*Bull. Nov. Comp. Center*, Num. Anal., 9 (2000), 25–28


# On a shape preserving interpolation by local *VP*-splines*

V.V. Bogdanov

The problem we discuss here is how to construct the local smooth *VP*-splines preserving the monotonicity of data from the point of view of the regions of their parameters. Also, we will give a practical recommendation how to satisfy these conditions if data and the slopes are given and fixed.

Let in every knot of the grid $a = x_0 < x_1 < \ldots < x_n$ be given the values $f_i$, $m_i \geq 0$, $i = 0, 1, \ldots, n$. We consider the interpolation by the generalized Hermite (cubic) spline $S(x)$, under the conditions $S(x_i) = f_i$, $S'(x_i) = m_i$:

$$S(x) = (1-t)f_i + tf_{i+1} + h_i\{C_i(\varphi_i(t) - t) + D_i(\psi_i(t) - 1 + t)\},$$

where $x \in [x_i, x_{i+1}]$, $h_i = x_{i+1} - x_i$, $t = (x - x_i)/h_i$, $\varphi_i(t), \psi_i(t) \in C^3[0,1]$, $\varphi_i^{(r)}(t) \geq 0$, $(-1)^{(r)}\psi_i^{(r)}(t) \geq 0$, $r = 0, 1, 2, 3$,

$$\begin{aligned} C_i &= -\{m_i - f[x_i, x_{i+1}] - (\psi_i'(0) + 1)(m_{i+1} - f[x_i, x_{i+1}])\}/\Delta_i, \\ D_i &= \{(\varphi_i'(1) - 1)(m_i - f[x_i, x_{i+1}]) + (m_{i+1} - f[x_i, x_{i+1}])\}/\Delta_i, \\ \Delta_i &= \varphi_i'(1)\psi_i'(0) + \varphi_i'(1) - \psi_i'(0), \quad i = 0, 1, \ldots, n-1. \end{aligned}$$

In the problem of monotone interpolation, it is necessary to satisfy the condition $S'(x) \geq 0$ on the whole segment $[a, b]$ if all the first divided differences $f[x_i, x_{i+1}] \geq 0$, $i = 0, 1, \ldots, n-1$.

It is interesting to describe the regions in $\boldsymbol{R}^2$ of the pairs $(m_i, m_{i+1})$, where $m_i = S'(x_i)$, ensuring the shape preserving properties. For the first time this question concerning the cubic splines appeared in 1980 and was successfully solved in [1]. In the terms of normalized moments $\overline{m}_i = m_i/f[x_i, x_{i+1}]$ and $\overline{m}_{i+1} = m_{i+1}/f[x_i, x_{i+1}]$, the boundary of the region consists of the axes and the elliptic curve in the first quarter (Figure 1). In 1986, the negative answer in the problem of the convexity for the Hermite interpolating cubic spline [2] stimulated the further development of the extending spline constructions. That is why in 1991 there was received the result on comonotonicity and coconvexity of the generalized (cubic) Hermite splines where the regions of $(m_i, m_{i+1})$ for ensuring monotonicity or convexity were described.

*Supported by SB RAS under Grant 1-2000.

**Theorem.** *A Hermite generalized interpolating spline $S(x)$ is monotone in the interval $[a,b]$ iff every point $E^0(m_i, m_{i+1})$, $i = 0,1,\dots,n-1$, belongs to the multitude $G_i$, where $G_i$ is a convex domain restricted by the axes ($m_i \geq 0$, $m_{i+1} \geq 0$) and by the curve $\Gamma_i$, defined as a solution to the system of linear equations*

$$S'(\zeta) = 0, \qquad S''(\zeta) = 0$$

*with respect to $m_i$ and $m_{i+1}$, where $\zeta \in [x_i, x_{i+1}]$.*

*If $\varphi_i^{(r)}(t) = 0$, $r = 0,1,2$, for $t = 0$ only, then the axis $m_i = 0$ is a tangent line for $\Gamma_i$. If $\psi_i^{(r)}(t) = 0$, $r = 0,1,2$, for $t = 1$ only, then the axis $m_{i+1} = 0$ is a tangent line for $\Gamma_i$.*

The case of the cubic spline is realized in (1) for $\varphi_i(t) = t^3$, $\psi_i(t) = (1-t)^3$.

For the spline with additional knots $\varphi_i(t) = \dfrac{(t-p_i)_+^3}{(1-p_i)^3}$, $\psi_i(t) = \dfrac{(q_i-t)_+^3}{{q_i}^3}$.

For the rational spline $\varphi_i(t) = \dfrac{t^3}{1+p_i(1-t)}$, $\psi_i(t) = \dfrac{(1-t)^3}{1+q_i t}$.

One more construction we regard now is the so-called variable power spline – *VP*-spline – considered in partial case by R.W. Soanes, Jr. in 1976.

We denote on the interval $[x_i, x_{i+1}]$ the basic functions as

$$\begin{aligned} \varphi_i(t) &= t^{p_i}, & p_i &\geq 3, \\ \psi_i(t) &= (1-t)^{q_i}, & q_i &\geq 3. \end{aligned}$$

Then for the *VP*-spline we have

$$S_{VP}(x) = (1-t)f_i + tf_{i+1} + h_i\{C_i(t^{p_i} - t) + D_i((1-t)^{q_i} - 1 + t)\},$$

where $t = (x - x_i)/(x_i - x_{i+1}) \in [0,1]$,

$$\begin{aligned} C_i &= -\frac{m_i - f[x_i, x_{i+1}] - (q_i - 1)(m_{i+1} - f[x_i, x_{i+1}])}{p_i + q_i - p_i q_i}, \\ D_i &= \frac{(p_i+1)(m_i - f[x_i, x_{i+1}]) + (m_{i+1} - f[x_i, x_{i+1}])}{p_i + q_i - p_i q_i}. \end{aligned}$$

In the Hermite representation, it has the form more suitable for the analysis:

$$\begin{aligned} S_{VP}(x) = {} & (1-t)f_i + tf_{i+1} + \\ & \frac{h_i}{p_i + q_i - p_i q_i}\Big\{\Big((t^{p_i} - t) + (p_i - 1)(1-t)^{q_i} + 1 - t\Big)m_i + \\ & \Big((1-q_i)(t^{p_i} - t) + (1-t)^{q_i} + 1 - t\Big)m_{i+1}\Big\}. \end{aligned}$$

Nonlocal construction of the *VP*-spline of the class $C^2$ with the same generating functions $\varphi_i(t)$, $\psi_i(t)$ is determined by the system

$$\frac{\lambda_i}{p_{i-1}-2}m_{i-1} + \left(1 + \frac{\lambda_i}{p_{i-1}-2} + \frac{\mu_i}{q_i-2}\right)m_i + \frac{\mu_i}{q_i-2}m_{i+1}$$
$$= \frac{\lambda_i p_{i-1}}{p_{i-1}-2} f[x_{i-1}, x_i] + \frac{\mu_i q_i}{q_i-2} f[x_i, x_{i+1}],$$

where $\lambda_i = h_i/(h_{i-1}+h_i)$, $\mu_i = 1-\lambda_i$, $i = 1, \ldots, n-1$.

The illustration in coordinates $(\overline{m}_i, \overline{m}_{i+1})$ of the regions ensuring the property of monotonicity for this type of *VP*-splines is given in Figures 1–3. Each coordinate is varying equally from zero to 10. In Figure 1, the lines corresponding to the values $p_i = q_i = 3, 4, \ldots, 10$ are shown. The elliptic curve corresponding to the case of the classic cubic spline tangent to the axes at the points (0,3) and (3,0) is obtained for $p_i = q_i = 3$.

In the symmetric case $p_i = q_i$ (see Figure 1), the character of insertion of the regions is monotone as $p_i$, $q_i$ increase. In Figure 2, $p_i = 3, 4, \ldots, 10$, $q_i = 3$. In Figure 3, $p_i = 3, 4, \ldots, 10$, $q_i = 7$.

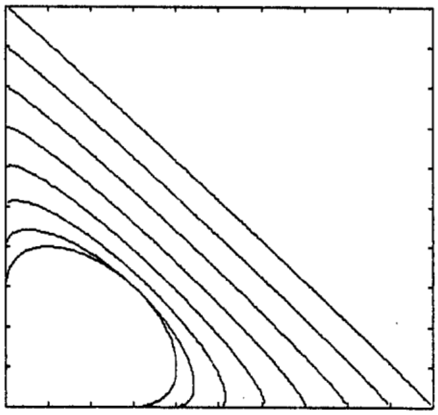

**Figure 1**

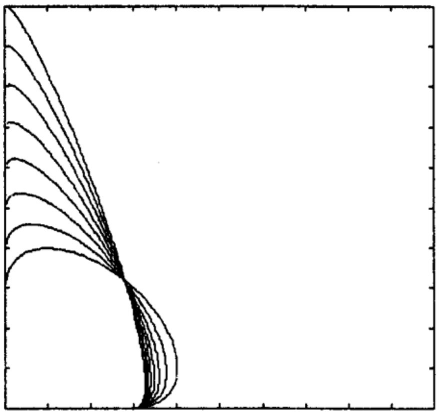

**Figure 2**

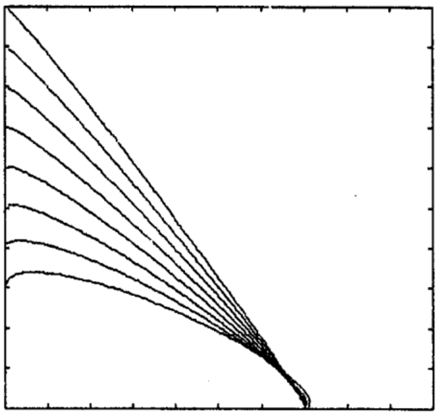

**Figure 3**

Another nonlocal construction of *VP*-splines – *VP*-spline with the equilibrium upon the second derivative was suggested by V.L. Miroshnichenko in 1997 [4]. The generating functions there have the form

$$\varphi_i(t) = \frac{t^{p_i}}{1 + A_i(1-t)^2}, \qquad A_i = \frac{p_i - 2}{2}\left(p_i - \frac{q_i}{q_i - 2}\right),$$
$$\psi_i(t) = \frac{(1-t)^{q_i}}{1 + B_i t^2}, \qquad B_i = \frac{q_i - 2}{2}\left(q_i - \frac{p_i}{p_i - 2}\right),$$
$$q_i \geq 3, \quad p_i \geq 3.$$

In Figure 4–5, we illustrate the regions of monotonicity for *VP*-spline with the equilibrium upon the second derivative. It is interesting that here even in the symmetric case the character of increasing of the regions is not monotone for small values of $p_i, q_i$.

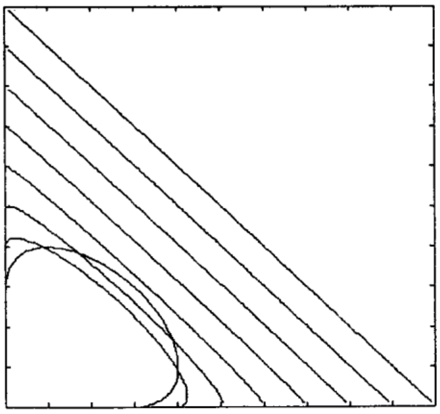

**Figure 4**

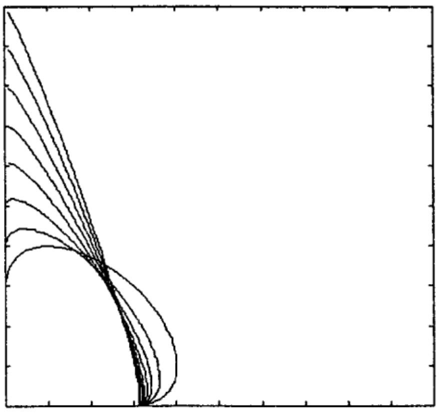

**Figure 5**

The regions of the monotonicity of all these splines for all permissible control parameters $p_i$, $q_i$ are convex. Moreover, they may be approximated by the triangles almost without essential loosing of the suitable values. So, the algorithm for choosing the suitable parameters ensuring the monotonicity on the whole interval became mostly simple.

## References

[1] Fritch F.N., Carlson R.E. Monotone piecewise cubic interpolation // SIAM J. Numer. Anal. – 1980. – Vol. 17, № 2. – P. 238–246.

[2] Shmidt J.W., Hess W. Schwach vercoppelte Ungleichungssysteme und konvexe Spline-Interpolation // El. Math. – 1984. – Vol. 39. – P. 85–95.

[3] Zav'yalov Ju.S., Bogdanov V.V. Monotone and convex Hermite interpolation by generalized cubic splines // Siberian Journal of Computer Mathematics. – 1993. – Vol. 2, № 1. – P. 1–30.

[4] Miroshnichenko V.L. Optimization of the form of the rational spline // Spline Functions and their Applications. – Novosibirsk, 1997. – № 159: Vychislitelnye sistemy. – P. 87–109 (in Russian).