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Simultaneous determination of source-time function and velocity via full wave field inversion*

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This paper describes a numerical method for the solution of the inverse problem of acoustics in the case of recording of the full wave field at the free surface. The solution is sought by means of minimization of the data misfit functional which is a mean square deviation of the given wave field from the wave field calculated for some "test" medium model. The main attention is given to the problem of simultaneous determination of the form of the sounding signal and the velocity structure of the vertically-inhomogeneous medium in the case of the point source and receivers located at the free surface.

Introduction

This work presents a numerical method for the solution of the inverse problem of acoustics in the case of the recording of full wave field at the free surface. It is an extension of the articles [1, 2].

The solution to the problem being considered is sought by means of the minimization of the data misfit functional which is a mean square deviation of the given wave field from the wave field calculated for some "test" medium model. This approach often called "inversion" was described in many works ([3–6]). The main difficulties associated with its application are well-known. This is the problem of determination of the trend component, i.e., the parameters of the medium under study cannot be reconstructed when there is no information on its low-frequency structure. The other problem is that it is necessary to know the exact form of the sounding signal.

The articles [5–6] are devoted to the investigation of the first problem, and therefore we shall not consider it in detail. The main attention will be given to the solution of the second problem – simultaneous determination of the form of the sounding signal and the velocity structure of the vertically-inhomogeneous medium in the case of the point source and the receivers located at the free surface.

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The problems in similar statements were investigated theoretically and numerically in [7–11]. The article of Cheverda and Voronina [11] where the theorem of uniqueness of the minimum point of the data misfit functional was proved should be pointed out. It proposes a new numerical algorithm for the solution of the inverse problem of vertical seismic profiling (VSP).

The present work describes the use of this algorithm in the case of recording of the full wave field at the free surface. It should be noted that here the proof of the theorem of uniqueness turns out to be problematic.

1. Statement of the inverse problem

The inverse problem we are studying here is of the following type: waves propagate in a three-dimensional medium, but the velocity distribution characterizing the medium is assumed to depend on only one variable z. Let the point source be applied at the surface point $(0,0,0)$. We denote the wave field caused by this excitation by $U(x,y,z)$. This function is the solution to the following initial boundary value problem:

$$\frac{\partial^2 U}{\partial t^2} = c_*^2(z)\Delta U, \tag{1}$$

$$U|_{t<0} \equiv 0, \tag{2}$$

$$\frac{\partial U}{\partial z}\Big|_{z=0} = f(t)\delta(x,y), \tag{3}$$

where $c_*(z)$ is a piecewise-continuous function characterizing the wave propagation velocity in a vertically inhomogeneous medium, $f(t)$ describes the influence in time of the source wavelet at the free surface. Let $z_1,\dots,z_n$ be boundaries of layers with the velocities $c_1,\dots,c_n$. For this model, we add the conditions at $z = z_i$ for relations (1)–(3):

$$[U]_{z=z_i} = \left[\frac{dU}{dz}\right]_{z=z_i} = 0, \qquad i = 1,\dots,N.$$

For system (1)–(3), let us consider the following inverse problem:

It is necessary to determine the functions $c(z)$ and $f(t)$ using the information on the oscillation regime of the observation surface $z = 0$

$$U|_{z=0} = U_0(x,y,t);\ 0 < t < T; \quad (x,y) \in S \subset \{(x,y,z):\ z = 0\}. \tag{4}$$

Following [6], we rewrite the initial statement (1)–(4) in the following way:

$$\frac{d^2 U}{dz^2} = (k^2 - \omega^2/c^2(z))U, \tag{5}$$

$$\left.\frac{dU}{dz}\right|_{z=0} = F(\omega), \tag{6}$$

$$U|_{z=0} = U_0(k,\omega), \tag{7}$$

where

$$F(\omega) = \int_0^T f(t)\exp(-i\cdot\omega t)dt,$$

and for the unique solution it is assumed that the principle of limit absorption, i.e.,

$$U(z,k,\omega) = \lim_{\varepsilon\to+0} U(z,k,\omega - i\varepsilon),$$

where, in turn, $U(z,k,\omega - i\varepsilon) \to 0$ as $z \to \infty$, is fulfilled (ε is the parameter of absorption).

Let us introduce some additional definitions.

The set of functions $\{c^2(z), F(\omega)\}$, where

$$0 < \delta \le c^2(z) \le c_0^2; \qquad c^2(z) = c_0^2, \quad z \ge H;$$
$$c^2(z) - c_0^2 \in L_2(0,\infty) \cap C^2(0,\infty),$$
$$F(\omega) \in L_2(\omega_1,\omega_2),$$

will be called the set of models M.

The operator $A[c^2, F](\omega, k)$ transferring the element (c^2, F) of the set M to the solution of the boundary value problem (5)–(6) at $z = 0$, will be called the operator of the solution to the direct problem. It can be shown that this operator is differentiable over Frechet with respect to the functions F and c^2.

Then the image of the set M under the action of the operator A will be called the set of acceptable data D ($D = A(M)$).

2. The method of solution

Solution of the inverse problem (5)–(7) will be sought as the minimum point of the following data misfit functional:

$$\Phi[c^2(z), F(\omega)] = \int_{\omega_1}^{\omega_2} d\omega \int_{k_1}^{k_2} |U_0(k,\omega) - A[c^2(z), F(\omega)](k,\omega)|^2 k\,dk, \tag{8}$$

where (ω_1,ω_2) and (k_1,k_2) are time and space frequency ranges determined by the spectral composition of the sounding signal $F(\omega)$ and the size of the observation domain S.

As shown in [6], realization of the search process of the minimum point of the data misfit functional in the frequency range (k, ω) enables a considerable reduction of computational resources necessary for the multiple solution of the direct problem. Moreover, it is possible to make detailed analysis of spectra of the wave fields at each step of calculations.

It can be easily shown that functional (8) is also differentiable over Frechet with respect to its variables $c^2(z)$ and $F(\omega)$. So, we can obtain the following expressions for its gradient:

$$\nabla_{c^2}\Phi[c^2(z), F(\omega)](\xi) = -2Re\int\limits_{\omega_1}^{\omega_2}(\omega+i\varepsilon)^2\bar{F}(\omega)d\omega \times$$
$$\int\limits_{k_1}^{k_2}\Big[U_0(k,\omega) - A[c^2(z), F(\omega)](k,\omega)\Big]\bar{G}^2(\xi,k,\omega)k\,dk, \tag{9}$$

$$\nabla_F\Phi[c^2(z), F(\omega)](\omega) =$$
$$-2\,\mathrm{Re}\int\limits_{k_1}^{k_2}\Big[U_0(k,\omega) - A[c^2(z), F(\omega)](k,\omega)\Big]G(\xi,k,\omega)k\,dk -$$
$$2i\cdot\mathrm{Im}\int\limits_{k_1}^{k_2}\Big[U_0(k,\omega) - A[c^2(z), F(\omega)](k,\omega)\Big]G(\xi,k,\omega)k\,dk, \tag{10}$$

where $G(\xi, k, \omega)$ is the solution to the problem (5), (6) at $F(\omega) = 1$.

Let there exist the point $(c_s^2, F_s) \in M$ where the gradient of the functional vanishes. Then, from (9) and (10), it is easy to obtain the following expression:

$$F_s(\omega) = \frac{\int_{k_1}^{k_2}\bar{G}(0,k,\omega)U_0(k,\omega)kdk}{\int_{k_1}^{k_2}\left|G^2(0,k,\omega)\right|^2kdk}, \tag{11}$$

where $G(0, k, \omega)$ is the solution to the boundary value problem (5), (6) at $F(\omega) = 1$, and $c^2(z) = c_s^2(z)$ taken at the point $z = 0$. In [11], it was proposed to use the formula similar to (11) for the calculation of the impulse $F_k(\omega)$ at the k-th iteration.

It is worth noting that [11] gives the proof of the theorem of uniqueness of the minimum point of the data misfit functional which is a square deviation of the wave fields recorded at several internal points of the medium from the wave fields calculated at the same points for the "test" medium model (VSP problem). In the case being considered, it is not possible to prove the theorem of uniqueness.

3. Numerical experiments

In order to conduct numerical experiments, two models of the vertically inhomogeneous medium were chosen. The first model is simple, without large differences of the velocity function. The second model is relatively complex. It contains waveguides and has sharp velocity variations. Reconstruction of the medium was made up to the depth of 1 km. Below this depth, the velocity was assumed to be constant and equal to the velocity value in the last layer. The whole medium, from its surface to the depth of 1 km, was divided into 10 equivalent layers with constant velocity.

In the calculation of the initial data, i.e., of the function $U_0(k,\omega)$, the function

$$F(\omega) = \left[\exp\Big(-\Big(\frac{\omega-2\pi f}{\pi f}\Big)^2\Big) + \exp\Big(-\Big(\frac{\omega+2\pi f}{\pi f}\Big)^2\Big)\right] \times \exp(-i\cdot 1.75\,\omega/f), \tag{12}$$

was taken as an input impulse. Here f is the dominant frequency equal to 20 Hz. The calculations were made with the help of the semi-analytical method described in [12]. The range of time frequencies was taken from 5 to 40 Hz.

For the search of the minimum point of the data misfit functional (8), we used the method of conjugate gradients in the following interpretation

$$\begin{gathered}
c_{j+1}(z) = c_j(z) - \alpha_j P_j(z), \\
\alpha_j = \arg\min_{a\geq 0} \Phi\left[c_j(z) - \alpha P_j(z), F_j(\omega)\right], \\
P_0(z) = \nabla_c \Phi\left[c_0(z), F_0(\omega)\right], \\
P_j(z) = \nabla_c \Phi\left[c_j(z), F_j(\omega)\right] - \beta_j P_j(z), \quad j \geq 1, \\
\beta_j = \left(\nabla_c\Phi[c_j(z), F_j(\omega)], \nabla_c\Phi[c_{j-1}(z), F_{j-1}(\omega)] - \nabla_c\Phi[c_j(z), F_j(\omega)]\right),
\end{gathered}$$

where the step α_j was chosen using the method of "golden section".

In order to calculate the impulse $F_j(\omega)$ at the j-th iteration, the condition of vanishing of the gradient of functional (8) with respect to the function F_j at the current velocity $c_j(z)$, i.e., expression (11) was used.

Figure 1 shows the first velocity model of the medium (solid line) and the initial approximation (dashed line). Figure 2 gives the form of the impulse of the input signal $f(t)$ and the first approximation for it obtained from (11). As a result of 35 iterations by the method of conjugate gradients, it was possible to reconstruct, with good accuracy, both the velocity distribution of the medium and the function $f(t)$. The results of calculations are presented in Figures 3–4. It should be noted that in this case we used the initial approximation which describes well the low frequency structure of the

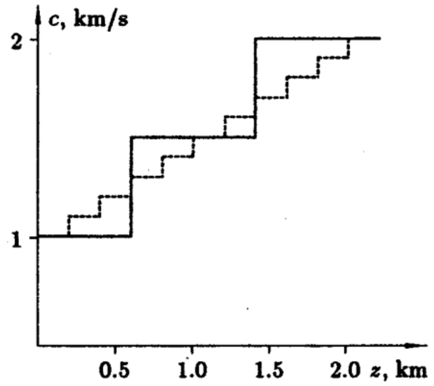

Figure 1

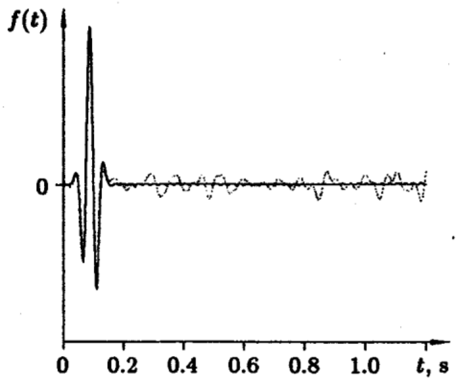

Figure 2

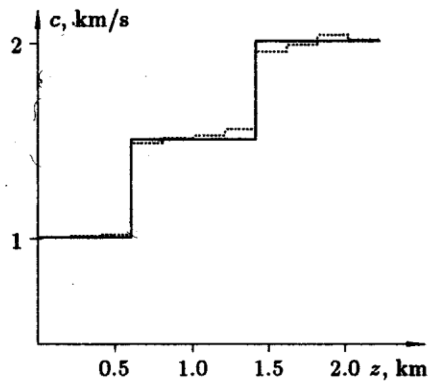

Figure 3

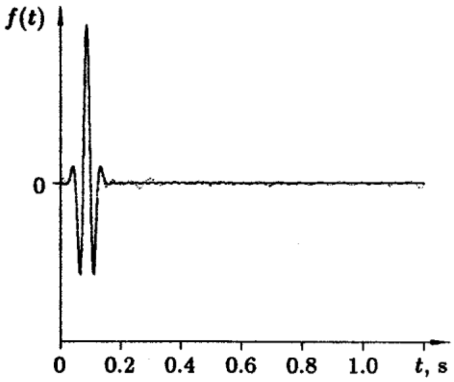

Figure 4

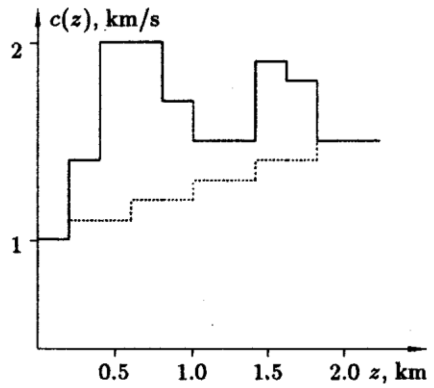

Figure 5

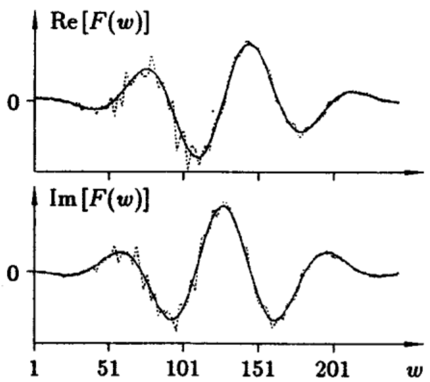

Figure 6

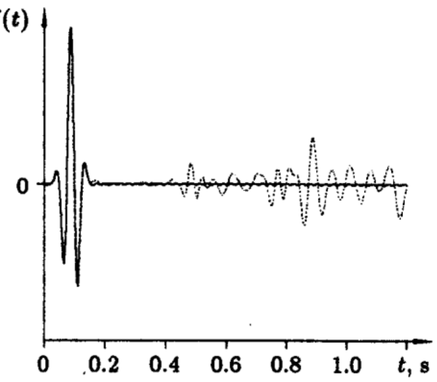

Figure 7

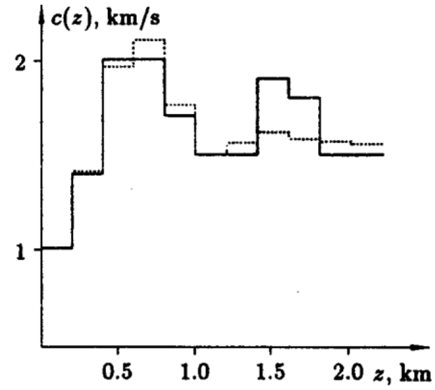

Figure 8

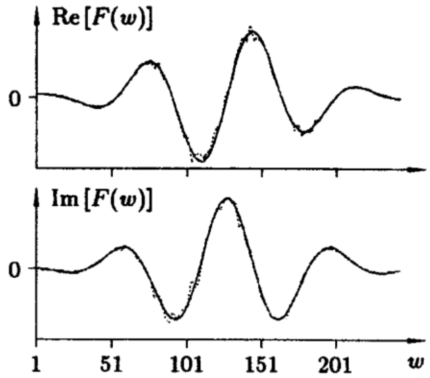

Figure 9

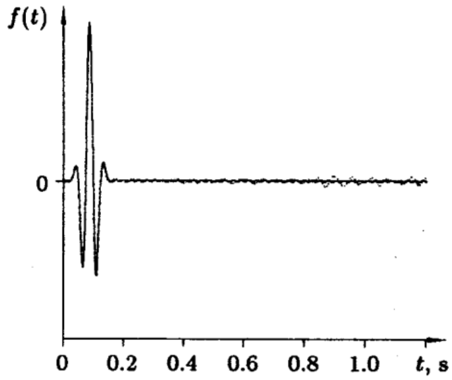

Figure 10

medium (i.e., the trend). Therefore, as expected, there were no difficulties with the reconstruction.

A more complex model was chosen for the other series of calculations. It is shown by the solid line in Figure 5. The initial approximation shown by the dashed line is far from the real velocity distribution. The exact value of the impulse form and the approximation calculated for it using (11) are presented in Figure 6 (in the frequency domain) and in Figure 7 (in the time domain).

In this case, there arises the well-known problem of trend component. Therefore, the calculations were made using the technique described in the article of Alekseev et. al. [6], i.e., the appropriate space frequency ranges k were chosen in order to compensate for the absence of low time frequencies ω in the spectrum of the recorded signal $U_0(k, \omega)$.

The final results are represented in Figures 8–10. It is seen that the impulse form was reconstructed sufficiently well. However, it was not possible to reconstruct the velocity function with a good accuracy.

4. Conclusion

The results of numerical experiments cited in the previous section showed the efficiency of the numerical method proposed for the solution of the inverse problem of acoustics in the case of the recording of the full wave field at the free surface.

They also confirmed the main difficulty associated with the use of the optimization approach (inversion), namely, the necessity to have information on the low velocity structure of the medium under study.

One way out of this situation is the search for a good initial approximation (i.e., containing low frequency components of the velocity function) with the help of the different scales' basis, as was done in the work of Cheverda and Voronina [11]. Another possible approach is the consideration of combined inverse problems [13, 14]. In this case, the solution is sought as the minimum point of the complex data misfit functional taking into account *a priori* connections between the parameters of the medium under study.

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