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The combined inverse problem of acoustics and geoelectrics: numerical approach*

A.V. Avdeev, E.V. Goryunov and V.V. Skazka

The paper deals with the problem of combined inversion of wave and electromagnetic fields. It is assumed that there exists the functional relation between the velocity and electric conductivity the character of which is known to us.

The distributions of velocity and conductivity in the medium under study are chosen such that the minimization process of data misfit functionals for the individual problems (acoustics and geoelectrics) could not bring to a satisfactory solution.

The complex functional containing a number of free parameters is used. Changing these parameters in the interactive regime, we manage to do away with the strong "ravine structure" of the individual functionals in the combined functional and reconstruct the distributions of velocity and conductivity in the medium under study with a good accuracy.

1. Introduction

Investigation of the inverse problems in combined statements began in the mid-50s. The interest of geophysicists to this theme was caused by the fact that there appeared a possibility to jointly analyse different geophysical fields in the study of the Earth's structure both at the qualitative and at the quantitative levels using computers.

Omitting here the geophysical aspect of the problem, we would like to point out the works [1–8] which are, in our opinion, of substantial mathematical and methodological importance. It turned out that, in essence, the inverse problems in combined statements provide a possibility of a more successful solution than the study of each of the inverse problems separately, taking into account the obtained data in order to get the general idea of the investigated medium.

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The property of individual inverse problems to "bring" more information and "of better quality" into the combined inverse problem that might appear at first glance, was called by A.S. Alekseev in [8] *the property of complementability*. He gave the mathematical definition of the combined inverse problem and showed its non-equivalence to a simple set of individual problems. Here it should be pointed out that term "joint inversion of some fields" is not the same as the solving of the combined inverse problem. The first term denotes the technical method, but the second one – a new mathematical problem. There exist many theoretical questions which are connected with the uniqueness, existence, and stability of solutions of combined inverse problems. These questions are more difficult as compared with the same questions concerning individual inverse problems. Here we have no possibility to discuss in detail their differences.

At present the most satisfactory results in the numerical solution of the inverse problems were obtained by the method of minimization of data misfit functional for the observed and calculated data. The work [6] should be mentioned here, in which the numerical study of a possibility for the reconstruction of the medium structure was carried out using the data of the wave and gravity fields known at the surface. In this case it was assumed that there exists the functional relation between the velocity and density. Two approaches were compared: minimization of the complex functional taking into account the relation between the parameters of the medium and sequential minimization of data misfit functionals for each of the problems with the recalculation of parameters.

The given work deals with the problem of combined inversion of wave and electromagnetic fields. It was assumed that there exists the functional relation between the velocity and electric conductivity the character of which is known to us. However, the constants of the correlational dependence are given not precisely. As shown in [9–10], such dependence can take place in porous media.

The distributions of velocity and conductivity in the medium under study were chosen to be characteristic for the real media and such that the minimization process of data misfit functionals for the individual problems (acoustics and geoelectrics) could not bring to a satisfactory solution.

The complex functional containing a number of free parameters was used. Changing these parameters in the interactive regime, we managed to do away with the strong "ravine structure" of the individual functionals in the combined functional and reconstruct the distributions of velocity and conductivity in the medium under study with a good accuracy.

2. Inverse problem of acoustics

Wave propagation in a vertically-inhomogeneous medium will be described by the following initial boundary value problem:

$$\frac{\partial^2 V}{\partial t^2} = c_*^2(z)\Delta V, \tag{1}$$

$$V|_{t<0} \equiv 0, \tag{2}$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = f(t)g(x,y), \tag{3}$$

where $c_*(z)$ is a piecewise continuous function characterizing the velocity of wave propagation in a vertically inhomogeneous medium, $f(t)$ and $g(t)$ describe the time influence and space distribution of the sources at the free surface, correspondingly. Let $z_1, \dots, z_n$ be the layers' boundaries with the velocities $c_1, \dots, c_n$. For such model, the following conditions at $z = z_i$:

$$[V]_{z=z_i} = \left[\frac{dV}{dz}\right]_{z=z_i} = 0, \quad i = 1, \dots, N,$$

should be added to (1)–(3).

For the system (1)–(3) let us consider the following inverse problem:

It is necessary to reconstruct the function $c(z)$ using the information

$$V|_{z=0} = V_0(x,y,t); \quad 0 < t < T; \quad (x,y) \in S \subset \{(x,y,z) : z = 0\} \tag{4}$$

on the regime of oscillations of the observation surface $z = 0$.

The solution to this problem will be sought as the minimum point of the following data misfit functional:

$$\Phi_1[c^2(z)] = \int_0^T dt \int_S \{V_0(x,y,t) - B_1[c^2(z)](x,y,t)\}^2 dx dy, \tag{5}$$

where $B_1[c^2(z)]$ is the non-linear operator transferring the function $c(z)$ into the solution of the direct problem (1)–(3) at $z = 0$ (the wave field on the set S).

Following [12], the initial statement (1)–(4) will be rewritten in the following way:

$$\frac{d^2V}{dz^2} = \nu^2 V, \tag{6}$$

$$\left.\frac{dV}{dz}\right|_{z=0} = F(\omega), \tag{7}$$

$$V|_{z=0} = V_0(k,\omega), \tag{8}$$

where

$$\nu^2 = k^2 - \omega^2/c^2(z), \qquad F(\omega) = \int\limits_0^T f(t)\exp(-i\cdot\omega t)dt,$$

and for the unique solution it is assumed that the principle of limit absorption, i.e.,

$$V(z,k,\omega) = \lim_{\varepsilon\to+0} V(z,k,\omega - i\varepsilon),$$

where, in turn, $V(z,k,\omega - i\varepsilon) \to 0$ as $z \to \infty$ is fulfilled.

As shown in [12], the realization of the process of search for the minimum point of the data misfit functional in the frequency domain (k,ω) enables substantial reduction of the computational resources necessary for the multiple solution of the direct problem and the detailed analysis of the spectra of wave fields at each step of calculations.

In this connection we assume that the recording time T and the size of the observation domain S at the free surface are sufficiently large and choose, instead of the functional (5), the following functional:

$$\Phi[n^2(z)] = \int\limits_{\omega_1}^{\omega_2} d\omega \int\limits_{k_1}^{k_2} \left|V_0(k,\omega) - \hat{B}_1[n^2(z)](k,\omega)\right|^2 kdk, \tag{9}$$

where (ω_1,ω_2) and (k_1,k_2) are the ranges of time and space frequencies determined by the spectral composition of the sounding signal $F(\omega)$ and the size of the observation domain S. $\hat{B}_1[n^2(z)]$ is non-linear operator transferring the function $n^2(z) = c^{-2}(z)$ ($c(z)$ is the current velocity of wave propagation) into the solution of the direct problem (6), (7) at $z = 0$.

It should be noted that the functional (9), at $\omega_1 = 0$, $\omega_2 = \infty$, $k_1 = 0$, $k_2 = \infty$, coincides with the functional (5) due to the Parseval equality.

Using the method of the conjugate Green function described in [13], we obtain the expression for the gradient of the data misfit functional (9) with respect to velocity:

$$\nabla_{n^2}\Phi_1[n^2(z)](\xi) = -2Re\int\limits_{\omega_1}^{\omega_2}(\omega+i\varepsilon)^2\bar{F}(\omega)d\omega$$
$$\times\int\limits_{k_1}^{k_2}\left[V_0(k,\omega)-\hat{B}[n^2(z)](k,\omega)\right]\bar{G_1}^2(\xi,k,\omega)kdk, \tag{10}$$

where $G_1(\xi,k,\omega)$ is the solution to the problem (6), (7) at $F(\omega)=1$.

3. Inverse problem of geoelectrics

The system of the Maxwell equations is in the basis of the theory of electric prospecting methods using the variable electromagnetic fields. In our case the solution to this system leads to the Helmholz equation [14] of the type

$$\Delta E + k^2E = i\mu_0\omega j_s, \tag{11}$$

where $k^2 = -i\mu_0\omega\sigma(z)$, $\omega = 2\pi f$, $\mu_0 = 4\pi\cdot 10^{-7}$ Hn/m, f is the current frequency, j_s is the current density of the source, $\sigma(z)$ is the conductivity of the medium, μ_0 is the magnetic constant.

Let us add the following boundary conditions to the equation (11):

$$\left.\frac{dE}{dz}\right|_{z=0} = 0. \tag{12}$$

Let $z_1,\dots,z_n$ be the boundaries of the layers with the conductivity $\sigma_1,\dots,\sigma_n$. Then it is necessary to add the conditions at $z=z_i$ to the relations (11), (12):

$$[E]_{z=z_i} = \left[\frac{dE}{dz}\right]_{z=z_i} = 0, \quad i=1,\dots,N.$$

Consider the following inverse problem:

It is necessary to reconstruct the conductivity function $\sigma(z)$ using the additional information

$$E|_{z=0} = E_0(x,y,\omega); \quad (x,y)\in S\subset\big\{(x,y,z): z=0\big\}, \tag{13}$$

given at the free surface $z=0$.

Assume that we have a point source located at the surface of the vertically-inhomogeneous medium. Passing to the cylindrical coordinate system (r,z,ϕ) and performing the Fourier–Bessel transform, we rewrite the initial statement (11)–(13) in the following form:

$$\frac{d^2\hat{E}}{dz^2} - (\lambda^2 + k^2)\hat{E} = \frac{i}{2\pi}\mu_0\omega M\delta(z), \tag{14}$$

$$\left.\frac{d\hat{E}}{dz}\right|_{z=0} = 0, \tag{15}$$

$$\hat{E}|_{z=0} = \hat{E}_0(\lambda,\omega), \tag{16}$$

where

$$\hat{E}(\lambda,\omega,z) = \int_0^\infty E(r,\omega,z)J_0(\lambda r)rdr.$$

Here j_s is rewritten in the form $j_s = M\delta(\cdot z)$, where M is the current moment.

The solution to the inverse problem will be sought as the minimum point of the following data misfit functional:

$$\Phi_2[\sigma(z)] = \int_0^\infty \left|\hat{E}_0(\lambda,\omega) - B_2[\sigma(z)](\lambda,\omega,z)\right|^2 \lambda d\lambda, \tag{17}$$

where $B_2[\sigma(z)]$ is the non-linear operator transferring the function $\sigma(z)$ (the "test" conductivity value) into the solution of the direct problem (14), (15) at $z = 0$.

The expression for the gradient of the data misfit functional (17) with respect to conductivity is written in the following way:

$$\nabla_\sigma\Phi_2[\sigma(z)](\xi) = 2\mu_0\omega \cdot \mathrm{Im}\int_0^\infty \left[\hat{E}_0(\lambda,\omega) - B_2[\sigma(z)](\lambda,\omega)\right] \times \bar{G_2}^2(\xi,\lambda,\omega)\bar{F}\lambda d\lambda, \tag{18}$$

where $F = \frac{i}{2\pi}\mu_0\omega M$, $G_2(\xi,\lambda,\omega)$ – the Green function of problem (14), (15).

4. Correlation between the electric and elastic properties of the medium

The dependence of conductivity and velocity of the elastic waves on porosity causes the existence of correlation between the electric and elastic properties of rocks.

In the works [9, 10] it is pointed out that the correlation between the velocity $c(z)$ and conductivity $\sigma(z)$ can be most reliably approximated by the dependence of the type

$$c(z) = a - b \cdot \ln \sigma(z), \quad a > 0,\ b > 0. \tag{19}$$

The coefficients a and b are constants and they are estimated empirically. The coefficient a for different types of rocks varies from 0.3 to 3.8, and b varies from 0.7 to 4.

5. Combined and sequential inversion

The works [12, 15] show the successful application of the optimization approach, i.e., the search for the minimum point of the data misfit functional for the solution of problems (1)–(4) and (11)–(13), correspondingly.

In attempting to give a complex interpretation of data of various geophysical methods, there arises the problem of choice of the best strategy. The work [6] describes two approaches to the complex processing.

Using the introduced terminology, **the joint inversion** is taken to mean minimization of the complex data misfit functional

$$\Phi[c(z), \sigma(z)] = \alpha\Phi_1\big[c(z)\big] + \beta\Phi_2\big[\sigma(z)\big] + \gamma\|c(z) - f(\sigma(z))\|^2, \tag{20}$$

where α, β, γ are some weight parameters that need to be selected, and the function $f(\sigma(z))$ describes the correlation of the form $c(z) = a - b \cdot \ln \sigma(z)$, $a > 0$, $b > 0$, $\|\cdot\|$ is some suitable norm.

The following procedure will be called *sequential inversion*: the result obtained after the minimization of the functional $\Phi_2[\sigma(z)]$ (or $\Phi_1[c(z)]$) is used as the initial approximation computed with the help of correlation (18) for the search of the minimum point of the functional $\Phi_1[c(z)]$ ($\Phi_2[\sigma(z)]$, correspondingly).

We think that the combination of these two approaches is most effective.

6. Numerical experiments

In order to carry out the numerical experiments, a program package was written in the language **Borland C++** with the extended graphic interface enabling the reconstruction of the functions $c(z)$ and $\sigma(z)$ in the interactive regime by means of the combined and sequential inversions.

The method of conjugate gradients in the following interpretation:

$$\begin{aligned}
&f_{j+1}(z) = f_j(z) - \alpha_j P_j(z),\\
&\alpha_j = \arg\min_{\alpha\geq 0} \Phi\big[f_j(z) - \alpha P_j(z)\big],\\
&P_0(z) = \nabla_f \Phi\big[f_0(z)\big], \quad P_j(z) = \nabla_f \Phi\big[f_j(z)\big] - \beta_j P_j(z), \quad j \geq 1,\\
&\beta_j = \big(\nabla_f \Phi[f_j(z)], \nabla_f \Phi[f_{j-1}(z)] - \nabla_f \Phi[f_j(z)]\big),
\end{aligned}$$

where $f(z) = c(z)$ (or $\sigma(z)$) and the step α_j is chosen using the "golden section" method and was used for the organization of the iterative process of the search for the minimum points of data misfit functionals.

Numerical experiments on the reconstruction of the velocity structure of the medium were carried out for a sufficiently complex model with wave guides and high-speed layers (Figure 1).

The impulse with the "bell-shaped enveloping curve" ($f = 20$ Hz)

$$F(\omega) = \left[\exp\left(-\left(\frac{\omega - 2\pi f}{\pi f}\right)^2\right) + \exp\left(-\left(\frac{\omega + 2\pi f}{\pi f}\right)^2\right)\right] \times \exp(-i \cdot 1.75\omega/f). \tag{21}$$

was chosen as the sounding signal.

The calculations were made for the time frequency range 5–40 Hz.

The full wave field (the data $V_0(k,\omega)$) was calculated by the algorithm described in [11].

The initial approximation shown in Figure 1 by the dotted line was chosen in the form of a linear function not containing any information on wave guides and high-velocity layers.

It was assumed that between the functions of velocity $c(z)$ and conductivity $\sigma(z)$ there exists the correlation of the form

$$c(z) = a - b \cdot \ln \sigma(z), \tag{22}$$

where $a = 1.8$, $b = 2.5$.

Some distribution of conductivity in the studied medium was obtained using this relation. It is shown by the dotted line in Figure 2. However, it is natural to consider that the relation (22) which was obtained empirically is not accurate. Therefore, the model was "spoiled" and as a result we obtained the "real function" $\sigma(z)$ (its graph is shown by the solid line in Figure 2).

The sounding frequency $F = 20$ Hz and the maximal value λ equal to 8 were chosen for the calculations.

The initial approximation for the function $\sigma(z)$ is shown in Figure 3 by a dotted line. It is evident that this approximation is rather far from the real distribution of conductivity in a medium.

The semi-analytical method [11] was also used for the calculation of the electromagnetic field strength $E_0(\lambda,\omega)$.

First stage. At the first stage of calculations it was reasonable to obtain some approximations for the functions $c(z)$ and $\sigma(z)$ separately, i.e., minimizing the functionals $\Phi_1[c(z)]$ and $\Phi_2[\sigma(z)]$.

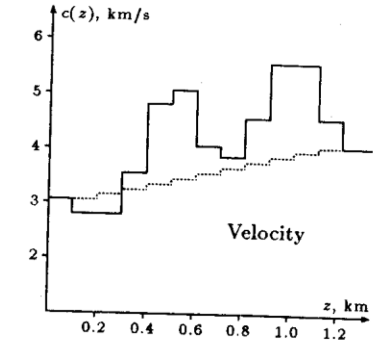

Figure 1

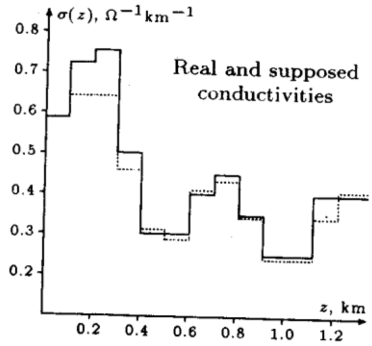

Figure 2

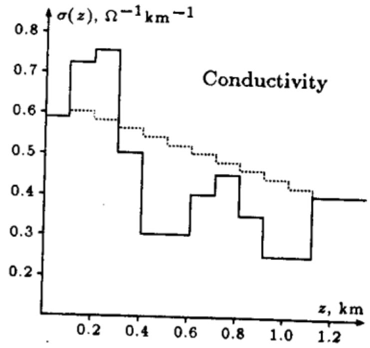

Figure 3

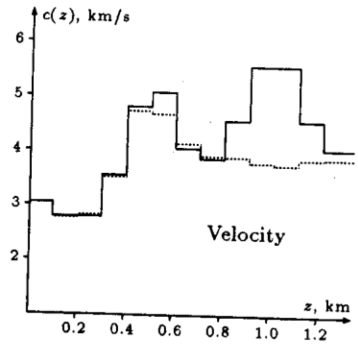

Figure 4

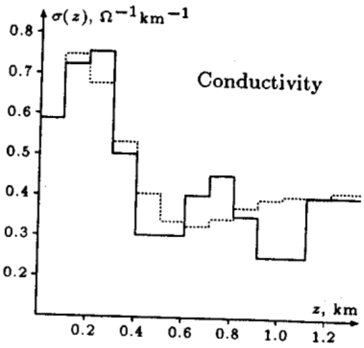

Figure 5

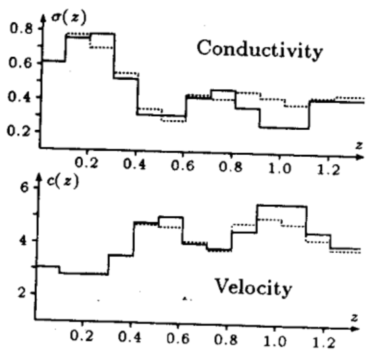

Figure 6

The approximation shown in Figure 4 was obtained for the function $c(z)$ as the result of 50 iterations.

The approximation for the function $\sigma(z)$ calculated in 80 iterations is given in Figure 5.

Here and below, giving the number of iterations made, we mean practically the complete stop of the iterative process at the considered stage. The quality of the approximations obtained was being estimated based on the closeness of the values of the corresponding functional to zero. The analysis of theoretical seismograms was also carried out in order to estimate the velocity distribution.

It should be noted that the minimization of the functional $\Phi_1[c(z)]$ gave better results than the minimization of the functional
$\Phi_2[\sigma(z)]$. But the approximation obtained is rather far from the real value also for the function $c(z)$, i.e., the separate inversions did not yield the desired result.

Second stage. The approximations that were obtained at the first stage of calculations were taken as initial approximations for the search of the minimum point of the complex functional

$$\Phi\big[c(z),\sigma(z)\big] = \alpha\Phi_1\big[c(z)\big] + \beta\Phi_2\big[\sigma(z)\big] + \gamma\|c(z) - f(\sigma(z))\|^2, \qquad (23)$$

where α, β, γ are some weight parameters that must be selected, and the function $f(\sigma(z))$ describes the "empirically determined" correlation of the form $c(z) = a - b \cdot \ln\sigma(z)$, $a = 1.8$, $b = 2.5$.

The minimization process of the functional (23) required the interactive regime. Approximately 150 iterations were carried out with the change of the parameters α, β and γ where necessary.

The parameters in the complex functional were selected mostly on the basis of intuitive considerations, namely, the functional was to:

- have the least pronounced "ravine structure" (values of the conditional number of the matrix of the second derivatives of the functional (22) were estimated).
- be convex, where possible, during the iterative process.

The results of the second stage of calculations are shown in Figure 6.

Third stage. At the third stage, attempts were again made to organize the iterative process of search for the minimum point separately, i.e., setting the parameters α and β in turn to zero and varying the parameter γ.

No improvement was achieved for the distribution of conductivity.

A good result was obtained for the velocity function using the method described in [12]. The resulting approximation is shown in Figure 7.

Fourth stage. As the obtained velocity distribution in the medium under study was rather good, an effort was made at this stage to specify the correlation between the functions $c(z)$ and $\sigma(z)$. For this, selection of the parameters a and b for the obtained values of the velocity function was made.

The calculated values of $\sigma(z)$ were substituted into the functional $\Phi_2[\sigma(z)]$. It turned out that this functional takes the minimal value at $a = 1.65$ and $b = 2.4$. These parameters were substituted into the complex functional $\Phi[c(z), \sigma(z)]$. The result of minimization of this functional at $\alpha = 1$, $\beta = 10^6$, $\gamma = 10^2 - 10^3$ is shown in Figure 8.

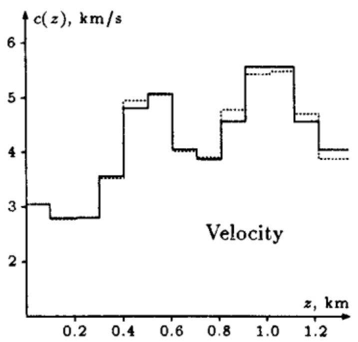

Figure 7

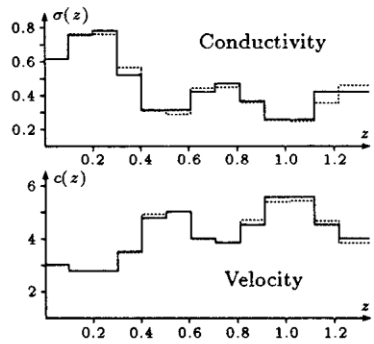

Figure 8

7. Conclusion

In attempting to solve numerically the combined inverse problems of acoustics and geoelectrics by the optimization technique it has been found that the minimization of the complex functional $\Phi[c(z), \sigma(z)]$ that we selected, with the variation of the parameters α, β and γ where necessary, gives much more satisfactory results than the minimization of individual functionals $\Phi_1[c(z)]$ and $\Phi_2[\sigma(z)]$. It should be noted that, as expected, this manifested itself to the highest degree in the reconstruction of the medium conductivity, i.e., in the inversion of the electromagnetic fields. Minimization only of the functional $\Phi_2[\sigma(z)]$ failed to lead to a solution more or less similar to the real model. Minimization of the functional $\Phi_1[c(z)]$, i.e., inversion of the wave fields, gave much better results. However, satisfactory approximation could not be obtained in this case either.

Minimization of the complex functional $\Phi[c(z), \sigma(z)]$ made it possible to overcome these difficulties. It should be noted that the part of the functional which was responsible for acoustics was most important, although it would be impossible to obtain the final result without the addition of the functionals responsible for the electromagnetic field and for the connection between the parameters of the problem.

Unfortunately, the minimization process required the interactive regime, because at each stage of calculations it was necessary to select the parameters of the complex functional in order to improve the convergence of the iterative process.

In this connection we would like to point out that at present the question of selection of the "optimal" data misfit functional for the solution of inverse problems, even in the case of their separate statements, is, in our opinion, far from being solved. This also concerns the question of relation between the points of global minimum of the data misfit functionals that are considered for the numerical solution of the inverse problems, and accurate solutions, i.e., the problem of stability. Unfortunately, the currently available theory is poorly applicable even to more or less complicated 1D problems.

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