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Two level iterative type explicit schemes*

P.V. Banushkina

In the paper, we continue the investigation carried out in [1], namely, another approach to design two level explicit schemes for solution of the boundary value parabolic problems is proposed. The paper is founded on the combination of approaches from works [1, 2].

1. The original family of explicit-implicit schemes

Let H_i, $i = 1, 2$, and $H = H_1 \times H_2$ be the real finite-dimensional Hilbert spaces with inner product $(\cdot,\cdot)_i$ and $(\cdot,\cdot)$, respectively. Let $A_{ii} : H_i \to H_i$, $i = 1, 2$, and $A_{12} : H_2 \to H_1$ be linear continuous operators, such that A_{ii} are self-conjugated and positive definite in H_i. Here $A_{12}^T : H_1 \to H_2$ is the operator conjugated to A_{12}. Define the operator $A : H \to H$ as a matrix operator of the following form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}.$$

A is a self-conjugated operator in H. In the sequel, it is supposed that A is positive semi-definite operator. It is well-known [3] that necessary and sufficient condition for that is positive semi-definiteness of the Sura complement $S_{22}(A) = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$:

$$(S_{22}(A)u_2, u_2)_2 \geq 0 \qquad \forall u_2 \in H_2. \tag{1.1}$$

Consider the following difference scheme in the spaces H_1 and H_2:

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} + A_{11}u_1^n + A_{12}u_2^n = f_1^n, \tag{1.2}$$

$$\frac{u_2^{n+1} - u_2^n}{\Delta t} + A_{12}^T u_1^{n+1} + A_{22}u_2^{n+1} = f_2^n, \tag{1.3}$$

where Δt is a time step. Reduce scheme (1.2), (1.3) to a canonical two-layer scheme in H [4]. For this end expand the second equation

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\begin{aligned}
(E+\Delta t A_{22})u_2^{n+1} - u_2^n + \Delta t A_{12}^T u_1^{n+1} - \Delta t f_2^n \\
= (E+\Delta t A_{22})(u_2^{n+1}-u_2^n) + \Delta t A_{12}^T(u_1^{n+1}-u_1^n) + \\
\Delta t A_{22} u_2^n + \Delta t A_{12}^T u_1^n - \Delta t f_2^n.
\end{aligned}
$$

Thus we obtain

$$\Delta t A_{12}^T \frac{u_1^{n+1}-u_1^n}{\Delta t} + (E+\Delta t A_{22})\frac{(u_2^{n+1}-u_2^n)}{\Delta t} + A_{12}^T u_1^n + A_{22}u_2^n = f_2^n. \quad (1.4)$$

Taking into account (1.2) and (1.4) write down the two-layer canonical scheme in H

$$B_1 \frac{u^{n+1}-u^n}{\Delta t} + Au^n = f^n, \quad (1.5)$$

where $f^n = (f_1^n, f_2^n)^T \in H$, $u^n = (u_1^n, u_2^n)^T \in H$, $B_1 : H \to H$ is the matrix operator of the form

$$B_1 = \begin{pmatrix} E_{11} & O_{12} \\ \Delta t A_{12}^T & E_{22} + \Delta t A_{22} \end{pmatrix}. \quad (1.6)$$

Let us investigate the stability of scheme (1.6).

Theorem 1. *Let the condition*

$$\frac{\Delta t}{2}\|A_{11}\|_{(1)} \le 1 \quad (1.7)$$

be satisfied, then for any $u \in H$ the inequality $(B_1 u, u) \ge 0.5\Delta t(Au, u)$ holds.

Proof. Verify positive semi-definiteness of the operator

$$D = B_1 - \frac{\Delta t}{2}A.$$

For this purpose represent D in the following form: $D = D^{(0)} + D^{(1)}$, where

$$D^{(0)} = \begin{pmatrix} E_{11} - \frac{\Delta t}{2}A_{11} & O_{12} \\ O_{21} & E_{22} + \frac{\Delta t}{2}A_{22} \end{pmatrix}, \qquad D^{(1)} = \begin{pmatrix} O_{11} & -\frac{\Delta t}{2}A_{12} \\ \frac{\Delta t}{2}A_{12}^T & O_{22} \end{pmatrix}.$$

Positive semi-definiteness of the operator $D^{(0)}$ follows from condition (1.7). Verify positive semi-definiteness of the operator $D^{(1)}$. Since $D^{(1)}$ is a skew-symmetric matrix, then $(D^{(1)}u, u) = 0$. Thus $D = D^{(0)} + D^{(1)} \ge 0$. □

2. Stability of iterative type schemes

Rewrite equation (1.3) in another form

$$(E_{22} + \Delta t A_{22}) u_2^{n+1} = u_2^n - \Delta t A_{12}^T u_1^{n+1} + \Delta t f_2^n \equiv g_2^n. \qquad (2.1)$$

Our aim is to replace problem (2.1) by some other problem with preserving approximation and stability of the method. Let C be some square $K \times K$ matrix, where K is the dimensionality of H. Introduce the vector v as

$$v = Cu^n + \Delta t f^n, \qquad (2.2)$$

where $v^T = (v_1^T, v_2^T)$. Seek the vector u_2^{n+1} in the following form:

$$u_2^{n+1} = Lv_2 + Qg_2^n, \qquad (2.3)$$

where L and Q are $K_2 \times K_2$ matrices, where K_2 is the dimensionality of H_2. The vector u_1^{n+1} as earlier is determined from (1.2). Further it is supposed that Q is definite matrix. From (1.2), (2.1), and (2.3) it is easy to obtain the system

$$\begin{aligned} &\frac{u_1^{n+1} - u_1^n}{\Delta t} + A_{11} u_1^n + A_{12} u_2^n = f_1^n, \\ &\Delta t A_{12}^T \frac{u_1^{n+1} - u_1^n}{\Delta t} + Q^{-1} \frac{u_2^{n+1} - u_2^n}{\Delta t} + \\ &\qquad A_{12}^T u_1^n + \frac{1}{\Delta t}(Q^{-1} - E_{22}) u_2^n - \frac{1}{\Delta t} Q^{-1} L v_2 = f_2^n, \end{aligned} \qquad (2.4)$$

or

$$B \frac{u^{n+1} - u^n}{\Delta t} + Ru^n - \frac{1}{\Delta t} \begin{pmatrix} O \\ Q^{-1} L v_2 \end{pmatrix} = f^n, \qquad (2.5)$$

where

$$B = \begin{pmatrix} E_{11} & O_{12} \\ \Delta t A_{12}^T & Q^{-1} \end{pmatrix}, \qquad R = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & \frac{1}{\Delta t}(Q^{-1} - E_{22}) \end{pmatrix}.$$

In the sequel, we will use the matrices D_0, D_1, and D of $K \times K$ of the form

$$D_0 = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & Q^{-1} L \end{pmatrix}, \quad D_1 = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & Q^{-1} - E_{22} - \Delta t A_{22} \end{pmatrix}, \quad D = E + D_0.$$

Note that

$$R = A + \frac{1}{\Delta t} D_1, \qquad \begin{pmatrix} O \\ Q^{-1} L v_2 \end{pmatrix} = D_0 (Cu^n + \Delta t f^n).$$

Using introduced notation rewrite (2.5):

$$B\frac{u^{n+1}-u^n}{\Delta t}+\left(A+\frac{1}{\Delta t}(D_1-D_0C)\right)u^n=Df^n. \qquad (2.6)$$

Define the matrix C and relation between matrices D_0 and D_1 by the following equalities:

$$C=E-\Delta tA, \qquad D_0=D_1.$$

The first of these equalities means that according to (2.2) the vector v is obtained as a result of using the explicit Euler scheme, namely, this equality provides approximation in spite of the choice of the matrix Q. Second equality means that

$$Q^{-1}L=Q^{-1}-E_{22}-\Delta tA_{22}, \qquad (2.7)$$

from where

$$L=E_{22}-Q(E_{22}+\Delta tA_{22}). \qquad (2.8)$$

Then scheme (2.6) takes the form

$$B\frac{u^{n+1}-u^n}{\Delta t}+DAu^n=Df^n.$$

Require that the matrix D is definite. Noting that $B=B_1+D_1=B_1+D-E$, we obtain

$$(E+D^{-1}(B_1-E))\frac{u^{n+1}-u^n}{\Delta t}+Au^n=f^n. \qquad (2.9)$$

Denote $\tilde{B}=(E+D^{-1}(B_1-E))$ and investigate the stability of the scheme

$$\tilde{B}\frac{u^{n+1}-u^n}{\Delta t}+Au^n=f^n \qquad (2.10)$$

with respect to initial data supposing that $f^n=0$ is a null element of the space H.

Theorem 2. *Let conditions*

$$\Delta t\|A_{11}\|_{(1)} \le 1, \qquad (2.11)$$

$$\|Q^{-1}L\|_{(2)} \le \frac{1}{\sqrt{2}} \qquad (2.12)$$

be satisfied. Then for any $u\in H$ the inequality

$$(\tilde{B}u,u)\ge\frac{\Delta t}{2}(Au,u), \qquad (2.13)$$

holds and the estimate

$$(Au^n,u^n)\le(Au^0,u^0), \quad n=1,2,\dots, \qquad (2.14)$$

takes place.

Proof. Since the space H is real so for any $u \in H$ $(\tilde{B}u, u) = (\tilde{B}_0 u, u)$, where $\tilde{B}_0 = \frac{1}{2}(\tilde{B} + \tilde{B}^T)$, where $\tilde{B}^T$ is the operator conjugated to $\tilde{B}$, i.e., B_0 is a self-conjugated operator. Thus, for the proof of (2.13) it is sufficient to verify positive semi-definiteness of the operator $\tilde{D} = \tilde{B}_0 - \frac{1}{2}\Delta t A$. Write down the operators $\tilde{B}$ and $\tilde{B}_0$:

$$\tilde{B} = \begin{pmatrix} E_{11} & O_{12} \\ \Delta t(Q^{-1} - \Delta t A_{22})^{-1} A_{12}^T & \Delta t(Q^{-1} - \Delta t A_{22})^{-1} A_{22} + E_{22} \end{pmatrix},$$

$$\tilde{B}_0 = \begin{pmatrix} E_{11} & \frac{\Delta t}{2} A_{12}^T (Q^{-1} - \Delta t A_{22})^{-1} \\ \frac{\Delta t}{2}(Q^{-1} - \Delta t A_{22})^{-1} A_{12}^T & \Delta t(Q^{-1} - \Delta t A_{22})^{-1} A_{22} + E_{22} \end{pmatrix}.$$

Then

$$\tilde{D} = \begin{pmatrix} E_{11} - \frac{\Delta t}{2} A_{11} & \frac{\Delta t}{2} A_{12}^T (Q^{-1}L + E_{22})^{-1} - \frac{\Delta t}{2} A_{12} \\ \frac{\Delta t}{2}(Q^{-1}L + E_{22})^{-1} A_{12}^T - \frac{\Delta t}{2} A_{12}^T & \Delta t(Q^{-1}L + E_{22})^{-1} A_{22} + E_{22} - \frac{\Delta t}{2} A_{22} \end{pmatrix}.$$

Here we used an equality $(Q^{-1} - \Delta t A_{22})^{-1} = (Q^{-1}L + E_{22})^{-1}$, which follows from (2.7). Represent $\tilde{D}$ in the form $\tilde{D} = D' + D''$

$$D' = \begin{pmatrix} E_{11} - \Delta t A_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix},$$

$$D'' = \frac{\Delta t}{2} \begin{pmatrix} A_{11} & A_{12} G \\ G A_{12}^T & 2 G A_{22} + \frac{2}{\Delta t} E_{22} + A_{22} \end{pmatrix},$$

where $G = (Q^{-1}L + E_{22})^{-1} - E_{22}$. From condition (2.11) positive semi-definiteness of the operator D' follows immediately. Furthermore, since A_{11} is a positive definite operator, then for positive semi-definiteness of D'' it is sufficient to verify positive semi-definiteness of the Sura complement [3]

$$S_{22}(D'') = 2GA_{22} + \frac{2}{\Delta t} E_{22} + A_{22} - G A_{12}^T A_{11}^{-1} A_{12} G.$$

Let Q and L be matrix polynomials with respect to A_{22}. In the sequel it will be shown how to construct them. Then Q, L, and G are self-conjugated and transposing with A_{22} operators. As follows from (1.1), it is sufficient to verify positive semi-definiteness of the operator

$$\widetilde{S}_{22} = 2GA_{22} + A_{22} - GA_{22}G = A_{22}(2G + E_{22} - G^2). \tag{2.15}$$

Since $A_{22} \geq 0$ hence for positive semi-definiteness of $\widetilde{S}_{22}$ it is sufficient

$$2G + E_{22} - G^2 \geq 0$$

or

$$G^2 - 2G - E_{22} \leq 0. \tag{2.16}$$

Since G is self-conjugated operator, its eigenvalues are real, and eigenvectors form the complete orthogonal system in H_2 [3]. If μ is an eigenvalue of G, then $\lambda = \mu^2 - 2\mu - 1$ is an eigenvalue of the operator $P(G) = G^2 - 2G - E_{22}$ Thus to satisfy (2.16) it is necessary $(1 - \sqrt{2}) \le \lambda \le (1 + \sqrt{2})$. Hence

$$(1 - \sqrt{2})E_{22} \le (Q^{-1}L + E_{22})^{-1} - E_{22} \le (1 + \sqrt{2})E_{22}.$$

It is valid under the condition

$$\|Q^{-1}L\|_{(2)} \le \frac{1}{\sqrt{2}}.$$

Hence $\widetilde{S}_{22} \ge 0$, from where positive semi-definiteness of the operator D'' follows. Thus inequality (2.13) is proved. And finally (2.14) is immediate consequence of (2.13) and [4, Theorem 1, p. 303]. □

Now investigate stability of scheme (2.10) with respect to the right-hand side. It is known [5, Theorem 5, p.172] that if to consider scheme (1.11) with uniform initial data and to require an equality

$$\tilde{B} \ge \frac{1}{2}(\varepsilon E + \Delta t A) \quad \forall u \in H, \quad \varepsilon > 0 \tag{2.17}$$

to hold, then an estimate

$$(Au^m, u^m) \le \frac{1}{\varepsilon} \sum_{n=0}^{m-1} \Delta t \|f^n\|^2 \tag{2.18}$$

takes place.

Theorem 3. *Let conditions* (2.12) *and*

$$\Delta t \|A_{11}\|_{(1)} \le 1 - \varepsilon, \quad \varepsilon \in (0, 1), \tag{2.19}$$

be satisfied. Then (2.17) *holds and in the case of the uniform initial data the inequality* (2.18) *takes place.*

Proof. Similarly to the proof of Theorem 2.1 represent the operator $\tilde{D} = \tilde{B}_0 - \frac{1}{2}(\varepsilon E + \Delta t A)$ in the form $\tilde{D} = D' + D''$ where

$$D' = \begin{pmatrix} (1 - \frac{\varepsilon}{2})E_{11} - \frac{1+\omega}{2}\Delta t A_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix},$$

$$D'' = \frac{\Delta t}{2} \begin{pmatrix} \omega A_{11} & A_{12}G \\ G A_{12}^T & 2GA_{22} + \frac{2}{\Delta t}E_{22} + A_{22} - \frac{\varepsilon}{\Delta t}E_{22} \end{pmatrix},$$

$\omega > 0$ is a number. Let $\omega = 1/(1 - \varepsilon)$. Then the equality

$$(1-\frac{\varepsilon}{2})E_{11}-\frac{1+\omega}{2}\Delta t A_{11}=\left(1-\frac{\varepsilon}{2}\right)\left(E_{11}-\frac{\Delta t}{1-\varepsilon}A_{11}\right)$$

takes place and $D' \geq 0$ under condition (2.19). Verify that $D'' \geq 0$. For that it is sufficient to show that the Sura complement

$$\begin{aligned} S_{22}(D'') &= 2GA_{22}+\frac{2}{\Delta t}E_{22}+A_{22}-\frac{\varepsilon}{2}E_{22}- \\ & GA_{12}^T(1-\varepsilon)A_{11}^{-1}A_{12}G+(1-\varepsilon)GA_{22}G \end{aligned}$$

is positive semi-definite. Taking into account (1.1) it is sufficient to show positive semi-definiteness of the operator

$$\widetilde{S}_{22}=2GA_{22}+A_{22}-GA_{22}G=A_{22}(2G+E_{22}-G^2)$$

In Theorem 2.1, it was proved that $\widetilde{S}_{22} \geq 0$ under condition (2.12). Thus $D'' \geq 0$, and inequality (2.17) is valid. The estimate (2.18) follows immediately from (2.17) and [5, Theorem 5, p. 172]. □

3. Choice of scheme parameters

In this section, consider the way of construction of the matrices L and Q. Let the vector u_2^{n+1} be calculated as a result of s steps of iterative process

$$\begin{aligned} & v_2^0 = v_2, \\ & \frac{v_2^k - v_2^{k-1}}{\tau_k} + (E_{22}+\Delta t A_{22})v_2^{k-1} - g_2^n = 0, \quad k=1,\dots,s, \\ & u_2^{n+1} = v_2^s, \end{aligned} \tag{3.1}$$

where v_2 and g_2^n are given according to (2.2) and (2.1). Denote

$$Z=(E_{22}+\Delta t A_{22}). \tag{3.2}$$

Expand (3.1) step by step

$$\begin{aligned} v_2^1 &= (E_{22}-\tau_1 Z)v_2^0+\tau_1 g_2^n, \\ v_2^2 &= (E_{22}-\tau_2 Z)v_2^1+\tau_2 g_2^n \\ &= (E_{22}-\tau_2 Z)(E_{22}-\tau_1 Z)v_2^0+\tau_1(E_{22}-\tau_2 Z)g_2^n+\tau_2 g_2^n, \\ & \dots\dots\dots, \\ v_2^s &= \left(\prod_{k=1}^{s}(E_{22}-\tau_k Z)\right)v_2^0+(\tau_s E_{22}+\tau_{s-1}(E_{22}-\tau_s Z)+ \\ & \tau_{s-2}(E_{22}-\tau_s Z)(E_{22}-\tau_{s-1}Z)+\dots+ \\ & \tau_1(E_{22}-\tau_s Z)\times\dots\times(E_{22}-\tau_2 Z))g_2^n. \end{aligned} \tag{3.3}$$

Write down an identity

$$\tau_k E_{22} = Z^{-1}(\tau_k Z - E_{22} + E_{22}) = Z^{-1} - Z^{-1}(E_{22} - \tau_k Z) \qquad (3.4).$$

Substituting (3.4) to (3.3), we obtain

$$v_2^s = \Big(\prod_{k=1}^{s}(E_{22} - \tau_k Z)\Big)v_2 + Z^{-1}\Big(E_{22} - \prod_{k=1}^{s}(E_{22} - \tau_k Z)\Big)g_2^n.$$

Then

$$u_2^{n+1} = L_s(Z)v_2 + Q_{s-1}(Z)g_2^n, \qquad (3.5)$$

where

$$L_s(Z) = \prod_{k=1}^{s}(E_{22} - \tau_k Z), \quad Q_{s-1}(Z) = Z^{-1}(E_{22} - L_s(Z)). \qquad (3.6)$$

The matrices $L_s(Z)$ and $Q_{s-1}(Z)$ are given by the polynomials $L_s(\lambda)$, and $Q_{s-1}(\lambda)$ ($Q_{s-1}(\lambda)$ is a polynomial because $L_s(0) = 1$). In the capacity of L and Q, let us take the matrix polynomials $L_s(Z)$ and $Q_{s-1}(Z)$. It is possible to do this since equality (3.5) corresponds to representation (2.3) and second part of equality (3.6) easy transforms to condition (2.7). Notice that for eigenvalues of the matrix $Z = E_{22} + \Delta t A_{22}$ it is held $\lambda \in [1, \beta]$ (A_{22} and $Z(A_{22})$ are self-conjugated operators, hence $\lambda(Z) = 1 + \Delta t \mu(A_{22})$), where the upper limit of the eigenvalue β can be easy calculated with the use of the Gershgorin circles. Then to satisfy the condition of stability (2.11) it is sufficient

$$-\frac{1}{\sqrt{2}} \le \frac{L_s(\lambda)}{Q_{s-1}(\lambda)} \le \frac{1}{\sqrt{2}}, \quad \lambda \in [1, \beta]. \qquad (3.7)$$

There arises the problem to find the polynomial of the form

$$L_s(\lambda) = \prod_{k=1}^{s}(1 - \tau_k \lambda) \qquad (3.8)$$

of minimal degree such that inequality (3.7) holds, where $Q_{s-1}(\lambda) = (1 - L_s(\lambda))/\lambda$. Below it will be pointed a polynomial which gives asymptotically the best result (when β is big) than the Chebyshev polynomial of first kind reducing to segment $[1, \beta]$. Notice that for implicit schemes we are interested in the case when $\tau \gg h^2$, i.e., the case of big $\beta = O(\tau/h^2)$. We will consider the polynomial introduced in [6] and used in a number of works. Consider a function

$$l_s(x) = \frac{1 - T_{s+1}(1 - 2x)}{2(s+1)^2 x}, \quad x \in [0, 1], \qquad (3.9)$$

where $T_{s+1}(y)$, $y \in [-1, 1]$, is the Chebyshev polynomial of first kind of the degree $s+1$. The function $l_s(x)$ is the polynomial of the degree s of the form

$(1 - x/x_1) \times \cdots \times (1 - x/x_s)$ with the roots $x_k = \sin^2 \frac{k\pi}{s+1} \in [0, 1]$ where $0 \leq l_s(x) \leq 1$, $l_s(0) = 1$. Further consider the function $q_{s-1}(x) = (1 - l_s(x))/x$. It is easy to note that $q_{s-1}(x)$ is the polynomial of the degree $s - 1$ and, furthermore, $q_{s-1}(x) \geq 1 - \frac{1}{(s+1)^2} > 0$ when $x \in [0, 1]$ [1]. Let $L_s(\lambda) = l_s(\lambda/\beta)$. According to the second equality of (3.6) $Q_{s-1}(\lambda) = \beta^{-1} q_{s-1}(\lambda/\beta)$. Notice that the matrix $L_s(Z)$ is positive semi-definite, the matrix $Q_{s-1}(Z)$ is positive definite and these matrices are permutable. Then there exists positive semi-definite matrix $R_s(Z) = Q_{s-1}^{-1}(Z) L_s(Z)$, i.e., corresponding rational function $R_s(\lambda) \geq 0$ in $[1, \beta]$, and hence the left-hand side of inequalities (3.7) holds. Clear up under what conditions the right-hand side of inequalities (3.7) takes place. Consider the function $r_s(x) = l_s(x)/q_{s-1}(x)$. We need the following

Lemma. *Let* $s \geq \left[\sqrt{(\sqrt{2} + 1)\beta}\right]$, *where* $[\cdot]$ *is an integer part of a number. Then*

$$r_s(x) \leq \frac{1}{\sqrt{2}\beta}, \quad x \in [1/\beta, 1].$$

Proof. Let $x = \sin^2 \alpha$, $\alpha \in [\alpha_0, \pi/2]$, where $\sin^2 \alpha_0 = 1/\beta$. Direct calculations give

$$r_s(x) = \frac{\sin^2 \alpha \sin^2(s+1)\alpha}{(s+1)^2 \sin^2 \alpha - \sin^2(s+1)\alpha}.$$

From condition of the lemma there follows $(s+1)^2 \geq (\sqrt{2} + 1)\beta$. Then

$$(s+1)^2 \sin^2 \alpha - \sin^2(s+1)\alpha \geq (s+1)^2 x - 1 \geq \frac{(s+1)^2}{\beta} - 1 \geq \sqrt{2}, \quad (3.10)$$

hence

$$r_s(x) \leq \frac{x}{(s+1)^2 x - 1}.$$

Notice that this inequality transforms to the equality, when $\sin^2(s+1)\alpha = 1$. Right-hand side of the above inequality decreases with respect to x and has maximum, when $x = 1/\beta$. Then according to (3.10)

$$r_s(x) \leq \frac{1/\beta}{(s+1)^2/\beta - 1} \leq \frac{1}{\sqrt{2}\beta}. \qquad \square$$

Since

$$\frac{L_s(\lambda)}{Q_{s-1}(\lambda)} = \frac{l_s(\lambda/\beta)}{q_{s-1}(\lambda/\beta)}\beta = r_s(\lambda/\beta)\beta,$$

hence

$$\frac{L_s(\lambda)}{Q_{s-1}(\lambda)} \leq \frac{1}{\sqrt{2}}.$$

Thus, inequality (3.7) holds that corresponds to holding the condition (2.12).

References

[1] Laevsky Yu.M., Banushkina P.V. The compound explicit schemes // Siberian J. of Numer. Mathematics / Sib. Branch of Russ. Acad. of Sci. – Novosibirsk, 2000. – Vol. 3, № 2. – P. 165–168.

[2] Laevsky Yu.M. On domain decomposition methods for grid parabolic problems // Russ. J. Numer. Anal. Math. Modeling. – The Netherlands, 1998. – Vol. 13, № 5. – P. 389–403.

[3] Voevodin V.V., Kuznetsov Yu.A. Matrices and computations. – Moscow: Nauka, 1984 (in Russian).

[4] Samarsky A.A. Introduction in difference schemes theory. – Moscow: Nauka, 1971 (in Russian).

[5] Samarsky A.A., Gulin A.V. The stability of difference schemes. – Moscow: Nauka, 1973 (in Russian).

[6] Lanczos C. Chebyshev polynomials in the solutions of Large-scale linear systems // Proc. Assoc. Comput. Mach. – Toronto, 1953. – P. 124–133.