Bull. Nov. Comp. Center, Num. Anal., 9 (2000), 7–17

Application of Monte Carlo method to determination of telegraph equation coefficients

Irina Belinskaya

In this paper, a probabilistic approach to solving some inverse and direct problems to the telegraph equation is presented. The multidimensional cases and specific features of the inverse problems, where it is commonly required to determine only the functional of solution, make the application of Monte Carlo method reasonable.

1. Introduction. The statistical simulation methods are developed to solve different problems of mathematical physics. It should be noted in this connection that by now the main attention was paid to the developing of Monte Carlo algorithms for the direct problems of mathematical physics while a number of complicated inverse problems remained unsolved.

The Monte Carlo method has a number of advantages over deterministic methods, namely, we can estimate certain functional without determination of the general solution which is essential while solving the inverse problems; to estimate error during the calculation; the insignificant dependence upon the dimension of a problem.

2. Posing the inverse problem to telegraph equation. First, let us describe the type of physical problems we are going to solve, giving an example arising in signal processing. The relationship between the shape of the vocal tract and its acoustical properties is of importance to speech research.

We apply Monte Carlo based on reducing the original problem to the integral Volterra equations of the second kind.

As the example of the 1D inverse problem let us consider the determining the shape of the vocal tract according to the acoustical measurements. As we know, this problem is ill-posed.

Let us consider the system of equations

$$p_x(x,t) = -L(x)u_t(x,t), \tag{1}$$

$$u_x(x,t) = -C(x)p_t(x,t). \tag{2}$$

Here $p(x,t)$ is pressure and $u(x,t)$ is volume velocity in a lossless vocal tract. If pressure is identified with voltage and volume velocity with current, then (1), (2) are telegraph equations for a lossless transmission line whose inductance and capacitance are such that $L(x) = 1/C(x) = A(x)$. This system can be readily written down as the equation of second order

$$\frac{\partial}{\partial x}A(x)\frac{\partial p(x,t)}{\partial x} = A(x)\frac{\partial^2 p(x,t)}{\partial t^2}.$$

We consider the problem of determining the function $A(x)$ from the above relations and the given impulse response function $h(t)$ to be inverse. In work [4], the solution of this inverse problem is presented by deriving an integral equation in relation of the area function to the impulse response $A(x)$:

$$f(a,t) + \frac{1}{2}\int_{-a}^{a} h(|t-s|)f(a,s)\,ds = 1, \quad |t| \le a. \tag{3}$$

There is unique dependence between the solution to the integral equation (3) and the function $A(x)$:

$$f(a,a) = \sqrt{A(a)}.$$

Also, sufficient and necessary conditions hold.

In this problem, we assume $f(a,t) = u(0,t)$. Suppose the vocal tract be quit at $t = t_0$, and the input of the volume velocity be applied at lips. At $t = t_0 + a$ the tract is undisturbed for all $x \ge 0$, because we assumed the velocity of sound to be equal to 1.

We propose to apply the Monte Carlo approach to solution of the integral equation (3). In this connection, we have to check the conditions of Monte Carlo Applicability, i.e., convergence of the Neumann series to the solution of this equation. Here we have the required conditions because of the Volterra type of the integral equation (3).

According to Monte Carlo theory [3], let us seek the solution to (3) in the space L_2 with appropriate norms.

As $h(|t-s|)$ is an even function, then its Fourier series can be written down with the cosines

$$h(x') = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x'}{2a}, \quad t \in (-a,a), \quad x' = |t-s|.$$

Here we take N_1 first members of the Fourier transform with the accuracy required which can be estimated with the help of Bernstein's lemma.

To find the local solution to (3) let us present this equation in the form $f = (k_x, f) + 1$, where $k_x(t,s) = h'(|t-s|)/2$ is the kernel of (3).

We construct the unbiased collision estimator ξ such that

$$M\xi = f(a) = M\sum_{n=0}^{N} Q_n k(x_n, a) + 1, \quad f(a) = f(a,a).$$

This means that we construct the Markov chain $x_0, x_1, \dots, x_N$ with random weights $Q_0, \dots, Q_N$ given by the formulas

$$Q_0 = \frac{1}{\pi(x_0)}, \qquad Q_n = Q_{n-1}\frac{k(x_{n-1}, x_n)}{p(x_{n-1}, x_n)(1 - g(x_{n-1}))}.$$

Here N is a random number of the final statement before breaking the chain, $k(x_{n-1}, x_n)$ is the kernel of (3), $g(x_n)$ is the probability of breaking the Markov chain in the statement x_n, $\pi(x_0)$ is the initial distribution density of the first statement $x_0 = -a$ of the Markov chain, $p(x_{n-1}, x_n)$ is the density of transition between any two neighbouring statements:

$$p(x_{n-1}, x_n) = \sum_{i=1}^{N_1} |b_i(a)| \cdot \left| \cos \frac{k\pi(x_{n-1}, x_n)}{2T} \right|,$$

or

$$p(x_{n-1}, x_n) = \sum_{i=1}^{N_1} |b_i| \theta\left(\cos \frac{k\pi(x_{n-1}, x_n)}{2T} \right),$$

where b_i are chosen according to $\int_{-a}^{a} p(t, s)\, ds = 1$.

Due to the Volterra type of integral equation (3) we have fulfilled the condition $\|K_1\| < 1$ which is sufficient for the convergence of Monte Carlo algorithm. Here K_1 is the integral operator with the kernel $k_1(t, s) = |k(t, s)|$.

Note that with these Markov chain characteristics we obtain the unbiased estimation [3] as well as finiteness of the mean number of the statements $M(N)$. This means our Markov chain will break after the finite number of transitions with probability 1.

As we may present the function $p(t, s)$ in the form

$$p = \sum_{i=1}^{N_1} p_i \left| \cos \frac{k\pi(|t - s|)}{2T} \right|, \qquad p_i \geq 0, \qquad \sum p_i = 1,$$

then we apply the well-known [3] superposition method to construct ξ from the formula $\int_{-a}^{\xi} p(t, s)\, ds = \eta$. Here $\eta \in U_{(0,1)}$ is the uniformly distributed in the interval $(0, 1)$ random value.

It is known [3], that if Neuman series converges to the variance equation

$$\chi(x) = \int_{-\infty}^{x} \frac{K^2(t, s)\chi(s)}{p(t, s)}\, ds + \frac{1}{m^2\pi(x)} \equiv K_p\chi(x) + \frac{1}{m^2\pi(x)},$$

the variance of the main estimation is finite. We also have this condition fulfilled because of the Volterra type of the above equation. The numerical results for the functions $f(t) = (1 + bt)^{-1}$ and $f(t) = 3 - \cos^2(bt)$, $b = \text{const}$ are presented in Figures 1 and 2.

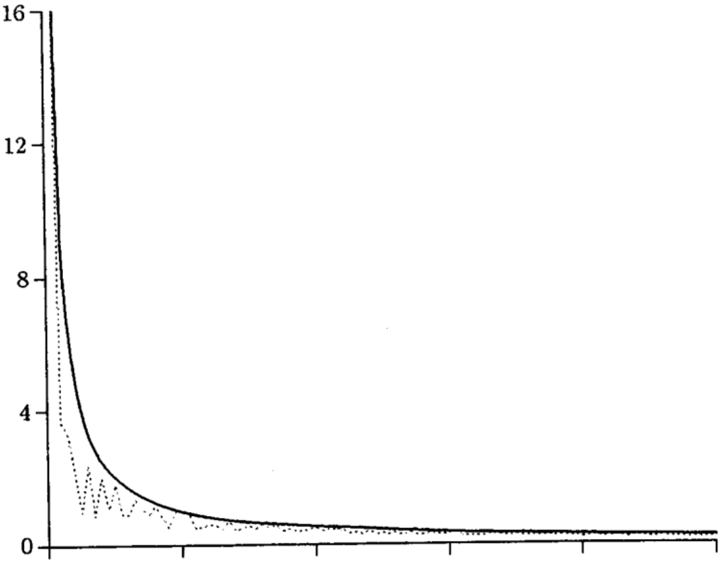

Figure 1

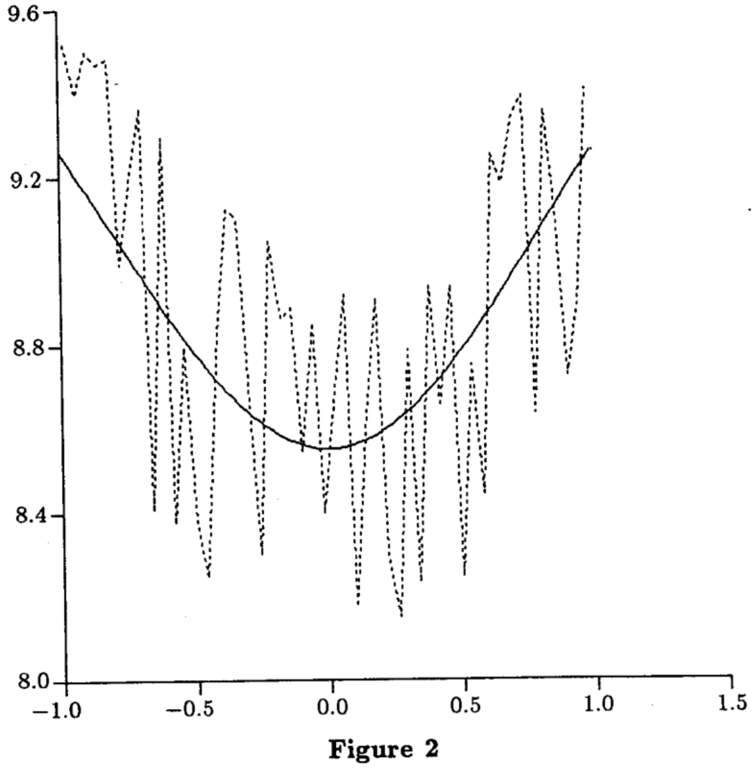

Figure 2

3. Numerical comparison. To prove the effectiveness of the method, it is important to compare it with other methods solving the same task. So, the numerical trials were made with the help of the finite element method and the inversion of the difference scheme method.

4. Finite element method. According to [7], we seek the solution to equation (3) in the space $L_2(-a,a)$ with some basis $\{\varphi_i\}$ in the form

$$\varphi_h = \sum_{i=0}^{N} u_i \varphi_i.$$

If an integral operator of (3) is positive definite, then we may approximate the solution to this equation by a linear combination of the piecewise linear functions of order $O(h^2)$.

5. The difference scheme inversion method was proposed and proved in [1]. The original difference problem is reduced to the appropriate finite difference problem. For this problem, a difference scheme of the nonlinear algebraic equations of order $O(h)$ is constructed. Its solution is considered to be an approximate solution to the original problem. This scheme has weak stability.

We have studied several model problems by the three above mentioned methods. In each case, a single test was used for comparison. Dividing the interval $(-T, T)$ into 2000 points, we obtain satisfactory (about 5–10 percent) accuracy for a relatively small x (here x is dimensionless variable named "depth" characterizing the method) by using finite element method. As x further increases, satisfactory accuracy is not attained. There is also a considerable growth of storage with the sampling.

In order to obtain the accuracy of 0.01–10% by Monte Carlo method, the averaging over 200,000 trajectories was used. The algorithm variance linearly increases as x increases, the average number of steps per trajectory increasing similarly. With x increasing, the estimate accuracy decreases slowly and remains satisfactory for relatively large x (100–150). The storage size for the Monte Carlo algorithm is comparatively small.

To attain the appropriate accuracy by the difference scheme inversion method a grid of 200×200 size is needed. In this case, the solution is accurately estimated only for small x from the interval $(0, 1)$.

There is a comparatively slow growth of variance, and the average number $M(N)$ of Markov chain statements in the Monte Carlo algorithm. In addition Monte Carlo method turns to be the most speedy among the three methods. The illustration of the comparative numerical results are shown in Figure 3. Tests for the function $f(t) = b + \sin ct$, $b, c = \text{const}$ are presented.

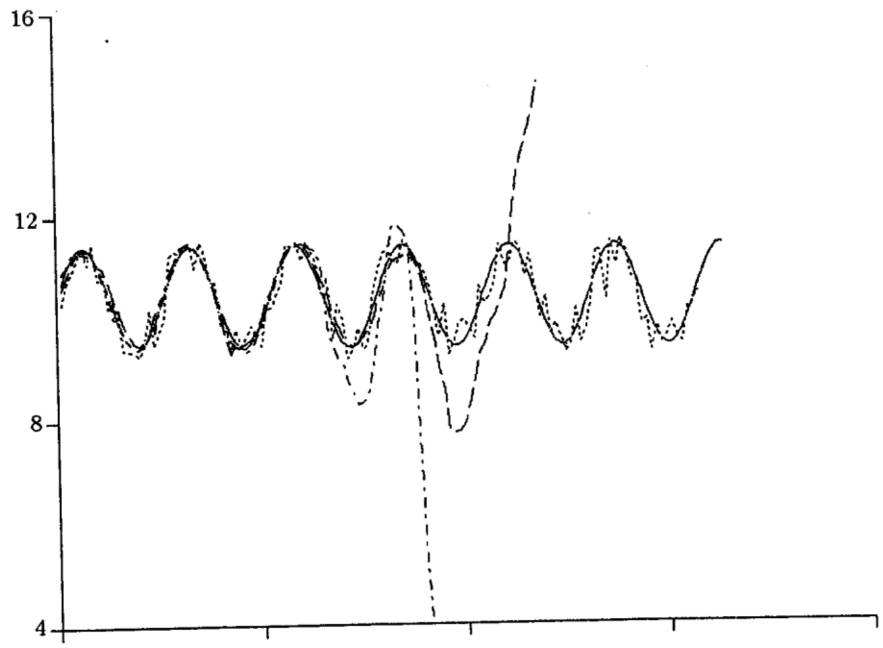

Figure 3

6. The 2D inverse problem to telegraph equation. The main object of investigation is the problem of determination of the coefficient $q(x, y)$ from the equation

$$L_q u \equiv v_{tt} + \Delta_{xy} v + qv = 0. \quad (4)$$

Here we know the solution of an infinite set of the Cauchy problems in $x = 0$. Note that q may depend on two or more variables. The case considered here, $q = q(x, y)$ may be extended to n-dimensional case without any serious changes. Let us do the exact formulations. Consider the set of the Cauchy problems for equation (4) depending on y_0 with the initial data

$$v|_{t=0} = 0, \quad v_t|_{t=0} = \delta(x)\delta(y - y_0). \quad (5)$$

Here δ is the Dirac δ-function, R is a set of real numbers, $R_+ = (t \in R : t > 0)$. Assume $\forall y \in R$, $\forall y_0 \in R$ the trace of the generalized solution $\bar{v}(x, y, t, y_0)$ of the Cauchy problem (1), (2) is given:

$$v|_{x=0} = g(y, t, y_0), \quad y, t, y_0 \in R. \quad (6)$$

We assume that $q(x, y)$ and $v(x, y, t, y_0)$ are respectively even with respect to x, and therefore,

$$v_x|_{x=0} = 0, \quad x, y, y_0 \in R. \quad (7)$$

The inverse problem is to determine the function $q(x, y)$ from relations (4)–(7).

With the appropriate renaming [6] there is constructed N*-approximation* of inverse problem in the form of the following matrix system of the 1D inverse problems

$$U_{tt} = U_{xx} + B(x)U, \qquad x \in R, \quad t \in R^+, \tag{8}$$

$$U|_{t=0} = 0, \qquad U_t|_{t=0} = E\delta(x), \tag{9}$$

$$U_x|_{x=0} = 0, \qquad U|_{x=0} = F(t), \quad t \in R^+, \tag{10}$$

where

$$B = -K + A, \quad K_{mj} = m^2\delta_{mj}, \quad E_{mj} = \delta_{mj}, \quad F_{mj}(t) = f_m^j(t),$$
$$A_{mj}(x) = \theta(N - |m-j|)a_{m-j}(x), \quad 0_{mj} = 0, \quad m, j = -N,_n +1, \ldots, N.$$

This differential system is reduced to the equivalent matrix system of the integral equation in the form

$$W(x,t) + \int_{-x}^{x} W(x,s)F'(t-s)\,ds = -\frac{1}{2}[F'(t+x) + F'(t-x)], \quad |t| < x. \tag{11}$$

Here $F'(t)$ is the derivation of the even continuation of $F(t)$ on $t < 0$.

There is unique dependence between $W(x,t)$ and matrix-coefficient $A(x)$ of the system (8)–(10): $A(x) = -K + 4W_x(x,x)$. It is sufficient to define only one column of $A(x)$ because it is of the Toeplitz type.

Equation (11) is the Volterra equation system of the second kind, depending on parameter x. In addition, we are interested only in obtaining the local estimation of the solution (11) at the point $t = x$. In this connection, it is convenient to use Monte Carlo method.

The theory of weight Monte Carlo methods [8] has been developed for estimating the system of Fredholm's equations of the second kind. We consider the solution to (11) in the space L_∞ of the matrix-valued functions with the norm $\|\Phi\| = \operatorname{vrai\,sup}_{x,i} \sum_j |\Phi_{ij}(x)|$. Let us rewrite equation (11) in a more convenient form, omitting the dependence $W(x,t)$ on x, as in this case x is a parameter:

$$W(t) = H(2t) + \int_{-x}^{x} W(s)F'(t-s)\,ds. \tag{12}$$

Here $\frac{\partial}{\partial t}F(t)$ is the derivation of the odd extension on $t < 0$ of the function $F(t)$ at the points of its continuity. In the operator form, equation (12) is written as $W = H + KW$, where K is a matrix integral operator with the components

$$k_{ij}(t-s)\varphi(s) = -\frac{1}{2}\int_{-x}^{x} \sum_{k=1}^{n} f'_{ik}(0)f'_{kj}(t-s)\varphi(s)\,ds.$$

Because $f'_{ij}(t)$ are even functions, on the interval $[0, 2T]$ we may present them in the form of the "cos"-segments of Fourier series, taking only several first terms of them with the required accuracy.

To obtain the local estimation of solution of equation (12) we construct the following Markov chain $\{x_n\}, n = 0, \dots, N$, with the transitional function, i.e., the density $p(t, x)$. In this connection, the value $1 - \int_{-x}^{x} p(t, s)\, ds \geq 0$ is assumed to be the probability of breaking the Markov chain at the point x_N, N is the number of the final state.

The matrix estimator of Monte Carlo method for $W(x, x)$ is constructed on a basis of relations:

$$W(x,x) = M\xi_x, \quad \xi_x = H + \sum_{n=1}^{N} Q_n K(x, x_n)$$

$$Q_0 = E, \quad Q_{n+1} = Q_n \frac{K(x_n, x_{n+1})}{p(x_n, x_{n+1})},$$

all functions in the relations are matrices.

From the point of view of smallness of variance, let us take the transitional density in the form

$$p(t,x) \approx \left(\frac{|a_0|}{2} + \sum_{k=1}^{n} |a_k| \cdot \left| \cos \frac{k\pi t}{2T} \right| \right)_{i,j},$$

where a_k are the Fourier series coefficients of the functions f_{ij} are the components of the matrix K.

The next point x_n in Markov's chain $\{x_n\}$ we simulate according to the superposition method [8].

We solve the inverse problem in two steps: (a) solution of the direct problem by obtaining additional information $u|_{x=0} = F(t)$ for the solution of the inverse problem; (b) solution of the inverse problem.

To solve the direct problem (8), (9), we reduce it to the corresponding equivalent Gursat problem:

$$U^k_{tt} = U^k_{xx} + B(x)U^k, \qquad U^k|_{t=|x|} = 0.5(0, \dots 1, \dots 0)^T,$$

here 1 is at the k-place, and $U^k = (U^k_{-N}, U^k_{-N+1}, \dots, U^k_N)^T$. To solve this problem, we use the explicit stable difference scheme "cross" of order $O(h^2)$, where h is the step with respect to time and space. When solving it we obtain the discrete analog of $U|_{x=0} = F(t)$.

7. Solution of the Cauchy problem to telegraph equation. Those who solve the inverse problems as a rule faces the problem of the solving the corresponding direct problem, which requires a lot of storage, especially in the multidimensional cases or when large matrices appear.

Let us present the numerical Monte Carlo realization of the probabilistic approach to the solution of the Cauchy problem for telegraph equation.

There are well-known presentations of the parabolic and elliptic equations solutions in the form of continual integrals in some probabilistic measure in the space of the diffusion process paths.

Note, that numerical algorithms, based on direct calculating of this integrals, are very cumbersome and time-consuming.

As for hyperbolic equations, then for the solving the Cauchy problem to the telegraph equation there also exists the probabilistic presentation in the form of integral in the Poisson measure [5].

We propose the algorithm based on the calculation of this integral as mean value in the path space of the Poisson process.

Consider the Cauchy problem for the telegraph equation

$$U_{tt}(x,t) = \Delta U(x,t) - 2aU_t(x,t), \qquad x \in R^3, \quad t \geq 0, \tag{13}$$

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x) \tag{14}$$

and the wave equation

$$V_{tt}(x,t) = \Delta V(x,t) \qquad x \in R^3, \quad t \geq 0, \tag{15}$$

$$V(x,0) = f(x), \qquad V_t(x,0) = g(x). \tag{16}$$

Assume that the functions f and g satisfy the conditions of the unique solvability of problems (13), (14), and (15), (16). For example, let us consider

$$\|f\|_{C^2(R^n)} \leq C_1, \quad \|g\|_{C^2(R^n)} \leq C_2, \quad C_1, C_2 = \text{const} < \infty.$$

There is a probabilistic Cauchy problems solution [5]:

$$U(x,t) = \mathrm{M}\left[V\left(x, \int_0^t (-1)^{N(s)}\, ds\right)\right], \tag{17}$$

provided its right-hand side exists. In (17), $N(s)$ is the Poisson process with a parameter $a > 0$, and $V(x,\cdot)$ is the solution to problem (15), (16); mathematical expectation here is the mean in the Poisson paths space $N(s)$.

In the 3D space, the solution to problem (15), (16) is defined with the Kirhgoff formula

$$V(x,t) = \frac{1}{4\pi t}\int_{|\xi - x| = t} g(\xi)\, d\xi + \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{|\xi - x| = t} f(\xi)\, d\xi\right].$$

It is not difficult to construct the unbiased estimation for problem (13), (14), using this formula, presentation (17) as well as the double randomization principle. Thus, consider the random estimation

$$\xi(x,t) = \eta(t)g(\omega) + f(\omega) + \eta(t)f_t(\omega),$$

where $\eta(t)$ is a random value defined with equality: $\eta(t) = \int_0^t (-1)^{N(s)} ds$. Here $\omega = x + \alpha\eta$, α is a unit isotropic vector.

We obtain the following algorithm. Construct the random variables according to the formula

$$\tau_k = -\frac{1}{a}\sum_{i=1}^{k} \ln \beta_i,$$

where τ_k are the Poisson process jumps, β_i are independent, uniformly distributed in $(0,1)$ random variables. Then

$$\eta(t) = \int_0^{\tau_1} (-1)^0 ds + \ldots + \int_{\tau_k}^{t} (-1)^k ds = (-1)^k t + 2\sum_{i=1}^{k} (-1)^{i-1}\tau_i, \quad (18)$$

where $k = N(t)$ are the numbers of the Poisson process jumps in the interval $(0,t)$. Because $N(t)$ in the $(0,t)$ has the Poisson distribution, namely

$$P\{N(t) = m\} = \frac{(at)^m \exp(-at)}{m!}, \qquad m = 0, 1, \ldots,$$

the mean number of the Poisson jumps at the time t is equal to $\mathrm{M}\,N(t) = at$. The random value $\eta(t)$, as can be seen from (18), is the sum of the independent random values. So, taking into account Theory of Renewal we may asymptotically estimate its mathematical expectation as $\mathrm{M}\eta(t) \le c(t - \mathrm{M}\tau_1)$ and variance as $D\eta(t) = \frac{\sigma^2}{a^3}t + o(t)$.

For each obtained $\eta(t)$ we construct the random value $\omega = x + \alpha\eta$, which is uniformly distributed on the surface of the unit sphere with the radius η and center at the point x.

Simple estimations show that the complexity of the algorithm has the order $O(t^4)$ with $t \to \infty$, because $D\eta(t) = O(t)$, and the functions f, g are limited.

References

[1] Alexeev A.S. Inverse dynamic problems of seismology // Methods and Algorithms for the Interpretation of Geophysical Data. – Moscow: Nauka, 1967. – P. 9–84 (in Russian).

[2] Blagoveshenskiy A.S. About the local method of solving the nonstationar inverse problem for inhomogenious string // W. MIAN. – 1971. – Vol. 115. – P. 28–38.

[3] Ermakov S.M., Mikhailov G.A. The Statistical Simulation. – Moscow: Nauka, 1982 (in Russian).

[4] Gopinath B., Sondhi M.M. Inversion of the telegraph equation and the synthesis of nonuniform lines // Bell Syst. Tech. J. – July, 1970. – P. 1195–1214.

[5] Hersh R., De Griego R. Theory of random evolutions with application to PDE // Trans. Am. Math.Soc. – 1971. – Vol. 156, № 2. – P. 405–418.

[6] Marchuk G.I., Agoshkov V.I. Introduction to Projection Grid Methods. – Moscow: Nauka, 1984 (in Russian).

[7] Kabanihin S.I. Projection Difference Methods of Hyperbolic Equations Coefficients Determination. – Novosibirsk: Nauka, 1988 (in Russian).

[8] Mikhailov G.A. The Optimization of Weight Monte Carlo Methods. – Novosibirsk: Nauka, 1987 (in Russian).

[9] Romanov V.G. The Inverse Problems of Mathematical Physics. – Novosibirsk: Nauka, 1984 (in Russian).