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Simulation of random variables in module Resonance of the package ACCORD*

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We present here general principles of organization of the module "Resonance" for stochastic simulation, which is a part of the package "ACCORD", and describe basic procedures of the module for simulation of random variables.

1. General principles

The module "Resonance" is planned as a set of procedures for stochastic simulation. We design to fulfil the following sections of the module: "pseudorandom numbers", "discrete distributions", "continuous distributions", "singular distributions", "multidimensional Gaussian distributions", "random processes and fields", "radiation transfer", and some others. Destination of the module is to simplify programming for Monte Carlo methods. The main programming language is assumed to be Pascal (in addition we plan to realize some of the procedures in C, Fortran, and Java). For developing the module we are guided by the following principles:

- Openness and availability: all sources will be available via Internet.
- Clearness: in addition to simulation procedures we present sources and executable files of demonstration programs, where a user can find examples for implementing the procedures.
- Independence and minimal connectivity: if possible, all the procedures of the module are not connected with each other, i.e., to run one procedure it is not necessary to add scores of other procedures. It enlarge the code but makes the usage more simple and comfortable. Procedures of the module "Resonance" are independent on other modules of the package "ACCORD".

Below we describe three sections of the module "Resonance" ("pseudorandom numbers", "discrete distributions", "continuous distributions") and

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present comparatively extensive bibliography [1–25] on simulation of random variables that we used developing the software. Most of the realized algorithms can be found in [24] (otherwise we give a reference where a description of a corresponding algorithm is presented).

2. Pseudorandom number generators

2.1. Implementation of pseudorandom number generators

Purpose: To simulate a sequence of random variables with uniform distribution in the interval $(0,1)$.

The range of values: $x \in (0,1)$.

Probability density: $p(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$

Header: `Function Random: Real;`

Parameters: Only global variables are used (see Remarks).

Algorithm: We realized three versions of the random number generator. Version 1 is multiplicative generator with modulus 2^{40} multiplier 5^{17} and period $2^{38} = 274,877,906,944$ (cf. [7]). Versions 2 and 3 realize chains of independent multiplicative generators, their periods are $2.10292729 \cdot 10^{18}$ and $2.0417656605 \cdot 10^{36}$, correspondingly (cf. [1, 3]).

Remarks: To use the function `Random` it is necessary to introduce global variables

```
Var rand1, rand2, rand3: Longint;
```

for Versions 1, 2 and

```
Var rand1D, rand2D, rand3D, rand4D, rand5D: Double;
```

for Version 3. Random number generators written by high-level languages are slower than built-in generators, but inspite of this fact we strongly recommend not to use unaudited built-in generators for reliable Monte Carlo computations.

2.2. The following procedure can be useful for implementation of dependent sampling technique.

Purpose: To simulate a sequence $a(0)$, $a(1)$, $a(2)$, $\ldots$ of random variables uniformly distributed in the interval $(0,1)$ with additional possibility to simulate subsequences $a(0)$, $a(K)$, $a(2K)$, $\ldots$ for different steps K.

Header: `Function Rand1_FROG(Ind, s1, s2, K: Longint): Double;`

Parameters: `Ind` – control variable with possible values from the set $\{0, 1, 2\}$ (Input); `s1, s2` – values to initialize generator, $0 \le$ `s1`, `s2` $\le 2^{20} - 1$ and `s2` should be odd (Input); `K` – the length of jumps, $1 \le$ `K` ≤ 2147483647 (Input).

Algorithm: Multiplicative generator with modulus 2^{40} and multiplier 5^{17} is implemented in this procedure. If `Ind` $= 0$, then `RAND1_FROG` $= a(i) = a(j * K) = a(0)$ for $i = j = 0$. The initial value $a(0)$ is determined by the values `s1` and `s2`. If `Ind` $= 1$, then it is assumed that $i = i + 1$ and `RAND1_FROG` $= a(i)$ (the values of `s1, s2, K` are ignored). If `Ind` $= 2$, then it is assumed that $j = j + 1$, $i = j *$ `K` and `RAND1_FROG` $= a(i) = a(j *$ `K`$)$ (the values of `s1, s2, K` are ignored). To change initial value $a(0)$ or step `K` it is necessary to call the function with `Ind` $= 0$ once again.

Remarks: Random number generator with jumps is convenient to realize the dependent sampling technique in Monte Carlo computations.

Some other realizations of pseudo-random number generators in Pascal, Fortran and C can be found on the Internet page of the second author `http://osmf.sscc.ru/~smp` (see also [18, 19]).

3. Discrete distributions

3.1. Discrete uniform distribution

The range of values: $k = a, a + 1, \ldots, a + n - 1$.

Probabilities: $p_k = 1/n$, where a and n are integer ($n \ge 2$).

Header: `Function DUniformRV(A, N: Integer): Integer;`

Parameters: `A` and `N` specify the values of a and n, respectively.

3.2. The Poisson distribution

The range of values: $k = 0, 1, 2, \ldots$

Probabilities: $p_k = \frac{\mu^k}{k!} e^{-\mu}$, where μ is the expectation ($\mu > 0$).

Headers: `Function PoissonRV(Mu: Real): Integer;`
`Function PoissonRV2(Mu: Real): Integer;`

Parameter: `Mu` specifies the expectation μ.

Remarks: Algorithm for Version 1 is the standard way of the simulation discrete random variables. Version 2 is based on a relation between the Poisson distribution, exponential, and the Erlang distribution.

3.3. The Bernoulli distribution

The range of values: $k = 0, 1$.

Probabilities: $p_k = p^k(1-p)^{1-k}$, where $0 < p < 1$.

Header: `Function BernoulliRV(P: Real): Integer;`

Parameter: `P` specifies the value of p.

3.4. Binomial distribution

The range of values: $k = 0, 1, \ldots, n$

Probabilities: $p_k = C_n^k p^k (1-p)^{n-k}$, $n \geq 1$ is integer, $0 < p < 1$.

Headers:
```
Function BinomialRV(P: Real; N: Integer): Integer;
Function BinomialRV2(P: Real; N: Integer): Integer;
Function BinomialRV3(P: Real; N: Integer): Integer;
```

Parameters: `P` and `N` specify the values of p and n, respectively.

Remarks: The algorithm of Version 1 is a standard way of the simulate discrete random variable. Version 2 is a sum of the number n of the independent Bernoulli random variables. Version 3 is based on a relation between binomial and geometric distributions.

3.5. Geometric distribution

The range of values: $k = 0, 1, 2, \ldots$

Probabilities: $p_k = p(1-p)^k$, $0 < p < 1$.

Header: `Function GeometricRV(P: Real): Integer;`

Parameter: `P` specifies the value of p.

3.6. Negative binomial distribution

The range of values: $k = 0, 1, 2 \ldots$

Probabilities: $p_k = C_{k+m-1}^k p^m (1-p)^k$, $m \geq 1$ is integer, $0 < p < 1$.

Headers:
```
Function NegBinomialRV(P: Real; M: Integer): Integer;
Function NegBinomialRV1(P: Real; M: Integer): Integer;
```

Parameters: `P, M` specifies the values of p, m, respectively.

Remarks: The standard method of simulation for discrete random variables has been used in first procedure and the algorithm for second procedure is based on a relation between negative binomial and geometric distributions.

3.7. The Pascal distribution

The range of values: $k = m, m+1, m+2, \ldots$

Probabilities: $p_k = C_{k-1}^{m-1} p^m (1-p)^{k-m}$, $m \geq 1$ is integer, $0 < p < 1$.

Header: `Function PascalRV(P: Real; M: Integer): Integer;`

Parameters: `P, M` specifies the values of p, m, respectively.

3.8. Hypergeometric distribution

The range of values: $k = a, a+1, \ldots, b$.

Probabilities: $p_k = C_M^k C_{L-M}^{n-k} / C_L^n$, where L, M, n are nonnegative integers, $M \leq L$, $n \leq L$, $a = \max(0, M+n-L)$, $b = \min(M, n)$.

Header: `Function HGeometricRV(L, M, N: Integer): Integer;`

Parameters: `L, M, N` specify the values of L, M, and n, respectively.

3.9. Negative hypergeometric distribution I

The range of values: $k = 0, 1, \ldots, L-M$.

Probabilities: $p_k = C_{k+n-1}^k C_{L-M-k}^{M-n} / C_L^M$, where L, M, n are nonnegative integers, $n \leq M \leq L$.

Header: `Function NegHGeometric1RV(L, M, N: Integer): Integer;`

Parameters: `L, M, N` specify the values of L, M, and n, respectively.

3.10. Negative hypergeometric distribution II

The range of values: $k = m, m+1, \ldots, L-M+m$.

Probabilities: $p_k = C_{k-1}^{n-1} C_{L-k}^{M-n} / C_L^M$, where L, M, n are nonnegative integers, $n \leq M \leq L$.

Header: `Function NegHGeometric2RV(L, M, N: Integer): Integer;`

Parameters: `L, M, N` specify the values of L, M, and n, respectively.

3.11. Logarithmic distribution I

The range of values: $k = 1, 2, \ldots$

Probabilities: $p_k = -\frac{1}{\ln q} \frac{p^k}{k}$, where $0 < p < 1$, $q = 1 - p$.

Header: `Function Log1RV(P: Real): Integer;`

Parameter: `P` specifies the value of p.

3.12. Logarithmic distribution II

The range of values: $k = 1, 2, \dots, m-1$

Probabilities: $p_k = \log_m \frac{k+1}{k}$, $m \geq 3$, m is integer.

Header: `Function Log2RV(M: Integer): Integer;`

Parameter: `M` specifies the value of m.

3.13. The Pólya distribution I

The range of values: $k = 0, 1, \dots, n$.

Probabilities:

$$p_0 = \frac{r(r+c)(r+2c)\dots(r+(n-1)c)}{(b+r)(b+r+c)(b+r+2c)\dots(b+r+(n-1)c)},$$

$$p_k = C_n^k \frac{b(b+c)\dots(b+(k-1)c)r(r+c)\dots(r+(n-k-1)c)}{(b+r)(b+r+c)\dots(b+r+(n-1)c)},$$

$$k = 1, 2, \dots, n-1,$$

$$p_n = \frac{b(b+c)(b+2c)\dots(b+(n-1)c)}{(b+r)(b+r+c)(b+r+2c)\dots(b+r+(n-1)c)},$$

$n > 0$, $b > 0$, $r > 0$, and c are integers. The parameter c can be negative, but it has to satisfy the following condition $b + r + (n-1)c > 0$.

Header: `Function Polya1RV(N, B, R, C: Integer): Integer;`

Parameters: `N, B, R, C` specify the values of n, b, r, and c, respectively.

3.14. The Pólya distribution II

The range of values: $k = 1, 2, \dots$

Probabilities:

$$p_0 = 1/(1-\alpha\mu)^{1/\alpha}, \qquad p_k = \frac{\mu^k}{k!}\frac{(1+\alpha)(1+2\alpha)\dots(1+(k-1)\alpha)}{(1+\alpha\mu)^{k+1/\alpha}},$$

where $\mu > 0$, $0 < \alpha < 1$.

Header: `Function Polya2RV(M, A: Real): Integer;`

Parameters: `M` and `A` specify the values of μ and α, respectively.

3.15. ζ distribution

The range of values: $k = 1, 2, \ldots$

Probabilities: $p_k = k^{-(\rho+1)}/\zeta(\rho+1)$, $\zeta(\alpha) = \sum_{k=1}^{\infty} k^{-\alpha}$ is Riman's zeta function; $\rho > 0$.

Header: `Function ZetaRV(R: Real): Integer;`

Parameter: `R` specifies the value of ρ.

3.16. The Borel and Tanner distribution

The range of values: $k = r, r+1, \ldots$

Probabilities: $p_k = \frac{r}{(k-r)!} k^{k-r-1} e^{-\alpha k} \alpha^{k-r}$, $r \geq 1$ is integer, $0 < \alpha < 1$.

Header: `Function BorelRV(R: Integer; A: Real): Integer;`

Parameters: `R` and `A` specify the values of r and α, respectively.

4. Continuous distributions

4.1. Normal distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ is the expectation and $\sigma^2 > 0$ is the variance, $\sigma > 0$.

Distribution function: $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, $\Phi(x)$ is the standard normal distribution function.

Headers: `Function NormalRV(M, D: Real): Real;`
`Function StdNormalRV: Real;`

Parameters: `M` and `D` specify the values of μ and σ, respectively.

Remarks: `StdNormalRV` simulates standard normal random variables ($\mu = 0$, $\sigma = 1$).

4.2. The Laplace distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{\lambda}{2} e^{-\lambda|x-\mu|}$, where $-\infty < \mu < \infty$, $\lambda > 0$.

Distribution function: $F(x) = \begin{cases} 0.5e^{\lambda(x-\mu)}, & x \leq \mu; \\ 1 - 0.5e^{-\lambda(x-\mu)}, & x \geq \mu \end{cases}$.

Header: `Function LaplaceRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.3. The Cauchy distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{\lambda}{\pi(\lambda^2+(x-\mu)^2)}$, where μ is median, $\lambda > 0$ is mean deviation.

Distribution function: $F(x) = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg}\frac{x-\mu}{\lambda}$.

Header: `Function CauchyRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.4. Distribution of minimum

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{1}{\lambda}\exp\left(\frac{x-\mu}{\lambda} - e^{(x-\mu)/\lambda}\right)$, where μ is mode, $\lambda > 0$.

Distribution function: $F(x) = 1 - \exp(-e^{(x-\mu)/\lambda})$.

Header: `Function MinRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.5. Distribution of maximum

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{1}{\lambda}\exp\left(-\frac{x-\mu}{\lambda} - e^{-(x-\mu)/\lambda}\right)$, where μ is mode, $\lambda > 0$.

Distribution function: $F(x) = \exp(-e^{-(x-\mu)/\lambda})$.

Header: `Function MaxRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.6. Dual exponential distribution

The range of values: $-\infty < x < \infty$

Probability density: $f(x) = \lambda\mu\exp(-\lambda\mu - \mu e^{-\lambda x})$, where $\mu > 0$, $\lambda > 0$.

Distribution function: $F(x) = \exp(-\mu e^{-\lambda x})$.

Header: `Function DualExpRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.7. Logistic distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{e^{(x-\mu)/\lambda}}{\lambda(1+e^{(x-\mu)/\lambda})^2}$, where $-\infty < \mu < \infty$, $\lambda > 0$.

Distribution function: $F(x) = \frac{1}{1+e^{-(x-\mu)/\lambda}}$.

Header: `Function LogisticRV(M, L: Real): Real;`

Parameters: `M` and `L` specify the values of μ and λ, respectively.

4.8. Exponential distribution

The range of values: $x > 0$.

Probability density: $f(x) = \lambda e^{-\lambda x}$, where $\lambda > 0$.

Distribution function: $F(x) = 1 - e^{-\lambda x}$.

Header: `Function ExpRV(L: Real): Real;`

Parameter: `L` specifies the value of λ.

4.9. Gamma distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, where $\lambda > 0$, $\alpha > 0$.

Distribution function: $F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\lambda x} t^{\alpha-1} e^{-t}\, dt$.

Headers:
```
Function GammaRV(L, A: Real): Real;
Function GammaRV2(L, A: Real): Real;
```

Parameters: `L` and `A` specify the values of λ and α, respectively.

4.10. Displaced gamma distribution

The range of values: $x < c$.

Probability density: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} (x-c)^{\alpha-1} e^{-\lambda(x-c)}$, where $\lambda > 0$, $\alpha > 0$.

Distribution function: $F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\lambda(x-c)} t^{\alpha-1} e^{-t}\, dt$.

Header: `Function DGammaRV(L, A, C: Real): Real;`

Parameters: `L, A, C` specify the values of λ, α, c, respectively.

4.11. M order Erlang distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{\lambda^m}{(m-1)!} x^{m-1} e^{-\lambda x}$, where $\lambda > 0$, m is a positive integer.

Distribution function: $F(x) = 1 - e^{-\lambda x} \sum_{i=0}^{m-1} \frac{(\lambda x)^i}{i!}$.

Header: `Function ErlangRV(L: Real; M: Integer): Real;`

Parameters: `L, M` specify the values of λ, m, respectively.

4.12. Normalized Erlang distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{(\lambda m)^m}{(m-1)!} x^{m-1} e^{-\lambda m x}$, where $\lambda > 0$, $m \geq 1$ is an integer.

Distribution function: $F(x) = 1 - e^{-\lambda m x} \sum_{i=0}^{m-1} \frac{(\lambda m x)^i}{i!}$.

Header: `Function NormErlangRV(L: Real; M: Integer): Real;`

Parameters: `L, M` specify the values of λ, m, respectively.

4.13. Generalized Erlang distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu x} - e^{-\lambda x})$, where $\lambda > 0$, $\mu > 0$.

Distribution function: $F(x) = 1 - \frac{1}{\lambda-\mu}(\lambda e^{-\mu x} - \mu e^{-\lambda x})$.

Header: `Function GenErlangRV(L, M: Real): Real;`

Parameters: `L, M` specify the values of λ, m, respectively.

4.14. The Weibull distribution

The range of values: $x > 0$.

Probability density: $f(x) = (c/a)(x/a)^{c-1} e^{-(x/a)^c}$, where $a > 0$, $c > 0$.

Distribution function: $F(x) = 1 - e^{-(x/a)^c}$.

Header: `Function WeibullRV(A, C: Real): Real;`

Parameters: `A, C` specify the values of a, c, respectively.

4.15. Displaced Weibull distribution

The range of values: $x > c$.

Probability density: $f(x) = \frac{b}{a}\left(\frac{x-c}{a}\right)^{b-1} e^{-((x-c)/a)^b}$, where $a > 0$, $b > 0$.

Distribution function: $F(x) = 1 - e^{((x-c)/a)^b}$.

Header: `Function DispWeibullRV(A, B, C: Real): Real;`

Parameters: `A, B, C` specify the values of a, b, c, respectively.

4.16. Hyperexponential distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = 2p^2\lambda e^{-2p\lambda x} + 2(1-p)^2\lambda e^{-2(1-p)\lambda x}$, where $\lambda > 0$, $0 < p < 0.5$.

Distribution function: $F(x) = 1 - pe^{-2p\lambda x} - (1-p)e^{-2(1-p)\lambda x}$.

Header: `Function HyperExpRV(L, P: Real;): Real;`

Parameters: `L, P` specify the values of λ and p, respectively.

4.17. Distribution of module of random vector

The range of values: $x > 0$.

Probability density: $f(x) = \frac{1}{2^{n/2-1}a^n\Gamma(n/2)} x^{n-1} e^{-x^2/(2a^2)}$, where n is a positive integer, $a > 0$.

Distribution function: $F(x) = \frac{\Gamma(x^2/(2a^2), n/2)}{\Gamma(n/2)}$.

Header: `Function ModuleRV(N: Integer; A: Real): Real;`

Parameters: `N, A` specify the values of n and a, respectively.

Remark: This is a distribution of module of n-dimensional Gaussian random vector with independent identically distributed components with zero mean and variance a^2.

4.18. Rayleigh distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{x}{a^2} e^{-x^2/(2a^2)}$, where $a > 0$ is a mode.

Distribution function: $F(x) = 1 - e^{-x^2/(2a^2)}$.

Header: `Function RayleighRV(A: Real): Real;`

Parameter: `A` specifies the value of a.

4.19. Generalized Rayleigh distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{x}{a^2} e^{-(x^2+h^2)/(2a^2)} I_0(xh/a^2)$, where $a > 0$ is a mode, $h > 0$, $I_0(x)$ is the modified zero-order Bessel function.

Distribution function: $F(x) = 1 - e^{x+\theta} \sum\limits_{m=0}^{\infty} \frac{\theta^m}{m!} \sum\limits_{k=0}^{m} \frac{x^k}{k!}$.

Headers: `Function GRayleighRV(A, H: Real): Real;`
`Function GRayleighRV2(A, H: Real): Real;`

Parameters: `A, H` specify the values of a, h, respectively.

4.20. Maxwell distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{2x^2}{a^3\sqrt{2\pi}} e^{-x^2/(2a^2)}$, where $a > 0$.

Distribution function: $F(x) = 2\left[\Phi_0(x/a) - \frac{x}{a\sqrt{2\pi}} e^{-x^2/(2a^2)}\right]$, Φ_0 is the Laplace function.

Header: `Function MaxwellRV(A: Real): Real;`

Parameter: `A` specifies the value of a.

4.21. Type 2 beta distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{1}{B(u,v)} \frac{x^{u-1}}{(1+x)^{u+v}}$, where $u > 0$, $v > 0$.

Distribution function: $F(x) = \frac{1}{B(u,v)} \int\limits_0^{x/(1+x)} t^{u-1}(1-t)^{v-1}\, dt$.

Header: `Function Beta2RV(U, V: Real): Real;`

Parameters: `U, V` specify the values of u, v, respectively.

4.22. Log-normal distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{1}{xa\sqrt{2\pi}} e^{-\frac{\ln^2(x/m)}{2a^2}}$, where m is a median.

Distribution function: $F(x) = \Phi(\ln(x/m)/a)$, Φ is the standard normal distribution function.

Header: `Function LogNormRV(M, A: Real): Real;`

Parameters: `M, A` specify the values of m, a, respectively.

Remark: Parameter $\ln m$ is used more often than m.

4.23. Pareto distribution

The range of values: $x > c$.

Probability density: $f(x) = (\alpha/c)(c/x)^{\alpha+1}$, where $c > 0$, $\alpha > 0$.

Distribution function: $F(x) = 1 - (c/x)^{\alpha}$.

Header: `Function ParetoRV(C, A: Real): Real;`

Parameters: `C, A` specify the values of c, α, respectively.

4.24. Reflected normal distribution

The range of values: $x \geq 0$.

Probability density: $f(x) = \frac{1}{a\sqrt{2\pi}}(e^{-(x-m)^2/(2a^2)} + e^{-(x+m)^2/(2a^2)})$, where m is a mean value of a source normal random variable, $a > 0$ is a variance of the source normal random variable.

Distribution function: $F(x) = \Phi_0((x-m)/a) + \Phi_0((x+m)/a)$, Φ_0 is the Laplace function.

Header: `Function RefNormalRV(M, A: Real): Real;`

Parameters: `M, A` specify the values of m, a, respectively.

4.25. Uniform distribution

The range of values: $a \leq x \leq b$.

Probability density: $f(x) = 1/(b-a)$.

Distribution function: $F(x) = (x-a)/(b-a)$.

Header: `Function UniformRV(A, B: Real): Real;`

Parameters: `A, B` specify the values of a, b, respectively.

4.26. Beta distribution

The range of values: $0 < x < 1$.

Probability density: $f(x) = \frac{x^{u-1}(1-x)^{v-1}}{B(u,v)}$, where $u > 0$, $v > 0$.

Distribution function: $F(x) = \frac{\int_0^x s^{u-1}(1-s)^{v-1}\,ds}{B(u,v)}$.

Headers:
```
Function BetaRV(U, V: Real): Real;
Function BetaRV2(U, V: Real): Real;
Function BetaRV3(U, V: Real): Real;
Function BetaRV4(U, V: Real): Real;
```

Parameters: `U, V` specify the values of u, v, respectively.

Remarks: Algorithms for Version 1 and 2 can be found in [24] (see also [13]). Algorithm for Version 3 is described in [7]. Algorithm for Version 4 was proposed in [25].

4.27. Generalized beta distribution

The range of values: $\alpha < x < \beta$.

Probability density: $f(x) = \frac{1}{B(u,v)} \frac{(x-\alpha)^{u-1}(\beta-x)^{v-1}}{(\beta-\alpha)^{u+v-1}}$, where $u > 0$, $v > 0$.

Distribution function: $F(x) = \frac{1}{B(u,v)} \int\limits_0^{(x-\alpha)/(\beta-\alpha)} s^{u-1}(1-s)^{v-1}\,ds$.

Header: `Function GBetaRV(U, V, A, B: Real): Real;`

Parameters: `U, V, A, B` specify the values of u, v, α, β, respectively.

4.28. Parabolic distribution

The range of values: $\alpha \le x \le \beta$.

Probability density: $f(x) = \frac{6(x-\alpha)(\beta-x)}{(\beta-\alpha)^3}$.

Distribution function: $F(x) = \frac{\alpha^2(3\beta-\alpha)-6\alpha\beta x+3(\alpha+\beta)x^2-2x^3}{(\beta-\alpha)^3}$.

Header: `Function ParabolicRV(A, B: Real): Real;`

Parameters: `A, B` specify the values of α, β, respectively.

4.29. Arcsine distribution

The range of values: $\mu - \lambda < x < \mu + \lambda$.

Probability density: $f(x) = \frac{1}{\pi\sqrt{\lambda^2-(x-\mu)^2}}$, where μ is a mean value, $\lambda > 0$.

Distribution function: $F(x) = \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{x-\mu}{\lambda}$.

Header: `Function ArcsineRV(L, M: Real): Real;`

Parameters: `L, M` specify the values of λ, μ, respectively.

4.30. Simpson distribution

The range of values: $\alpha \le x \le \beta$.

Probability density: $f(x) = \frac{2}{\beta-\alpha}\left[1 - \frac{|\alpha+\beta-2x|}{\beta-\alpha}\right]$.

Distribution function:

$$F(x)=\begin{cases}\dfrac{2(x-\alpha)^2}{(\beta-\alpha)^2}, & \alpha\le x\le\dfrac{\alpha+\beta}{2};\\[2mm] 1-\dfrac{2(\beta-x)^2}{(\beta-\alpha)^2}, & \dfrac{\alpha+\beta}{2}\le x\le\beta\end{cases}$$

Header: `Function SimpsonRV(A, B: Real): Real;`

Parameters: `A, B` specify the values of a, b, respectively

4.31. Chi-square distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$, where n is a positive integer (number of degrees of freedom).

Distribution function: $F(x) = \frac{\Gamma(x/2,n/2)}{\Gamma(n/2)}$.

Headers: `Function ChiSquareRV(N: Integer): Real;`
`Function ChiSquareRV2(N: Integer): Real;`

Parameter: `N` specifies the value of n.

4.32. Chi distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{1}{2^{n/2-1}\Gamma(n/2)}x^{n-1}e^{-x^2/2}$, where n is a positive integer (number of degrees of freedom).

Distribution function: $F(x) = \frac{\Gamma(x^2/2,n/2)}{\Gamma(n/2)}$.

Headers: `Function ChiRV(N: Integer): Real;`
`Function ChiRV2(N: Integer): Real;`

Parameter: `N` specifies the value of n.

4.33. Student's distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}$, where n is a positive integer (number of degrees of freedom).

Header: `Function StudentRV(N: Integer): Real;`

Parameter: `N` specifies the value of n.

4.34. F distribution

The range of values: $x > 0$.

Probability density: $f(x) = \frac{1}{B(u/2,v/2)}(u/v)^{u/2}x^{(u/2)-1}\left(1+\frac{u}{v}x\right)^{-(u+v)/2}$, where u and v are positive integers (numbers of degrees of freedom for numerator and denominator).

Distribution function: $F(x) = \frac{1}{B(u/2,v/2)} \int\limits_0^{ux/(v+ux)} s^{(u/2)-1}(1-s)^{(v/2)-1}\,ds$.

Header: `FRV(U, V: Integer): Real;`

Parameters: `U, V` specify the values of u, v, respectively.

4.35. Z distribution

The range of values: $-\infty < x < \infty$.

Probability density: $f(x) = \frac{2u^{u/2}v^{v/2}}{B(u/2,v/2)}e^{ux}(v+e^{2x})^{(u+v)/2}$, where u and v are positive integers.

Distribution function: $F(x) = I_\zeta(u/2, v/2)$, where $\zeta = \frac{ue^{2x}}{v+ue^{2x}}$, $I_x(u,v) = \frac{1}{B(u,v)}\int\limits_0^x s^{u-1}(1-s)^{v-1}\,ds$.

Header: `Function ZRV(U, V: Integer): Real;`

Parameters: `U, V` specify the values of u, v, respectively.

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