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Spectral analysis and synthesis of logical functions for special-purpose VLSI implementation

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Some results of investigation into the spectral properties of k-valued ($k \geq 2$) logical functions are presented. The main goal of the investigation is to show the power of spectral methods in implicant extraction and recognition of some useful logical function properties. The relation between the sum-of-products form of a logical function and its spectrum is established by means of a group representation of implicants. It allowed to attract some results of abstract harmonic analysis of discrete-valued functions. A number of theorems which states the necessary and sufficient conditions for the existence or absence of an implicant and for recognition of monotonicity and symmetry are formulated. The application of the results is shown by presenting an algorithm for deriving an irreduntant sum-of-product form of a logical function.

1. Introduction

The spectral theory of logical functions studies the properties of logical functions in the domain of their Fourier transforms (spectra). Investigation techniques of this theory are based on the powerful tools of harmonic analysis and have the following properties.

1. The algorithms they generate have natural concurrency at both fine-grained (calculation of a spectrum) and coarse-grained (analysis of spectral properties) levels.

2. They are applicable to k-valued logical functions with any $k \geq 2$, the Boolean functions being a particular case. Despite the fact that fundamental works on the spectral theory of logical functions were written long ago [5-7], it has not become widespread. This is attributed to the following facts. First, a great complexity of computation of a spectrum constitutes a large portion of the total complexity of function analysis. Second, because of the poorly developed multi-valued hardware there is no urgent need in the techniques for the synthesis and analysis of multi-valued functions to which spectral methods are more suitable than the conventional ones.

Due to these reasons the architecture and hardware of nowadays computers will go as computers of new generations emerge. Even now there exist very fast special purpose processors which perform the Fourier transforms [12]. Multivalued programmable logic arrays are also anticipated to appear in the near future [14]. In view of the development of optical methods of computation, spectral methods are to become most useful. The reason for that is that computing a spectrum is a single-step operation in optical processor [3,4]. Besides, light invariance to magnetic and electrical fields makes light switching reliable and attracts multi-valued data representation.

Hence the assurance that progress in the spectral theory of logical functions will contribute not only to theoretical computer science by enriching our knowledge, but also to practical efforts of creating computers of new generation. This paper presents the results of investigation into the spectral properties of k-valued logical functions. The first section gives the necessary definitions. A number of theorems on spectral properties of logical functions are stated and proved in the second section. The third section is dedicated to a particular case of the Boolean functions. The fourth section reports on practical applications of the results by considering an algorithm for deriving an irredundant sum-of-products form of function representation. This contribution is an evolution of the classical work by Lechner [7] in terms of extending the class of the functions under investigation and elaborating the methods for their analysis and synthesis [2,10,11].

Because of the lack of space, complete proofs of theorems are left out. The underlying postulates are pointed out instead.

2. Group representation of a-implicants of logical functions

2.1. Sum-of-products representation of logical functions

A logical function (function of k-valued logic, $k \geq 2$) of n variables $f(X) = f(x_1, x_2, \dots, x_n)$ is assumed to be a mapping of the form

$$f(X) : X^n \to E^k,$$

where $E^k = \{1, 2, \dots, k-1\}, X^n = \{X_i : i = 0, 1, \dots, k^n - 1\}$ is a set of ordered n-dimensional vectors. The vector $X_i = (x_1, x_2, \dots, x_n)$, $x_m \in E^k$, $m = 1, 2, \dots, n$, is a decimal i in a k-valued code. The set of vectors such that $f(X_i) = a$, $a \in E^k$, is specified as $f^{-1}(a)$. There exists a number of ways of specifying logical functions in the form of a sum of products of unary functions. Here the Su and Cheung representation [16] based on algebra from [1,17] is used, because of its similarity to the Boolean functions in the form of implicants.

Su and Cheung representation includes three operations: AND *(logical product)*, OR *(logical sum)* and *compound literal.*

AND *(max)* and OR *(min)* are binary operations in the ring E^k such that for any a and b from E^k

$$\begin{aligned} a \cdot b &= \min(a, b), \\ a + b &= \max(a, b). \end{aligned}$$

Compound literal for variable x is a literal or product of literals of the same variable x

$$x(Q) = x(a_1, b_1)^* x(a_2, b_2)^* \dots x(a_p, b_p)^*,$$

where Q, $Q \in E^k$, is a set of values of variable x at which the product $x(Q)$ takes the value equal to $(k-1)$, $a_i \leq b_i$ for all $i = 1, 2, \dots, p$, $x(a_i, b_i)^*$ is an *uncomplemented literal*

$$x(a_i, b_i) = \begin{cases} k-1, & \text{if} \quad a_i \leq x \leq b_i, \\ 0, & \text{otherwise} \end{cases}$$

or a complement literal

$$x(a_i, b_i)^- = \begin{cases} k-1, & \text{if} \quad a_i > x > b_i, \\ 0, & \text{otherwise.} \end{cases}$$

An example of a compound literal $x(Q) = x(0,5)x(2,3)^-$, $k = 7$, is given in Figure 1. Here the set Q of values of x is equal to $\{0, 1, 4, 5\}$. Let the power of Q be referred to as the *cost* of a compound literal and be denoted by $\hat{Q}$.

A function of the following form

$$h(X) = x_{i_1}(a_{i_1}, b_{i_1})^* x_{i_2}(a_{i_2}, b_{i_2})^* \dots x_{i_p}(a_{i_p}, b_{i_p})^* \tag{1}$$

is called a ***multiplicative*** term. Variables x_{i_j} in (1) form a set of *bound* variables $X' \subset X$. A set of vectors X_i such that $h(X_i) = a$ is called a *set of constituents* of a of the function $h(X)$ and is denoted by $V(h)$.

In [14] a theorem is proved which states that any logical function can be expressed by a logical sum of multiplicative terms, i.e.,

$$f(X) = \sum_{l=1}^{\lambda} h_l(X). \tag{2}$$

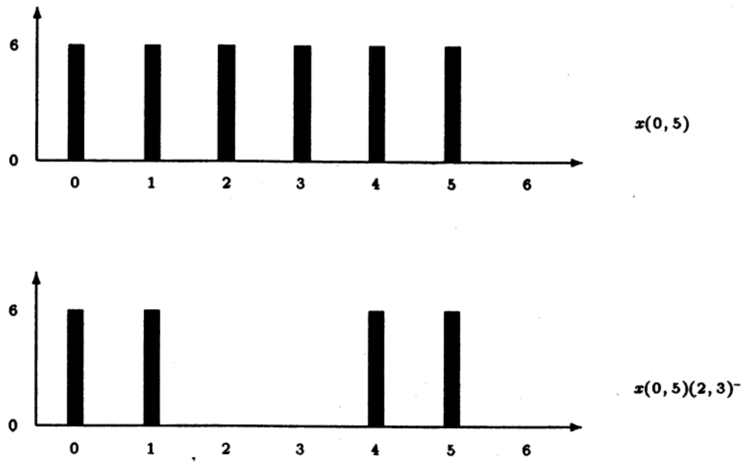

Figure 1

It is referred to as a *sum-of-products form.*

Definition 1. Function (1) is called an a-implicant of the logical function $f(X)$ iff

$$h(X_i) \leq f(X_i) \quad \text{for all} \quad X_i \in V(h).$$

An a-implicant may be represented by a 3-tuple $< a, X', Q >$. Here $X' = \{x_{i_1}, \ldots, x_{i_p}\}$ is a set of bound variables. The power p of the set X' is called the *dimension* of the a-implicant. $Q = \{Q_{i_1}, \ldots, Q_{i_p}\}$ is a set of subsets of E^k, where each Q_{i_j} is a set of values which x_m takes in the vectors $X_i \in V(h)$. For example, the implicant

$$h(X) = 3x_1(0,3)x_1(1,1)^{-}x_2(1,2)x_3(1,3)$$

corresponds to the 3-tuple $< 3, \{x_1, x_2, x_4\}, \{\{0,2,3\}, \{1,2\}, \{1,3\}\} >$.

It follows from (2) that any logical function can be expressed by a logical sum of a-implicants. Let's say that the a-implicant of the function $f(X)$ *covers* the vector $X_i \in V(h)$.

Definition 2. A logical sum of a-implicants of a logical function $f(X)$ is called the irredundant cover of $f(X)$ iff it has the following properties:

1) a-implicants cover those and only those vectors X_i at which $f(X) \neq 0$;

2) no a-implicant can be removed from the logical sum so that property 1) holds.

Example 1. The function $f(X), k = 3, n = 3$, (see Table 1) can be expressed by a sum of two implicants $h_1(X) = 2x_1(0,2)x_1(1,1)^{-}x_2(1,2)$ and $h_2(X) = x_1(0,0)x_2(0,0)x_3(1,2)$, which may be represented also by 3-tuples:

$$
\begin{aligned}
h_1(X) &= \ < 2, \{x_1, x_2\}, \{\{0,2\}, \{1,2\}\} >, \\
h_2(X) &= \ < \{x_1, x_2, x_3\}, \{\{0\}, \{0\}, \{1,2\}\} > .
\end{aligned}
$$

The sum of these implicants is an irredundant cover of $f(X)$.

Table 1

$x_1x_2x_3$	$f(x_1,x_2,x_3)$	$x_1x_2x_3$	$f(x_1,x_2,x_3)$	$x_1x_2x_3$	$f(x_1,x_2,x_3)$
000	0	100	0	200	0
001	1	101	0	201	0
002	1	102	0	202	0
010	2	110	0	210	2
011	2	111	0	211	2
012	2	112	0	212	2
020	2	120	0	220	2
021	2	121	0	221	2
022	2	122	0	222	2

3. Group representation of a-implicants of logical functions

The set of vectors $\{X_i : i = 0, 1, \ldots, k^n - 1\}$ forms the Abelian group X^n under (mod k) addition. The zero of the group is the element $X_0 = (0, 0, \ldots, 0)$, the element $(-X_i)$ inverse to X_i is estimated by $(-X_i) \oplus X_i = X_0$.

Let us now return to the definition of a-implicant. We will consider the case of all bound variables of a-implicant being equal to zero. It means that $V(h)$ includes these and only these elements X_i in which the values of all bound variables are zero. The remaining variables $x_m \in X''$, where $X'' = X \backslash X'$, $\mid X'' \mid = n - p$, take all the values from the E^k and are called *free* variables. Such a set of elements $\{X_i : X_i \in V(h)\}$ forms a subgroup of order k^{n-p} of the group X^n with the maximum element $b = (b_1, b_2, \ldots, b_n)$ whose components have the following values

$$
b_m = \begin{cases} 0, & \text{if} \quad x_m \in X', \\ k-1, & \text{if} \quad x_m \in X''. \end{cases}
$$

Subgroups V such that $V = \{X_i : X_i \leq b\}$ are now denoted by V_b. Thus, element b partitions a set of variables x into two subsets: of bound and free variables X' and X'' respectively. The weight of the vector b (the number of nonzero components) $\hat{b} = p$.

Now let an a-implicant consist of a single compound literal $x_m(Q_m)$. Let us first consider a simple case of $\hat{Q}_m = 1$, $Q_m = \{d\}$, $d \in E^k \backslash 0$. Then $V(h)$ is a coset of the subgroup V_b, i.e., $V(h) = \{X_i \oplus c : X_i \leq b\}$. Here the vector $c = (c_1, c_2, \ldots, c_n)$, $c \in X^n$, is a *leader* of a coset of the subgroup V_b. Its components are evaluated as follows:

$$c_m = \begin{cases} d, & \text{if} \quad d \in Q_m, \\ 0, & \text{otherwise.} \end{cases}$$

Secondly, let us have $\hat{Q}_m > 1$, i.e., $Q_m = \{d_1, d_2, \ldots, d_r\}$, $r = \hat{Q}_m$. Then

$$V(h) = \bigcup_{c^i \in \Gamma} (V_b \oplus c^i),$$

where $\Gamma = \{c^1, c^2, \ldots, c^r\}$ is a set of leaders of cosets of V_b, $r = \mid \Gamma \mid = \hat{Q}_m$. For all $c^i \in \Gamma$, $c^i = (c^i_1, c^i_2, \ldots, c^i_n)$, their components are defined as follows:

$$c^i_m = \begin{cases} d_i, & \text{if} \quad d_i \in Q_m, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let an implicant $h(X) = < a, X, Q >$ contain several compound literals with $\hat{Q}_m \geq 1$ and let D be a scalar product of all $Q_m \in Q$, $D = \{D^1, D^2, \ldots, D^r\}$, where $r = \prod_{Q_m \in Q} \hat{Q}_m$. Then the set of the leaders $\Gamma = \{c^1, c^2, \ldots, c^r\}$ is isomorphic to D, the components of each vector c^i are evaluated as follows:

$$c^i_m = \begin{cases} d, & \text{if} \quad d \in D^i \cap Q_m, \\ 0, & \text{otherwise.} \end{cases}$$

The set of all leaders of the cosets of the subgroup V_b under (mod k) addition forms a subgroup. The maximal element of this subgroup is the vector $\bar{b} = (\bar{b}_1, \ldots, \bar{b}_n)$, whose all significant components correspond to the bound variables of the a-implicant and are

$$\bar{b}_m = \begin{cases} k-1, & \text{if} \quad b_m = 0, \\ 0, & \text{if} \quad b_m = k-1. \end{cases}$$

Hence, the order of $V_{\bar{b}}$ is k^p, $\Gamma \subset V_{\bar{b}}$.

From the above it follows that an a-implicant $h(X)$ in terms of groups may be represented with a 3-tuple $< a, b, \Gamma >$.

Example 2. Let $h(X) =< 2, \{x_1, x_2\}, \{0,1,4\}, \{1,2\} >, k = 5, n = 4$. A set of bound variables $X' = \{x_1, x_2\}$ specifies the vectors b and $\bar{b}$, i.e., $b = (0,0,4,4)$, $\bar{b} = (4,4,0,0)$, $V_{\bar{b}} = \{W_i : W_i \leq (4,4,0,0)\}$. The set of leaders Γ is constructed from the Cartesian product of $Q_1 = \{0,1,4\}$ and $Q_2 = \{1,2\}$.

$$Q_1 \times Q_2 = \{\{0,1\}, \{0,2\}, \{1,1\}, \{1,2\}, \{4,1\}, \{4,2\}\}.$$

So, the group representation of $h(X)$ is

$$\begin{aligned} h(X) &= < 2, (0,0,4,4), \{(0,1,0,0), (0,2,0,0), (1,1,0,0), \\ &\quad (1,2,0,0), (4,1,0,0), (4,2,0,0)\} > . \end{aligned}$$

4. Main theorems of the harmonic analysis of logical functions

4.1. Abelian group characters

It is known [13] that a set of functions $\chi(X^n) = \{\chi_j(X) \colon j = 0, 1, \ldots, k^n - 1\}$ called *a group of characters* may be constructed for any Abelian group. Elements of the group $\chi(X^n)$, $\chi_j(X) = (\chi_j(X_0), \chi_j(X_1), \ldots, \chi_j(X_{k^n-1})$ called *characters* are mappings $\chi_j(X) : X^n -> C^*$ such that

$$\chi_j(X_i \oplus X_p) = \chi_j(X_i)\chi_j(X_p) \quad \text{for all} \quad X_i, X_p \in X^n,$$

where $C^*$is a cyclic subgroup generated by a primitive route $\sqrt[k]{1}$. Values of $\chi_j(X_i)$ are estimated as

$$\chi_j(X_i) = \epsilon^{X_i W_j^t} \quad \text{for all} \quad X_i \in X^n, \tag{3}$$

where W_j is an element from the transform domain W^n, identical to X^n, $X_i W_j^t$ is a vector product with (mod k) addition.

Further, we consider the characters of the group X^n, which are the Cartesian products of n cycle of k-order groups. The set of the Vilenkin-Chrestenson functions ordered according to Kronecker will be used. as a group of characters of X^n. An example of the group of characters with $k = 3$, $n = 2$ is a set of columns of the matrix $\bar{\chi}$, which is called *a transform matrix* (Table 2). Here $C^* = \{1, \epsilon, \epsilon^2\}$, $\epsilon = \sqrt[3]{1}$, vectors X_i and W_j are ordered correspondingly to their decimal values. Each element χ_{ij}

Table 2

$$
\chi = \begin{array}{c|ccccccccc|}
 & 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\
 & \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 & \chi_8 \\
\hline
00 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
01 & 1 & \epsilon & \epsilon^2 & 1 & \epsilon & \epsilon^2 & 1 & \epsilon & \epsilon^2 \\
02 & 1 & \epsilon^2 & \epsilon & 1 & \epsilon^2 & \epsilon & 1 & \epsilon^2 & \epsilon \\
10 & 1 & 1 & 1 & \epsilon & \epsilon & \epsilon & \epsilon^2 & \epsilon^2 & \epsilon^2 \\
11 & 1 & \epsilon & \epsilon^2 & \epsilon & \epsilon^2 & 1 & \epsilon^2 & 1 & \epsilon \\
12 & 1 & \epsilon^2 & \epsilon & \epsilon & 1 & \epsilon^2 & \epsilon^2 & \epsilon & 1 \\
20 & 1 & 1 & 1 & \epsilon^2 & \epsilon^2 & \epsilon^2 & \epsilon & \epsilon & \epsilon \\
21 & 1 & \epsilon & \epsilon^2 & \epsilon^2 & 1 & \epsilon & \epsilon & \epsilon^2 & 1 \\
22 & 1 & \epsilon^2 & \epsilon & \epsilon^2 & \epsilon & 1 & \epsilon & 1 & \epsilon^2
\end{array}
$$

$$
T = \begin{array}{|ccccccccc|}
\theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\
0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\
0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2
\end{array}
$$

$\theta_0 = 0$ $\quad$ $\theta_1 = x_2$ $\quad$ $\theta_2 = x_2 \oplus x_2$

$\theta_3 = x_1$ $\quad$ $\theta_4 = x_1 \oplus x_1$ $\quad$ $\theta_5 = x_1 \oplus x_2 \oplus x_2$

$\theta_6 = x_1 \oplus x_1$ $\quad$ $\theta_7 = x_2 \oplus x_1 \oplus x_1$ $\quad$ $\theta_8 = x_2 \oplus x_1 \oplus x_1 \oplus x_2$

of the matrix c is equal to the value of j-th character for X_i and may be evaluated by (3). From the definition of the character it follows that χ is a symmetric matrix, i.e., $\chi_j(X) = \chi(X_j)$.

Let the matrix T be associated with the matrix χ so that each element of T $\tau_{ij} = X_i W_j^t$. Since $\tau_{ij} \in E^k$, the columns of T may be interpreted as logical functions $\theta_j(X)$, $j = 0, 1, \ldots, k^n - 1$. These functions may be analyzed in accordance with the partition on the set of W^n. $W^n = W^{(0)} \cup W^{(1)} \cup \ldots \cup W^{(n)}$. Each element of the partition $W^{(l)}$, $l = 0, 1, \ldots, n$, is called *a level.* An l-th level contains those $W_j \in W^n$ whose weight is l, i.e., $W^{(l)} = \{W_j : \hat{W}_j = l\}$. Hence, $W^{(0)} = \{W_0\}$, $W^{(1)} = W_j$, the m-th component of W_j being equal to d, all others – to 0. Each subsequent level $W^{(l)}$ contains $W_j \in W^n$, which may be represented as a *(mod k)* sum of l different vectors from $W^{(l)}$. The partition of W^n induces a similar partition on the set θ of logical functions $\theta_j(X)$, so that $\theta = \theta^{(1)} \cup \theta^{(2)} \cup \ldots \cup \theta^{(n)}$. A function $\theta_j(X) \in \theta^{(l)}$, if $W_j \in W^{(l)}$. Evidently, $\theta^{(0)} = \{\theta_0(X)\}$ and $\theta_0(X)$ is an identical function equal to 0. Any function from $\theta^{(1)}$ is a function of one variable since for any $X_i \in X^n$ $\quad \theta_i(X) =$

$r_{ij} = (x_1, \ldots, x_m, \ldots, x_n)(0, \ldots, d, \ldots, 0)^t = dx_m$.

Similarly, $\theta_i(X) \in \theta^{(l)}$ is a function of l variables. Since it may be obtained as a (mod k) sum of different $\theta_j(X) \in \theta^{(1)}$.

4.2. Spectra of logical functions

The character group $\chi(X^n)$ is known [13] to form a complement orthogonal basis. It means that any logical function $f(X)$ may be expanded in a spectrum (the Fourier transform). Each coefficient of the spectrum is evaluated as

$$f^*(X_j) = f(X) \cdot \bar{\chi}_j(X) = \sum_{X_i \in X^n} f(X_i)\epsilon^{X_i W_j^t}, \tag{4}$$

where $\bar{\chi}_j(X)$ and $\chi_j(X)$ are componentwise complex conjugate, i.e., $\chi(X_i) = \bar{\chi}_j(X_i) = \chi_j(-X_i)$ for all $X_i \in X^n$. From the logical interpretation it follows that each coefficient $f^*(W_j)$ reflects the dependence of $f(X)$ on the subset of variables from $\theta_j(X)$.

In the matrix form the Fourier transform pair is defined as follows:

$$\begin{aligned} f^*(W) &= f(X) \cdot \bar{\chi}, \\ f(X) &= k^{-n} f^*(W) \cdot \chi. \end{aligned}$$

Example 3. Let $f(X) = 1x_1(2,2) + 2x_1(0,0)x_2(0,0) + 2x_2(1,1)x_2(0,0)$, $n = 2$, $k = 3$. Then $f^*(W) = (1,0,0,2,0,0,1,1,1) \cdot \bar{\chi} = (5,3,3,\epsilon^2 + 2\epsilon, 1 + \epsilon^2, 1 + 2\epsilon^2, \epsilon + 2\epsilon^2, 1 + 2\epsilon)$.

Theorem 1 [10]. *For any logical function $f(X)$*

$$f^*(W_0) = \sum_{X_i \in X^n} f(X_i). \tag{5}$$

Each coefficient $f^*(W_j)$ may be represented in the form of a polynomial

$$f^*(W_j) = \sum_{\tau \in E^k} s_\tau(W_j), \tag{6}$$

where $s_\tau(W_j)$ is the arithmetical sum of those $f(X_i)$ for which $X_i(-W_j)^t = \tau$.

Let $L(W_j)$ be a set of powers of ϵ of polynomical (6). Then the following theorem holds.

Theorem 2. *Let $f^*(W)$ be a spectrum of the logical function $f(X)$. Then*

$$\sum_{\tau \in L(W_j)} s_\tau(W_j) \leq f^*(W_0) \quad \textit{for all} \quad W_j \in W^n.$$

The proof is straightforward from (5) and (6).

4.3. Spectral properties of a-implicant of logical functions

As it was stated above the implicant $h(X) =< a, X', \Gamma >$ in the term of group theory is considered as a function which is constant and equal to a for all elements of a quotient group $X^n/V_b, V_b$ being a subgroup of X^n.

The proof of theorems which determine a-implicant behavior in the transform range is based on the very important property of the *quotient group characters.* The abstract proof of this property is given in [8]. In [7] it is proved for the special case of X^n with (mod 2) addition, $k = 2$. For $k > 2$ it is formulated here in the form of the following lemma.

Lemma 1. *Quotient group of characters X^n/V_b is isomorphic to a subgroup of the group of characters $\chi(X^n)$, this subgroup contains all characters that are constant on cosets of V_b, i.e.,*

$$\chi(X^n/V_b) = \{\chi_j(X) : (X_i \oplus c)W_j^t = 0, \quad \textit{for all } X_i \in V_b \quad \textit{and every } c \in V_{\bar{b}}\},$$

where $V_{\bar{b}} = \{W_j : W_j \leq \bar{b}\}$.

This lemma is the basis for the following theorem.

Theorem 3 (on the a-implicant spectrum) [9] . *Let $h(X) =< a, b, \Gamma >$ be an implicant. Then*

$$h^*(W_j) = \begin{cases} ak^p \sum\limits_{c \in \Gamma} \epsilon^{c(-W_j)^t}, & \textit{for all} \quad W_j \in V_{\bar{b}}, \\ 0, & \textit{otherwise.} \end{cases}$$

From the preceding theorem it follows that the coefficients of the a-implicant spectrum are nonzero only on elements of the subgroup $V_{\bar{b}}$. Relations between a, X' and Q determining the Su and Cheung representation of an implicant and the coefficients of its spectrum $h^*(W)$ are stated by the following theorem.

Theorem 4. *Let $h(X) =< a, X', Q >$ be an implicant. Then for all coefficients $h^*(W_j)$ such that $W_j \in W^{(1)}$ and $W_j \leq \bar{b}$ the following holds:*

$$\sum_{\tau \in L(W_j)} s_\tau(W_j) = ak^p \prod_{Q_m \in Q} \hat{Q}_m = A,$$

$$S\tau(W_j) = s_{\tau'}(W_j) = A/\hat{Q}_m \quad \textit{for all pairs} \quad \tau, \tau' \in L(W_j).$$

Let E be a set of vectors $W_j \in W^n$ such that $W_j \leq \bar{b}$ and $W_j = (0, \ldots, k-1, \ldots, 0)$, $k-1$ being equal to the m-th component of W_j. Then the following corollary is straightforward from Theorem 4.

Corollary 1. *Let* $h(X) = < a, X', Q >$. *Then*

$$L_j = Q_m \quad \textit{for all} \quad W_j \in E,$$

where $Q_m \in Q$.

The relations between the sum of values of $f(X)$ on the cosets of the subgroup V_b of X^n and the sum of coefficients of $f^*(W)$ on the subgroup $V_{\bar{b}}$ of W^n are stated by the *summation theorem*. For the continuous functions this theorem was proved in [8] (the Poisson theorem). For the case of the Boolean functions it is formulated and proved in [7]. Here this theorem is given for logical k-valued functions ($k \geq 2$) in the form of the following lemma.

Lemma 2. *Let* V_b *be a subgroup of* X^n *of order* k^p, $V_{\bar{b}}$ *be a subgroup of* W^n, *and let* Γ *be a set of these coset leaders. Then for any logical function* $f(X)$

$$\sum_{c \in \Gamma} \sum_{X_i \in V_b} f(X_i \oplus c) = k^{p-n} \sum_{c \in \Gamma} \sum_{W_j \in V_{\bar{b}}} f^*(W_j)\, \epsilon^{cW_j^t}.$$

4.4. Conditions for a-implicant extraction

It follows from Lemma 2 that a-implicant extraction may be done through the analysis of spectral coefficients on the subgroup $V_{\bar{b}}$ of W^n. The following theorems give the criteria for this analysis.

Theorem 6 [9]. *Let a logical function* $f(X)$ *contain no* a' *-implicant,* $a' = k-1, k-2, ..., a+1$. *Then if for any subgroup* $V_{\bar{b}}$

$$\sum_{W_j \in V_{\bar{b}}} | f^*(W_j) | < ak^n,$$

then there is no coset of V_b *on which* $f(X)$ *is constant and equal to a, i.e., here is no leader* $c \in V_{\bar{b}}$ *such that* $(V_b \oplus c) \subseteq f^{-1}(a)$. *Here* $| f^*(W_j) |$ *is a module of a complex number* $f^*(W_j)$.

Theorem 7 [10]. *Let* $f(X)$ *be a logical function and* $(V_b \oplus c) \subseteq f^{-1}(a)$. *Then for any subgroup* $V_{b'}$ *of* W^n *such that* $\bar{b}'\bar{b}^t = 0$ *and any* $c \in V_{\bar{b}'}$ *coset* $(V_{b'} \oplus c)$ *is not included in* $f^{-1}(a)$.

Theorem 8 [9]. *Let a logical function* $f(X)$ *contain no* a'*-implicant,* $a' = k-1, k-2, ..., a+1$. *Then if for any* $c \in V_{\bar{b}}$

$$\sum_{W_j \in V_{\bar{b}}} f^*(W_j)\epsilon^{cW_j^t} = ak^t, \quad \textit{then} \quad (V_b \oplus c) \subseteq f^{-1}(a), \tag{7a}$$

$$\sum_{W_j \in V_{\bar{b}}} f^*(W_j)\epsilon^{cW_j^t} = 0, \quad then \quad (V_b \oplus c) \subseteq f^{-1}(0). \tag{7b}$$

This theorem states the existence of an a-implicant $h(X) = < a, b, c >$ (condition $7a$) or of a 0-implicant $h(X) =< 0, b, c >$ (condition $7b$) in $f(X)$. The test for these conditions may be done as follows. The vector $f_{\bar{b}}^*$ is formed with a subset of spectral coefficient $\{f^*(W_j) : W_j \leq \bar{b}\}$, being allocated in $f_{\bar{b}}^*$ in the same order as in $f^*(W)$. This vector is then multiplied by the matrix $\bar{\chi}$ minor generated by the subsets of rows and columns corresponding to the subgroup $V_{\bar{b}}$. If the resulting vector $f'(X)$ has a component $f'(X_i) = ak^n$ ($f'(X_i) = 0$), then $f(X)$ contains the implicant $h(X) =< a, b, c >$ (or $h(X) =< 0, b, c >$) with $c = X_i$.

The following rule helps to determine the analytic representation of the excluded by Theorem 8 a-implicant.

Rule. *Let condition (7a) hold for a set of the leaders Γ of W^n and let $|\Gamma| = \prod_{W_j \in E} | L_j |$. Then logical function $f(X)$ contains an implicant $h(X) =< a, X', Q >$, where*

$$\begin{aligned} X' &= \{x_m : b_m = k-1\}, \\ Q &= \{Q_m : Q_m = L_j \quad for \quad W_j \in E\} \quad if \quad x_m \in X'. \end{aligned}$$

Theorem 9 [10]. *A logical function $f(X)$ may be represented in the form $f(X) = x_m(Q_m)g(X)$ iff for all $W_j = (0, ..., d, ...0)$ having b as the m-th component, $d \in E^k \backslash 0$,*

$$\sum_{\tau \in L_j} s_\tau(W_j) = f^*(W_0).$$

Subset $Q_m = L_j$ for $W_j \in E$.

Theorem 10 [10]. *Let $h(X) =< a, b, c >$ be an implicant of a logical function $f(X)$. Then any other implicant $h'(X) =< a', b', c' >$ such that $\bar{b}'\bar{b}^t = 0$, $c' \in V_{\bar{b}}$, has a nonempty intersection with $h(X)$.*

The above theorems present the following conditions helpful for spectral analysis:

1) the sufficient condition of the absence of an a-implicant of the dimension equal or greater than a given value (Theorem 5);

2) the sufficient condition of the absence of an a-implicant wiht a given et of bound variables (Theorems 6,7);

3) the necessary and sufficient condition of the existence of an a-implicant with a given set of bound variables (Theorem 8);

4) the necessary condition of factorization (Theorem 9) and implicant intersection (Theorem 10).

Example 4. Let a function $f(X)$, $n = 2$, $k = 3$, be given with the spectrum $f^*(W) = (13, \epsilon^2, \epsilon, 5 + 5\epsilon, \epsilon, 1, 5 + 5\epsilon, 1, \epsilon^2)$. Determine all 2-implicants of the dimention $p = 1$. Since the condition of Theorem 5 does not hold, the search for implicants in $V_{\bar{b}} = (0, 2)$ (Theorem 8) is negative, since

$$\sum_{W_j \in V_{\bar{b}}} | f^*(W_j) |= 15 < ak^n.$$

A similar test for $\bar{b} = (2, 0)$ gives

$$\sum_{W_j \in V_{\bar{b}}} | f^*(W_j) |= 29 > ak^n.$$

Hence, $f(X)$ has a 2-implicant with $X' = \{x_1\}$. For finding the leader of a coset of $V_{\bar{b}}$, the vector $f^*_{\bar{b}}$, $\bar{b} = (2, 0)$, is multiplied by the minor of $\bar{\chi}$, constructed on rows and columns indexed as $\{(0,0), (1,0), (2,0)\}$

$$(13, 5 + 5\epsilon, 5 + 5\epsilon^2) \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{vmatrix} = (18, 0, 18).$$

From the resulting vector it is seen that $f(X)$ has the implicant with $a = 2$, $b = (0, 2)$ and $\Gamma = \{(0,0), (2,0)\}$. With the help of Rule the following analytic representation is obtained:

$$h(X) =< a, X', Q >=< 2, \{x_1\}, \{0, 2\} > \quad \text{or} \quad h(X) = 2x_1(0, 2)x_1(1, 1)^-.$$

5. Recognition of the Boolean function properties using the spectrum

The Boolean functions spectra are studied in detail in [5-7]. Their distinguishing features are the following. 1) The spectrum is an integer-valued vector. 2) The group of characters is a set of the Walsh functions, ordered according to the Hadamard transform. Hence $X_i = -X_i$, $\chi^{-1}(X_i) = \chi(X_i)$, $C^* = \{-1, 1\}$. The coefficients of a spectrum are evaluated by the formula:

$$f^*(W_j) = \sum_{X_i \in X^n} f(X_i)(-1)^{X_i W_j^t} \quad \text{for all} \quad W_j \in W^n.$$

5.1. Condition of the Boolean function monotonicity

The Boolean function is called *monotonous* iff there exists its sum-of-products representation, which contains no variables in the inverse form. From this definition it follows that a monotonic function may have an implicant only on one of the cosets of any subgroup V_b of X^n with the leader $c_{\bar{b}}$. Hence follows the lemma.

Lemma 3 [10]. *Let $h(X) =< 1, b, \bar{b} >$ be an implicant of a monotonous Boolean function $f(X)$. Then*

$$sign f^*(W_j) = sign(-1)^{\hat{W}_j} \quad \textit{for all} \quad W_j \in V_{\bar{b}},$$

where $\hat{W}_j$ is the weight of the vector W_j.

Theorem 11 [10]. *A Boolean function $f(X)$ is monotonous iff there exists a set of subset U such that*

$$\bigcup_{V_b \in U} (V_b \oplus \bar{b}) = f^{-1}(1)$$

and for all $V_b \in U$ meet the following conditions:

$$\operatorname{sign} f^*(W_j) = \operatorname{sign}(-1)^{\hat{W}_j} \quad \textit{for all} \quad W_j \in V_{\bar{b}},$$

$$\sum_{W_j \in V_b} | f^*(W_j) | = 2^n.$$

The algorithm for monotonicity recognition of the Boolean function, using its spectrum, is given in [11].

5.2. Condition of the Boolean function symmetry

A Boolean function $f(X)$ is called *symmetric* with respect to a pair of variables x_m and x_k iff

$$f(X_i) = f(X_i P_{m,k} \oplus c) \quad \text{for all} \quad X_i \in X^n,$$

where $c \in X^n$, $P_{m,k}$ is a permutation matrix of $(n \times n)$ order which permutes x_m and x_k in X_i.

The necessary spectral conditions for the Boolean function to be symmetric with respect to a pair x_m, $x_k \in X^n$ are obtained in [1]. Necessary and sufficient conditions are stated by the following theorem.

Theorem 12 [11]. *A Boolean function $f(X)$ is symmetric with respect to a pair of variable x_m, $x_k \in X^n$, iff there exists a matrix $P_{m,k}$ and a vector $c \in X^n$ such that for all levels $W^{(l)}$, $l = 1, 2, ..., n-1$, the following holds:*

$$f^*(W_j) = (-1)^{cW_i^t} f^*(W_i) \quad \textit{for all} \quad W_j, W_i \in W^{(l)},$$

such that $W_i = W_j P_{m,k}$.

A Boolean function $f(X)$ is *completely symmetric* (or *symmetric*) iff $f(X_i) = f(X_i P \oplus c)$ for all $X_i \in X^n$ any permutation matrix P and any $c \in X^n$. It is well-known that any permutation matrix may be expressed as the product of matrices of $P_{m,k}$ type. Hence follows a corollary from Theorem 12.

Corollary 2. *A Boolean function $f(X)$ is symmetric iff there exists a vector $c \in X^n$ such that for all level $W^{(l)}$, $l = 1, 2, \ldots, n-1$,*

$$f^*(W_j) = (-1)^{cW_q^t} f^*(W_q) \quad \textit{for any pair} \quad W_j, W_q \in W^{(l)}.$$

6. Spectral algoritm for finding an irridundant cover of logical function

The spectral algorithm for irredundant cover constructing has the following features.

1. Unlike the classical two-stage scheme (first: finding the complete set of implicants, second: constructing a cover) the irredundant cover is formed by means of extracting a-implicants by turns, and immediately excluding their spectra from the spectrum of the original function.

2. The search for implicants starts with the maximal values of a and p. The decrementing of values of a and p is stipulated by two reasons. First, the sufficient conditions for the existence of an a-implicant are obtained only for the case when the function has no a'-implicants with $a' > a$ (Theorem 8). The decrement of a is not a disadvantage of the algorithm. It is shown by computer experiments [15] with the great number of algorithms for multi-valued functions minimization that the algorithms where a is decremented are more efficient than those starting with $a = 1$. Second, decrementing the dimention of implicants permits us to reduce significantly the algorithm complexity, since the amount of computations increases with the decrease of implicant, and also due to the fact that the absent conditions (Theorems 7 and 10) for large implicants exclude amounts of subgroups from the subsequent test.

Another advantage of this algorithm is the possibility of extracting all implicants of the same dimention in parallel.

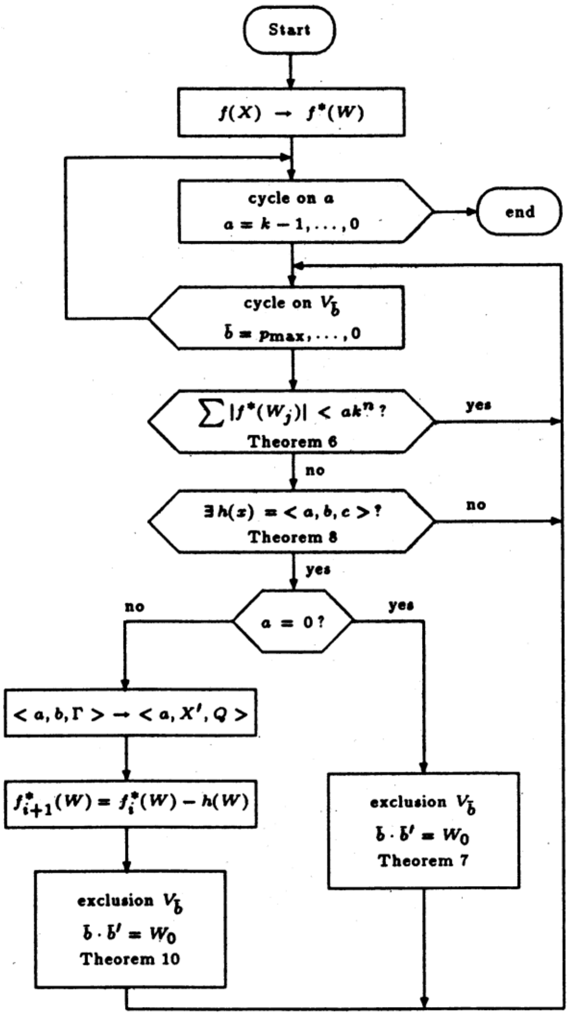

Figure 2

For simplicity the algorithm for obtaining an irredundant cover with orthogonal implicants is presented here (Figure 2). The algorithm has two nested loops: on the external loop all a-implicants for a signal value of a are extracted; on the internal loop one a subgroup $V_{\bar{b}}$ of W^n is tested for having cosets in $f^{-1}(a)$. Let $f^*(W)$ be a current spectrum, $p_{\max}$ the maximal dimention of the implicant under extraction, determined according to Theorem 5. Then six steps of the internal loop are as follows:

1. The condition of the absence of a-implicant on the cosets of V_b

(Theorem 6) is tested. If it does not hold then go to step 2. Otherwise take a new subgroup $V_{b'}$ with $\hat{b} = \hat{b}'$ and go to step 1.

2. For the set $\{f^*(W_j) : W_j \leq \bar{b}\}$ the existent condition of an a-implicant on the cosets of V_b is tested (Theorem 8). If it does not hold, then go to step 1, otherwise - to step 3.

3. If $a > 0$, then the Rule is used to determine the analystic form of the extracted a-implicant, which is then included in the cover. If $a = 0$, then go to step 6.

4. The spectrum $f_i^*(W)$ is modified so that $f_{i+1}^*(W) = f_i^*(W) - h^*(W)$.

5. Subgroups $V_{\bar{b}'}$ such that $\bar{b}'\bar{b}^t = W_0$ are excluded from the subsequent analysis (Theorem 10).

6. Subgroups $V_{\bar{b}'}$ such that $\bar{b}'\bar{b}^t = W_0$ are excluded from the subsequent analysis (Theorem 7).

The algorithms stops when $f_i^*(W)$ is a zero vector.

7. Conclusion

Presenting here some results in studying of logical functions, the authors yearned to show that the implementation of spectral techniques opens a new area in combinatorics. There are two reasons for that. The first is that examining the spectrum of the function we may see the properties which are hidden in its analytic representation. The second reason is based on the possibility of using the most powerful methods for solving combinatorial problems transfers the computational complexity from the procedures of searching fo. best solutions to those which compute spectra and manipulate with vectors and matrices. The prospect of matrix vector fast processors would reduce total computer cosets of spectral techniques and stimulate their development.

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