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Fictitious domain method for fourth-order elliptic problem*

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In this paper, we study the convergence of the fictitious domain method for solving a system of grid equations for the finite element method that approximates the third boundary problem for the differential equation $\Delta^2 u + au = f$ in the piecewise bicubic Hermit interpolations subspace of $W_2^2(\Omega)$ on a rectangular grid. The main operation on each step of the method is a double inversion of the FEM operator for the trivial boundary value Dirichlet problem for the Poisson equation in a rectangle, which interior includes the rectangle composed domain Ω. For this purpose an inner iterational process is built. The speed of the convergence of two-level iterative method does not depend on the grid parameter, so this method gives the solution with $O(h^{-2}(\ln h^{-1})^2 \ln \varepsilon^{-1})$ arithmetic operations, where ε is an accuracy of the solution.

1. Definition and convergence of the fictitious domain method

Let us consider the fourth-order equation

$$Lu \equiv \Delta^2 u + a\,u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + a\,u = f, \qquad (1.1)$$
$$a > 0, \quad f \in L_2(\Omega),$$

in the domain $\Omega \subset R^2$ and boundary conditions

$$\begin{aligned}&\Big[\Delta u + (1-\sigma)\Big(2n_1n_2\frac{\partial^2 u}{\partial x\partial y} - n_2^2\frac{\partial^2 u}{\partial x^2} - n_1^2\frac{\partial^2 u}{\partial y^2}\Big)\Big]_{\partial\Omega} = 0,\\ &\Big[\frac{\partial}{\partial n}\Delta u + (1-\sigma)\frac{\partial}{\partial s}\Big(n_1n_2\Big(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\Big) + (n_1^2 - n_2^2)\frac{\partial^2 u}{\partial x\partial y}\Big)\Big]_{\partial\Omega} = 0,\end{aligned} \qquad (1.2)$$

where $\sigma \in (0, 1)$, $n = (n_1, n_2)$ is the vector of outward normal and $s = (n_2, -n_1)$ is the tangent.

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The projective definition of this problem has a form of [1]:

$$u \in W_2^2(\Omega): \quad M_\Omega(u,v) = f_\Omega(v) \equiv \int_\Omega f\,v\,d\Omega \qquad \forall\, v \in W_2^2(\Omega), \tag{1.3}$$

where

$$\begin{aligned} M_\Omega(u,v) &\equiv \Delta_\Omega(u,v) + (1-\sigma)L_\Omega(u,v), \\ \Delta_\Omega(u,v) &= \int_\Omega \Delta u \Delta v\, d\Omega, \\ L_\Omega(u,v) &= \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy} + a\,uv)\, d\Omega. \end{aligned} \tag{1.4}$$

Following a strategy of [2, 3] we are going to build a fictitious component method for solution of (1.3).

Let the open rectangle $\Pi \subset R^2$ include the closure of Ω. Now we define functional spaces

$$\begin{aligned} H &= W_2^2(\Pi) \cap \overset{\circ}{W}{}_2^1(\Pi), \\ H_{0\Omega} &= \{v \in H: \ v(x,y) = 0 \quad \forall\,(x,y) \in \Omega\}, \\ H_{1,M} &= \{v \in H: \ M_\Pi(v,w) = 0 \quad \forall\, w \in H_{0\Omega}\}. \end{aligned} \tag{1.5}$$

We define inner products $M_\Omega(u,v)$ and $M_\Pi(u,v)$ in the spaces $W_2^2(\Omega)$ and H respectively. Let us assume that there is bounded linear operators of the function prolongation from Ω to $\Pi \setminus \Omega$:

$$p_r : W_2^2(\Omega) \to H, \tag{1.6}$$

and of the function restriction from $\Pi \setminus \Omega$ to Ω:

$$r_\Omega : H \to W_2^2(\Omega). \tag{1.7}$$

Let $\Delta_\Pi(u,v)$ be an inner product in H and $M_\Omega(r_\Omega u, r_\Omega v)$ be an inner product in $H_{1,M}$, so we note that

$$\begin{aligned} \|p_r\|^{-2} M_\Pi(v,v) \le M_\Omega(r_\Omega v, r_\Omega v) \le M_\Pi(v,v) \quad &\forall\, v \in H_{1,M}, \\ c_0 \Delta_\Pi(v,v) \le M_\Pi(v,v) \le c_1 \Delta_\Pi(v,v) \quad &\forall\, v \in H. \end{aligned} \tag{1.8}$$

Statement 1. *The sequence* $\{r_\Omega u^k\}$ *of iterative process*

$$\begin{aligned} u^{k+1} \in H: \ &\forall\, v \in H \\ &\Delta_\Pi(u^{k+1} - u^k, v) = -\tau[M_\Omega(r_\Omega u^k, r_\Omega v) - f_\Omega(r_\Omega v)] \end{aligned} \tag{1.9}$$

converges to solution of (1.3) *when* $\tau \in (0, 2/c_1)$.

Proof. We denote an orthogonal projector from H to $H_{1,M}$ with respect to the inner product $M_\Pi(v,w)$ by $P_{1,M}$. Then $\forall\, u,v \in H$

$$\begin{aligned} M_\Omega(r_\Omega v, r_\Omega w) &= M_\Omega(r_\Omega P_{1,M}v, r_\Omega P_{1,M}w) = M_\Pi(AP_{1,M}v, P_{1,M}w) \\ &= M_\Pi([A^{1/2}P_{1,M}]v, [A^{1/2}P_{1,M}]w), \end{aligned}$$

where the operator $A : H_{1,M} \to H_{1,M}$ is self-adjoint and positive definite one.

Further, the bilinear form $M_\Pi(v, w)$ defines a self-adjoint and positive definite in the inner product $\Delta_\Pi(v, w)$ operator $B : H \to H$:

$$M_\Pi(v, w) = \Delta_\Pi(Bv, w).$$

Then $\forall\, v, w \in H$

$$\begin{aligned} M_\Omega(r_\Omega v, r_\Omega w) &= M_\Pi([A^{1/2}P_{1,M}]v, [A^{1/2}P_{1,M}]w) \\ &= \Delta_\Pi([A^{1/2}P_{1,M}]^* B[A^{1/2}P_{1,M}]v, w) = \Delta_\Pi(Tv, w), \end{aligned}$$

where $T : H \to H$ is a self-adjoint and positive semi-defined in $\Delta_\Pi(v, w)$ operator and $\ker T = H_{0\Omega}$.

So, the iterative process has an operator form

$$u^{k+1} - u^k = -\tau T(u^k - u), \tag{1.10}$$

and to research its convergence we should note that from (1.8) it follows that

$$\Delta_\Pi(Tv, v) = M_\Omega(r_\Omega v, r_\Omega v) \le M_\Pi(v, v) \le c_1 \Delta_\Pi(v, v) \quad \forall\, v \in H. \tag{1.11}$$

Introduce a subspace

$$H_{1,\Delta} = \{v \in H : \ \Delta_\Pi(v, w) = 0 \ \ \forall\, w \in H_{0\Omega}\} = \operatorname{Im} T,$$

consisting of functions $v \in W_2^2(\Omega)$ prolonged to $\Pi \setminus \Omega$ with the minimal norm $\|v\|_\Delta = \sqrt{\Delta_\Pi(v, v)}$. If we denote by $P_{1,\Delta}$ the orthogonal projector from H to $H_{1,\Delta}$ with respect to the inner product $\Delta_\Pi(v, w)$, then from (1.8) it follows that $\forall\, v \in H$

$$\begin{aligned} \Delta_\Pi(Tv, v) = M_\Omega(r_\Omega v, r_\Omega v) &\ge \|p_r\|^{-2} M_\Pi(P_{1,M}v, P_{1,M}v) \\ &\ge \|p_r\|^{-2} c_0 \Delta_\Pi(P_{1,M}v, P_{1,M}v) \\ &\ge \|p_r\|^{-2} c_0 \Delta_\Pi(P_{1,\Delta}v, P_{1,\Delta}v). \end{aligned} \tag{1.12}$$

Let us now rewrite the iterative process (1.11) in the form

$$\begin{aligned} (u^{k+1})_{0,\Omega} - (u^k)_{0,\Omega} &= 0, \\ (u^{k+1})_{1,\Delta} - (u^k)_{1,\Delta} &= -\tau T[(u^k)_{1,\Delta} - (u)_{1,\Delta}], \end{aligned} \tag{1.13}$$

where new designations have been introduced: $(v)_{1,\Delta} = P_{1,\Delta}v$, $(v)_{0,\Omega} = v - P_{1,\Delta}v \in H_{0,\Omega}$. From (1.11) and (1.12) it follows that in $H_{1,\Delta}$

$$\|E - \tau T\|_\Delta \le \max\Big\{\Big|1 - \tau\|p_\tau\|^{-2}c_0\Big|,\ \Big|1 - \tau c_1\Big|\Big\} = q_\tau < 1 \qquad (1.14)$$

when $\tau \in (0, 2/c_1)$. Therefore, due to $r_\Omega v = r_\Omega P_{1,\Delta} v \ \ \forall\, v \in H$, the following formulae hold true:

$$\Big\|(u^k)_{1,\Delta} - (u)_{1,\Delta}\Big\|_\Delta \le q_\tau^k \Big\|(u^0)_{1,\Delta} - (u)_{1,\Delta}\Big\|_\Delta \to 0,$$

$$\Big\|r_\Omega u^k - r_\Omega u\Big\|_{M_\Omega} \le \Big\|(u^k)_{1,\Delta} - (u)_{1,\Delta}\Big\|_{M_\Pi} \le c_1 q_\tau^k \Big\|(u^0)_{1,\Delta} - (u)_{1,\Delta}\Big\|_\Delta \to 0.$$

The iterative process (1.9) is called *the fictitious domain method for solving* (1.3). □

2. Fictitious domain method for FEM scheme

Let us assume that Ω is composed of rectangles and it is possible to make a square grid Ω_h on Ω with a step h. Moreover, we assume that it is possible to extend this grid to a square grid Π_h on the rectangle Π. We denote by H_h a subspace of H which consists of all functions v continuous in Ω with their derivatives v_x, v_y, v_{xy} and bicubic in every rectangle of Π_h. The finite element method for (1.3) is formulated as follows:

$$u_h \in H_h : \ \ M_\Omega(r_\Omega u_h, r_\Omega v) = f_\Omega(r_\Omega v) \equiv \int_\Omega f\, r_\Omega v \, d\Omega \quad \forall\, v \in H_h. \qquad (2.1)$$

The iterative process (1.9) is approximated by the process

$$\begin{aligned} u_h^{k+1} \in H_h : \ \ & \forall\, v \in H_h \\ & \Delta_\Pi(u_h^{k+1} - u_h^k, v) = -\tau[M_\Omega(r_\Omega u_h^k, r_\Omega v) - f_\Omega(r_\Omega v)]. \end{aligned} \qquad (2.2)$$

The proof of the convergence of the sequence $\{r_\Omega u_h^k\}$ to $r_\Omega u_h$ is analogous to the proof of Statement 1, so the speed of the convergence does not depend on h. The main operation of the fictitious domain method (2.2) is to solve problems like

$$v_h \in H_h : \ \ \Delta_\Pi(v_h, w) = g(w) \quad \forall\, w \in H_h. \qquad (2.3)$$

We assign each function $v \in H_h$ to the vector $\bar v \in R^{N(h)}$ of values of the function and its derivatives v_x, v_y, v_{xy} in grid nodes of Π_h excluding values of the function and its tangent derivative in the boundary points because they are equal to zero. We define symmetric matrices and a vector of the dimension $N(h)$:

$$M_\Omega(r_\Omega u_h, r_\Omega v) = (M_h \bar v,\, \bar w), \quad f_\Omega(r_\Omega w) = (\bar f,\, \bar w), \quad \Delta_\Pi(v, w) = (B_h \bar v,\, \bar w),$$

$$\nabla_\Pi(v, w) \equiv \int_\Pi \nabla v \nabla w \, d\Pi = (D_h \bar v,\, \bar w),$$

$$(v, w)_0 \equiv \int_\Pi v w \, d\Pi = (E_h \bar v,\, \bar w), \qquad \forall\, v, w \in H_h. \qquad (2.4)$$

Theorem 1. *There exist positive constants γ_0 and γ_1 independent on h such that for all $\bar{v} \in R^{N(h)}$ the following inequality is true:*

$$\gamma_0(D_h[E_h]^{-1}D_h\bar{v},\, \bar{v}) \le (B_h\bar{v},\, \bar{v}) \le \gamma_1(D_h[E_h]^{-1}D_h\bar{v},\, \bar{v}). \qquad (2.5)$$

From (2.5) it follows that the iterative process (2.2), a matrix form of which is

$$B_h(\bar{u}^{k+1} - \bar{u}^k) = -\tau(M_h\bar{u}^k - \bar{f}), \qquad (2.6)$$

can be changed to the iterative method

$$D_h[E_h]^{-1}D_h(\bar{u}^{k+1} - \bar{u}^k) = -\tau(M_h\bar{u}^k - \bar{f}), \qquad (2.7)$$

which converges when $\tau \in (0, 2\gamma_0/c_1)$ and the convergence speed of which does not depend on h. The main operation of the method (2.7) is the solution of the variational-differences Dirichlet problem for the Poisson equation. Therefore, the rectangle Π is divided into square cells by grid lines $x = x_i$, $i = 0, 1, \dots, n+1$, and $y = y_j$, $j = 0, 1, \dots, m+1$. We define the polynomials $\varphi(t) = (1-t)^3 + 3(1-t)^2 t$, $\psi(t) = (1-t)^2 t$, and the functions [1]

$$\varphi_i^{(x)}(x) = \begin{cases} \varphi([x - x_i]/h), & x_i \le x \le x_{i+1}, \\ \varphi([x_i - x]/h), & x_{i-1} \le x \le x_i, \\ 0, & x \notin (x_{i-1},\, x_{i+1}); \end{cases}$$

$$\psi_i^{(x)}(x) = \begin{cases} h\,\psi([x - x_i]/h), & x_i \le x \le x_{i+1}, \\ -h\,\psi([x_i - x]/h), & x_{i-1} \le x \le x_i, \\ 0, & x \notin (x_{i-1},\, x_{i+1}). \end{cases}$$

The functions $\varphi_j^{(y)}(y)$ and $\psi_j^{(y)}(y)$, $j = 0, 1, \dots, m+1$, are to be defined analogously.

3. Equivalence of matrices $D_h[E_h]^{-1}D_h$ and B_h in one-dimensional case

We assign $v(x)$ of

$$H_h^{(x)} = \mathrm{Span}\{\varphi_1^{(x)}, \varphi_2^{(x)}, \dots, \varphi_n^{(x)}, \psi_0^{(x)}, \psi_1^{(x)}, \dots, \psi_n^{(x)}, \psi_{n+1}^{(x)}\}$$

to a vector $\bar{v} = (v_1, v_2, \dots, v_n, v_0', v_1', \dots, v_n', v_{n+1}')^T$, where $v_i = v(x_i)$, $v_i' = v'(x_i)$. Define the one-dimensional analogs $E_h^{(x)}$, $D_h^{(x)}$, and $M_h^{(x)}$ of the matrices E_h, D_h, and B_h:

$$
\begin{aligned}
(E_h^{(x)}\bar{v},\bar{w}) &= \int_{x_0}^{x_{n+1}} v(x)w(x)\,dx, \\
(D_h^{(x)}\bar{v},\bar{w}) &= \int_{x_0}^{x_{n+1}} v'(x)w'(x)\,dx, \qquad\qquad (3.1)\\
(B_h^{(x)}\bar{v},\bar{w}) &= \int_{x_0}^{x_{n+1}} v''(x)w''(x)\,dx, \qquad \forall\, v,w\in H_h^{(x)},
\end{aligned}
$$

where $(\bar{v},\bar{w}) = \sum_{i=0}^{n+1}[v_iw_i + v_i'w_i']$ is an inner product in R^{2n+2} ($v_0 = v_{n+1} = 0$).

Statement 2. *For each vector $\bar{v}\in R^{2n+2}$ the following equations are true:*

$$
(E_h^{(x)}\bar{v},\,\bar{v}) = \frac{h}{420}\sum_{i=1}^{n+1}\left(\begin{bmatrix}156 & 54 & 22 & -13\\ 54 & 156 & 13 & -22\\ 22 & 13 & 4 & -3\\ -13 & -22 & -3 & 4\end{bmatrix}\begin{bmatrix}v_{i-1}\\ v_i\\ hv_{i-1}'\\ hv_i'\end{bmatrix},\begin{bmatrix}v_{i-1}\\ v_i\\ hv_{i-1}'\\ hv_i'\end{bmatrix}\right), \qquad (3.2)
$$

$$
c_{0,E}(I_h^{(x)}\bar{v},\bar{v}) \le (E_h^{(x)}\bar{v},\bar{v}) \le c_{1,E}(I_h^{(x)}\bar{v},\bar{v}), \qquad (3.3)
$$

where $I_h^{(x)} = \mathrm{diag}\{2h,\dots,2h,h^3,2h^3,\dots,2h^3,h^3\}$, $c_{0,E}\approx 0.2/420$, $c_{1,E}\approx 216/420$;

$$
(D_h^{(x)}\bar{v},\,\bar{v}) = \frac{1}{60h}\sum_{i=1}^{n+1}\left(\begin{bmatrix}72 & 3\sqrt{2} & 3\sqrt{2}\\ 3\sqrt{2} & 4 & -1\\ 3\sqrt{2} & -1 & 4\end{bmatrix}\begin{bmatrix}v_{i-1}-v_i\\ \sqrt{2}hv_{i-1}\\ \sqrt{2}hv_i\end{bmatrix},\begin{bmatrix}v_{i-1}-v_i\\ \sqrt{2}hv_{i-1}\\ \sqrt{2}hv_i\end{bmatrix}\right), \qquad (3.4)
$$

$$
(B_h^{(x)}\bar{v},\,\bar{v}) = \sum_{i=1}^{n+1}\left\{\frac{(v_{i-1}'-v_i')^2}{h} + 3\left[\frac{v_{i-1}'+v_i'}{\sqrt{h}} - 2\frac{v_i-v_{i-1}}{h\sqrt{h}}\right]^2\right\}. \qquad (3.5)
$$

These equations follow from definitions of matrices (3.2). From (3.3) it follows that matrices $D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}$ and $D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}$ are spectral equivalence:

$$
\begin{aligned}
&c_{0,E}\Big(D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\,\bar{v}\Big)\\
&\qquad\le \Big(D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\,\bar{v}\Big) \le c_{1,E}\Big(D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\,\bar{v}\Big), \qquad (3.6)
\end{aligned}
$$

so we will proof the equivalence of matrices $B_h^{(x)}$ and $D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}$. Let

$$
\begin{aligned}
r_i &= \frac{v_{i-1}'-v_i'}{h} \qquad t_i = \frac{v_{i-1}'+v_i'}{h} - 2\frac{v_i-v_{i-1}}{h^2},\\
\alpha_i &= \frac{3t_i-5r_i}{\sqrt{h}}, \quad \beta_i = \frac{3t_i+5r_i}{\sqrt{h}}, \quad i=1,\dots,n+1.
\end{aligned}
$$

From equations of Statement 2 it follows that

$$(B_h^{(x)}\bar{v},\bar{v}) = \sum_{i=1}^{n+1}(r_i^2 + 3\,t_i^2)h = \frac{1}{25}\sum_{i=1}^{n+1}[0.3(\alpha_i^2+\beta_i^2) + 1.1(\alpha_i+\beta_i)^2], \quad (3.7)$$

$$30^2(D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\bar{v}) = \frac{9}{4}\sum_{i=1}^{n}(3\alpha_{i+1} - \alpha_i + \beta_{i+1} - 3\beta_i)^2 + \frac{1}{4}\alpha_1^2 + \frac{1}{4}\sum_{i=1}^{n}(\alpha_{i+1}+\beta_i)^2 + \frac{1}{4}\beta_{n+1}^2. \quad (3.8)$$

It is easy to check the following identity:

$$\sum_{i=1}^{n}(3\alpha_{i+1} - \alpha_i + \beta_{i+1} - 3\,\beta_i)^2 + 18\alpha_1^2 + 9\sum_{i=1}^{n}(\alpha_{i+1}+\beta_i)^2 + 18\beta_{n+1}^2 \equiv 12(\alpha_1^2+\beta_1^2) + 6\sum_{i=2}^{n}(\alpha_i^2+\beta_i^2) + 12(\alpha_{n+1}^2+\beta_{n+1}^2) + \Phi^2, \quad (3.9)$$

where

$$\Phi^2 = \sum_{i=1}^{n}(\beta_{i+1}-\alpha_i)^2 + 3(\alpha_1+\beta_1)^2 + 6\sum_{i=2}^{n}(\alpha_i+\beta_i)^2 + 3(\alpha_{n+1}+\beta_{n+1})^2 + 3\sum_{i=2}^{n}[(\alpha_{i+1}-\alpha_i)^2 + (\beta_{i+1}-\beta_i)^2].$$

From (3.7) it follows that

$$\frac{3}{250}\sum_{i=1}^{n+1}(\alpha_i^2+\beta_i^2) \le (M_h^{(x)}\bar{v},\bar{v}) \le \frac{1}{10}\sum_{i=1}^{n+1}(\alpha_i^2+\beta_i^2).$$

From (3.8) and (3.9) we have

$$\frac{1}{12}\sum_{i=1}^{n+1}(\alpha_i^2+\beta_i^2) \le 30^2(D_h^{(x)}[I_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\bar{v}) \le 180\sum_{i=1}^{n+1}(\alpha_i^2+\beta_i^2).$$

From these inequalities and (3.6) it follows

Theorem 2. *There exist positive constants γ_0 and γ_1 independent of h such that*

$$\gamma_0 \le \frac{(B_h^{(x)}\bar{v},\bar{v})}{(D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}\bar{v},\bar{v})} \le \gamma_1 \quad \forall\,\bar{v}\in R^{N(h)},\ \ \bar{v}\ne 0, \quad (3.10)$$

and $\gamma_0 \ge \frac{7}{60}$, $\gamma_1 \le 2268000$.

Remark. Estimations for γ_0 and γ_1 obtained in this section are very rough. From the experiment it follows that $\gamma_0 = 1$, $\gamma_1 = 1.45$.

Denote by $E_h^{(y)}$, $D_h^{(y)}$, and $B_h^{(y)}$ the matrices of $(2m+1)$-th order that are one-dimensional (with respect to y) analogs of the matrices E_h, D_h, and B_h. Obviously that all the results of this section are also valid for them.

4. Spectral equivalence of $D_h[E_h]^{-1}D_h$ and B_h

Each function $v(x,y) \in H_h$:

$$\begin{aligned} v(x,y) = & \sum_{j=1}^{m}\sum_{i=1}^{n} v_{i,j}\,\varphi_i^{(x)}\varphi_j^{(y)} + \sum_{j=1}^{m}\sum_{i=0}^{n+1}(v_x)_{i,j}\psi_i^{(x)}\varphi_j^{(y)} + \\ & \sum_{i=1}^{n}\sum_{j=0}^{m+1}(v_y)_{i,j}\varphi_i^{(x)}\psi_j^{(y)} + \sum_{i=0}^{n}\sum_{j=0}^{m+1}(v_{xy})_{i,j}\psi_i^{(x)}\psi_j^{(y)} \end{aligned}$$

corresponds to a vector of the dimension $(2m+2)(2n+2)$:

$$\bar{v} = \left[\{\bar{v}_1^{(x)}\}^T, \ldots, \{\bar{v}_m^{(x)}\}^T, \{\overline{(v_y)}_0^{(x)}\}^T, \{\overline{(v_y)}_1^{(x)}\}^T, \ldots, \{\overline{(v_y)}_{m+1}^{(x)}\}^T\right]^T,$$

where

$$\begin{aligned} \bar{v}_j^{(x)} &= [v_{1,j}, \ldots, v_{n,j}, (v_x)_{0,j}, (v_x)_{1,j}, \ldots, (v_x)_{n+1,j}]^T, \\ \overline{(v_y)}_j^{(x)} &= [(v_y)_{1,j}, \ldots, (v_y)_{n,j}, (v_{yx})_{0,j}, (v_{yx})_{1,j}, \ldots, (v_{yx})_{n+1,j}]^T. \end{aligned}$$

We introduce a matrix of the dimension $2n+2$

$$Q_n^{(x,k)} = \left[\begin{array}{c|c} (\varphi_i^{(x)}, \varphi_j^{(x)})_k\Big|_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} & (\varphi_i^{(x)}, \psi_j^{(x)})_k\Big|_{\substack{i=1,\ldots,n \\ j=0,\ldots,n+1}} \\ \hline (\psi_i^{(x)}, \varphi_j^{(x)})_k\Big|_{\substack{i=0,\ldots,n+1 \\ j=1,\ldots,n}} & (\psi_i^{(x)}, \psi_j^{(x)})_k\Big|_{\substack{i=0,\ldots,n+1 \\ j=0,\ldots,n+1}} \end{array}\right],$$

where

$$(v^{(x)}, w^{(x)})_k = \int_{x_0}^{x_{n+1}} \frac{d^k v^{(x)}(x)}{dx^k} \cdot \frac{d^k w^{(x)}(x)}{dx^k}\, dx.$$

Then

$$\begin{aligned} E_h^{(x)} &= Q_n^{(x,0)}, & D_h^{(x)} &= Q_n^{(x,1)}, & B_h^{(x)} &= Q_n^{(x,2)}, \\ E_h^{(y)} &= Q_m^{(y,0)}, & D_h^{(y)} &= Q_m^{(y,1)}, & B_h^{(y)} &= Q_m^{(y,2)}. \end{aligned}$$

$$\begin{aligned}
E_h &= E_h^{(y)} \otimes E_h^{(x)}, \\
D_h &= (E_h^{(y)} \otimes D_h^{(x)}) + (D_h^{(y)} \otimes E_h^{(x)}), \\
B_h &= (E_h^{(y)} \otimes B_h^{(x)}) + 2(D_h^{(y)} \otimes D_h^{(x)}) + (B_h^{(y)} \otimes E_h^{(x)}), \qquad (4.1)\\
D_h[E_h]^{-1}D_h &= (E_h^{(y)} \otimes D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}) + 2(D_h^{(y)} \otimes D_h^{(x)}) + \\
&\quad (D_h^{(y)}[E_h^{(y)}]^{-1}D_h^{(y)} \otimes E_h^{(x)}),
\end{aligned}$$

where $\otimes$ is a matrix tensor product symbol ([4]).

Statement 3.

$$\mathrm{Sp}\{(D_h[E_h]^{-1}D_h)^{-1}B_h\} \in [\gamma_0, \gamma_1], \qquad (4.2)$$

where the constants γ_0, γ_1 *are spectral bounds of the matrix*

$$(D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)})^{-1}B_h^{(x)}$$

from (3.10).

Proof. Let $\bar{v}$ be an eigenvector and λ be an eigenvalue of the matrix $(D_h[E_h]^{-1}D_h)^{-1}B_h$. Then from (3.10) it follows that

$$\begin{aligned}
\lambda &= \frac{(B_h\bar{v}, \bar{v})}{(D_h[E_h]^{-1}D_h\bar{v}, \bar{v})} \\
&\le \max\left\{\frac{([E_h^{(y)} \otimes B_h^{(x)}]\bar{v}, \bar{v})}{([E_h^{(y)} \otimes D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}]\bar{v}, \bar{v})}, \frac{([B_h^{(y)} \otimes E_h^{(x)}]\bar{v}, \bar{v})}{([D_h^{(y)}[E_h^{(y)}]^{-1}D_h^{(y)} \otimes E_h^{(x)}]\bar{v}, \bar{v})}, 1\right\} \\
&\le \max\left\{\rho([D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}]^{-1}B_h^{(x)}, 1\right\} \le \gamma_1.
\end{aligned}$$

The lower estimation of λ is proved similarly:

$$\lambda \ge \lambda_{\min}([D_h^{(x)}[E_h^{(x)}]^{-1}D_h^{(x)}]^{-1}B_h^{(x)}) \ge \gamma_0. \qquad \square$$

This statement finishes the proof of Theorem 1.

5. Inversion of the matrix D_h

We introduce a matrix

$$\tilde{D}_h = (\tilde{E}_h^{(y)} \otimes \tilde{D}_h^{(x)}) + (\tilde{D}_h^{(y)} \otimes \tilde{E}_h^{(x)}), \qquad (5.1)$$

where

$$\tilde{E}_h^{(x)} = h \begin{bmatrix} 112E_n & 0 \\ 0 & h^2\widehat{E}_{n+2} \end{bmatrix}, \qquad \tilde{E}_h^{(y)} = h \begin{bmatrix} 112E_m & 0 \\ 0 & h^2\widehat{E}_{m+2} \end{bmatrix},$$
$$\tilde{D}_h^{(x)} = \frac{1}{h} \begin{bmatrix} 0.3A_n & 0 \\ 0 & \frac{h^2}{30}\widehat{E}_{n+2} \end{bmatrix}, \qquad \tilde{D}_h^{(y)} = \frac{1}{h} \begin{bmatrix} 0.3A_m & 0 \\ 0 & \frac{h^2}{30}\widehat{E}_{m+2} \end{bmatrix}, \tag{5.2}$$

E_k is the identity matrix of the dimension k, $\widehat{E}_{k+2} = \mathrm{diag}\{1, 2, \ldots, 2, 1\}$ is a diagonal matrix of the dimension $k+2$, A_k is a tridiagonal matrix of the dimension k: $(A_k)_{i,i} = 2$, $(A_k)_{i,i+1} = (A_k)_{i+1,i} = -1$.

Statement 4. *The matrix D_h is spectral equivalent to $\tilde{D}_h$:*

$$d_0(\tilde{D}_h\bar{v}, \bar{v}) \le (D_h\bar{v}, \bar{v}) \le d_1(\tilde{D}_h\bar{v}, \bar{v}) \quad \forall\, \bar{v} \in R^{(2n+2)(2m+2)}, \tag{5.3}$$

where $d_0 \ge 0.56/420$, $d_1 \le 103/420$.

Proof. One can prove that

$$\begin{aligned} e_0(\tilde{E}_h^{(x)}\bar{v}, \bar{v}) &\le (E_h^{(x)}\bar{v}, \bar{v}) \le e_1(\tilde{E}_h^{(x)}\bar{v}, \bar{v}) \quad \forall\, \bar{v} \in R^{2n+2}, \\ e_0(\tilde{E}_h^{(y)}\bar{v}, \bar{v}) &\le (E_h^{(y)}\bar{v}, \bar{v}) \le e_1(\tilde{E}_h^{(y)}\bar{v}, \bar{v}) \quad \forall\, \bar{v} \in R^{2m+2}, \\ 2(\tilde{D}_h^{(x)}\bar{v}, \bar{v}) &\le (D_h^{(x)}\bar{v}, \bar{v}) \le 5(\tilde{D}_h^{(x)}\bar{v}, \bar{v}) \quad \forall\, \bar{v} \in R^{2n+2}, \\ 2(\tilde{D}_h^{(y)}\bar{v}, \bar{v}) &\le (D_h^{(y)}\bar{v}, \bar{v}) \le 5(\tilde{D}_h^{(y)}\bar{v}, \bar{v}) \quad \forall\, \bar{v} \in R^{2m+2}, \end{aligned}$$

where $e_0 \approx 0.14/420$ and $e_1 \approx 10.3/420$. So, from the properties of the tensor product of matrices ([4]) we can obtain inequality (5.3). □

Statement 5. *The exact solution to a system with the matrix $\tilde{D}_h$ can be obtained for $O(h^{-2}\ln h^{-1})$ arithmetic operations.*

Proof. From the careful analysis of the system $\tilde{D}_h\bar{v} = \bar{g}$ it follows that its solution can be computed by performing the following steps:

1. To solve the 5-points finite-differences scheme with constant coefficients by, for example, the cyclic reduction method for $O(h^{-2}\ln h^{-1})$ operations [5, 6];

2. To solve $n+2$ systems with tridiagonal matrices of m dimension and $m+2$ systems with tridiagonal matrices of n dimension (by the sweep method for $O(h^{-2})$ operations [5, 6]);

3. To compute two multiplications of $(2n+2)(2m+2)$ dimensional vectors and diagonal matrices (for $O(h^{-2})$ operations). □

Statement 6. *The solution to the system $D_h\bar{v} = \bar{g}$ can be found with an accuracy ε in the energetic norm for $O(h^{-2}\ln h^{-1}\ln\varepsilon^{-1})$ arithmetic operations by the iterative method*

$$\begin{aligned} &\tilde{D}_h(\bar{v}^{k+1} - \bar{v}^k) = -\tau(D_h\bar{v}^k - \bar{g}), \quad \bar{v}^0 = 0, \\ &\bar{v}^N = (D_h^{-1} - T_N)g, \quad N = O(\ln\varepsilon^{-1}), \\ &T_N = \tilde{D}_h^{-1/2}[\tilde{D}_h^{-1/2}D_h\tilde{D}_h^{-1/2}]^{-1}(E - \tau[\tilde{D}_h^{-1/2}D_h\tilde{D}_h^{-1/2}])^N\tilde{D}_h^{-1/2}. \end{aligned} \tag{5.4}$$

This statement is a corollary of Statements 4 and 5.

Statement 7. *There exists $N = O(\ln h^{-1})$ such that $\forall\, \bar{v} \in R^{(2n+2)(2m+2)}$*

$$0.5([D_h^{-1} - T_N]\bar{v}, \bar{v}) \le (D_h^{-1}\bar{v}, \bar{v}) \le 1.5([D_h^{-1} - T_N]\bar{v}, \bar{v}), \tag{5.5}$$

From this statement and Theorem 1 follows

Theorem 3. *The solution to system $M_h\bar{u}^k = \bar{f}$ (the matrix formulating of (2.1)) can be found with precision ε for $O(h^{-2}(\ln h^{-1})^2\ln\varepsilon^{-1})$ arithmetic operations by two-level iterative method* ([7])

$$\bar{u}^{k+1} = \bar{u}^k - \tau_k[D_h^{-1} - T_N]E^h[D_h^{-1} - T_N](M_h\bar{u}^k - \bar{f}), \tag{5.6}$$

where the parameters τ_k can be chosen on variational principles [8].

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