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Mathematical statement of the inverse kinematic problem of seismics for 3D inhomogeneous medium. Part I

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In [1], the inverse kinematic problem (IKP) for 2D inhomogeneous half-plane with a 2D hodograph of refracted waves given at its boundary was reduced to the initial problem for the non-classical evolutional partial differential equation of the first order. In the present paper, the method from [1] is extended to the 3D case of the IKP. It is necessary to determine the velocity distribution of elastic waves in a 3D spatial subdomain adjacent to the aperture for the 3D inhomogeneous half-space using the 4D hodograph of refracted waves given at the 2D aperture of its boundary. In the derivation of the equation, the differentiation rule of hodograph (used heuristically in [1]) is rigorously substantiated for the case, when the coordinates of the source and the receiver coincide.

It should be noted that the problem being considered differs fundamentally from its 2D statement. In the 2D statement, the function of two variables is reconstructed using 2D information, and in our problem the function of three variables is reconstructed using 4D information. This can provide a substantially higher stability of the solution to the 3D problem on a computer.

Let the inhomogeneous isotropic elastic medium, in which the kinematics of wave processes is considered, fill the half-space $z \geq 0$ of the Cartesian system of coordinates x, y, z. The sought-for function describing the propagation velocity of elastic longitudinal or transverse waves is denoted by $v(x,y,z)$ and assumed to be a positive function of the class C^r, $r \geq 2$. Below v is explicitly and implicitly subject to some additional limitations of global character.

Let Δ denote some convex domain of the variables x, y. We assume that, for any two points $M_1 = (x_1, y_1, 0)$, $M_2 = (x_2, y_2, 0)$ of the daily surface, where $(x_i, y_i) \in \Delta$, $i = 1, 2$, there exists one and only one geodesic $L(M_1, M_2)$ of the metrics

$$dr^2 = n(x,y,z)(dx^2 + dy^2 + dz^2), \quad n(x,y,z) = (v(x,y,z))^{-1}.$$

It connects the points M_1 and M_2. Here the curve $L(M_1, M_2)$ (seismic ray), with the exception of its ends M_1 and M_2, is in the half-space $z > 0$.

The distance $\tau(M_1, M_2)$ between the points M_1 and M_2 in the metrics $d\tau^2$ ($\tau(M_1, M_2)$ is the travel time of the seismic wave along the ray $L(M_1, M_2)$) is denoted by $\tau(x_1, y_1; x_2, y_2)$. The function $\tau(x_1, y_1; x_2, y_2)$ which is called a 4D hodograph of refracted waves given on the aperture $\Delta \times \{0\}$ is assumed to be the known function $\varphi(x_1, y_1; x_2, y_2)$ of the variables $(x_1, y_1; x_2, y_2) \in \Delta \times \Delta$.

Let us next assume that the set-theoretic union of the internal points of the rays $L(M_1, M_2)$, $M_1, M_2 \in \Delta \times \{0\}$, is a convex 3D domain Ω ("adjacent to" the aperture $\Delta \times \{0\}$) and that, for any points $P_1, P_2 \in \Omega$, there exists the only ray $L(M_1, M_2)$ with the ends M_1 and M_2 on the aperture $\Delta \times \{0\}$ passing through the points P_1 and P_2. It is clear that the part of the ray $L(M_1, M_2)$ between the points P_1 and P_2 is the only geodesic $L(P_1, P_2)$ (of the metrics $d\tau^2$) connecting P_1 and P_2. And, finally, we assume that every plane $z = \text{const} > 0$ intersects $L(M_1, M_2)$ at no more than two points. It should be noted that all above assumptions are implicit limitations on the function v, and only the latter admits a constructive sufficient condition associated with the monotonicity of $v(x, y, z)$.

For the arbitrary points $P_i = (x_i, y_i, z_i) \in \Omega$, $i = 1, 2$, let $\tau_1(P_1, P_2) = \tau_1(x_1, y_1, z_1; x_2, y_2, z_2)$ denote the wave travel time along the ray $L(P_1, P_2)$, that is

$$\tau_1(P_1, P_2) = \int_{P_1}^{P_2} n(x, y, z)\, dz, \tag{1}$$

where dz is the length element of the curve arch $L(P_1, P_2)$. The function $\tau_1 : (P_1, P_2) \to \tau_1(P_1, P_2)$ satisfies for any $P_1, P_2 \in \Omega$ the eikonal equation:

$$|\nabla_{p_1} \tau_1(P_1, P_2)|^2 = n^2(P_1), \quad |\nabla_{p_2} \tau_1(P_1, P_2)|^2 = n^2(P_2), \quad P_1 \neq P_2, \tag{2}$$

$$\nabla_p = \frac{\partial}{\partial x_s}\vec{i} + \frac{\partial}{\partial y_s}\vec{j} + \frac{\partial}{\partial z_s}\vec{k}, \quad s = 1, 2;$$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors of the axes x, y, z, correspondingly. As in [1], we determine the function τ of the variables x_1, y_1, z_1, x_2, y_2, z_2, assuming that

$$\tau \equiv \tau(x_1, y_1, z_1, x_2, y_2, z_2) = \tau_1(x_1, y_1, z, x_2, y_2, z), \tag{3}$$

$$z_1 = z_2 = z, \quad (x_1, y_1, z), (x_2, y_2, z) \in \Omega.$$

As in [1], we use the identity

$$\frac{\partial \tau}{\partial z} = \left.\frac{\partial \tau_1}{\partial z_1}\right|_{z_1 = z} + \left.\frac{\partial \tau_1}{\partial z_2}\right|_{z_2 = z}, \tag{4}$$

from which we find, using (2), (3), that

$$\frac{\partial\tau}{\partial z} = \operatorname{sgn}\left(\frac{\partial\tau_1}{\partial z_1}\bigg|_{z_1=z}\right)\sqrt{n^2(x_1,y_1,z) - \left(\frac{\partial\tau}{\partial x_1}\right)^2 + \left(\frac{\partial\tau}{\partial y_1}\right)^2} +$$
$$\operatorname{sgn}\left(\frac{\partial\tau_1}{\partial z_2}\bigg|_{z_2=z}\right)\sqrt{n^2(x_2,y_2,z) - \left(\frac{\partial\tau}{\partial x_2}\right)^2 + \left(\frac{\partial\tau}{\partial y_2}\right)^2}. \qquad (5)$$

Relation (5) is the initial relation for derivation of the differential equation for the function $\tau(x_1, y_1, x_2, y_2, z)$.

In accordance with the assumptions that $v(x_1, y_1, z)$, every ray $L(P_1, P_2)$ at $z_1 = z_2$ lies in the half-space of the points (x, y, z) with the values $z \geq z_1 = z_2$. Therefore, the angles α_1 and α_2 of the vectors tangent to the ray at the points P_1, P_2, with the axis z, are obtuse. Hence, on the basis of the equalities

$$\frac{\partial\tau_1(P_1,P_2)}{\partial z_i}\bigg|_{z_i=z} = n(P_i)\cos\alpha_i, \quad i = 1,2,$$

we find that

$$\frac{\partial\tau_1}{\partial z_i}\bigg|_{z_i=z} \leq 0, \quad i = 1,2,$$

and, therefore, (5) takes the form:

$$\frac{\partial\tau}{\partial z} = -\sqrt{n^2(x_1,y_1,z) - \left(\frac{\partial\tau}{\partial x_1}\right)^2 + \left(\frac{\partial\tau}{\partial y_1}\right)^2} -$$
$$\sqrt{n^2(x_2,y_2,z) - \left(\frac{\partial\tau}{\partial x_2}\right)^2 + \left(\frac{\partial\tau}{\partial y_2}\right)^2}, \qquad (6)$$
$$\tau \equiv \tau(x_1, y_1; x_2, y_2, z).$$

The relation between the functions $n(x_1, y_1, z)$, $n(x_2, y_2, z)$ and the one-sided partial derivatives of the function τ for the case, when the values x_1 and x_2, y_1 and y_2 coincide, is determined using some facts of the calculus of variations in the small. We have

$$n(x_1,y_1,z) = \frac{\partial\tau(x_1,y_1,x_2,y_2,z)}{\partial x_1}\bigg|_{x_2=x_1+0,\ y_2=y_1}$$
$$= -\frac{\partial\tau(x_1,y_1,x_2,y_2,z)}{\partial y_1}\bigg|_{x_2=x_1,\ y_2=y_1+0}, \qquad (7)$$

$$n(x_2,y_2,z) = \frac{\partial\tau(x_1,y_1,x_2,y_2,z)}{\partial x_2}\bigg|_{x_1=x_2-0,\ y_1=y_2}$$
$$= \frac{\partial\tau(x_1,y_1,x_2,y_2,z)}{\partial y_2}\bigg|_{x_1=x_2-0,\ y_1=y_2}. \qquad (8)$$

Differential relation (6), where $n(x_1, y_1, z)$, $n(x_2, y_2, z)$ are expressed by formula (7) or (8), and the "initial" condition

$$\tau(x_1, x_2, x_2, y_2, z)|_{z=0} = \varphi(x_1, y_1, x_2, y_2), \quad (x_i, y_i) \in \Delta, \ \ i = 1, 2, \qquad (9)$$

form the mathematical statement of the IKP promised.

The numerical solution to problem (6, 7, 8, 9) can be obtained, for example, by the finite-difference method with the help of the schemes used for the evolutional equations. The stability of the solution to the problem can be investigated, as in [1], using computational experiments.

References

[1] Belonosova A.V., Alekseev A.S. On one statement of the inverse kinematic problem of seismics for 2D inhomogeneous medium // Some Methods and Algorithms for Interpretation of Geophysical Data. – Moscow: Nauka, 1967. – P. 137–154.