverdlovsk, 1988 (in Russian).

[2] P.E. Kloeden and E. Platen, *The Numerical Solution of Stochactic Differential Equations*, Berlin – Heidelberg – New York, Springer, 1992.

[3] D.B. Hernandez and R. Spigler, *Convergence and Stability of Implicit Runge-Kutta Methods for Systems with Multiplicative Noise*, BIT, Vol. 33, 1993, 654–669.

[4] Y. Saito and T. Mitsui, *T-stability of Numerical Scheme for Stochastic Differential Equations*, WSSIAA, Vol. 2, 1993, 333–344.

[5] S.S. Artemiev, *Numerical Solution of the Cauchy Problem for Systems of Ordinary and Stochastic Differential Equations*, Computing Center, Novosibirsk, 1993 (in Russian).

[6] S.S. Artemiev, *The Mean-square Stability of Numerical Methods for Solving Stochastic Differential Equations*, Ros. Math. Dokl., Vol. 333, No. 4, 1993, 421–422 (in Russian).

[7] A.M. Lyapunov, *The General Problem of the Movement Stability*, Gostehizdat, Moscow, 1950 (in Russian).

[8] R.Z. Hasminski, *Stability of Systems of Differential Equations under Random Perturbations of their Parameters*, Nauka, Moscow, 1969 (in Russian).

[9] K. Dekker and J.G. Verwer, *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*, North Holland, Amsterdam – New York – Oxford, 1984.