Bull. Nov. Comp. Center, Num. Model. in Atmosph., etc., 6 (2000), 1–8

Model of dynamics of the atmosphere with monotone numerical schemes*

E.A. Boyarshinova

On the base of integral identity numerical schemes, monotonic and conditional monotonic schemes, for the solution of a three-dimensional problem of thermodynamics of the atmosphere have been constructed. The models have been realized with respect to these two schemes, as well the numerical research of qualities of schemes has been made. The numerical experiments showed that the schemes worked satisfactorily in the main.

One of the most productive ideas, used at the construction of algorithms for the realization of three-dimensional problems of thermodynamics, is the method of splitting. In works [1–3], a series of finite difference schemes of numerical realization is developed on the base of this method in a combination with a variational approach for the problems of mesometeorology.

In the work by V.V. Penenko [4], a method of construction of monotonic finite difference schemes is offered for advective–diffusion equations on the base of the variation difference approach in a combination with the method of weighed discrepancies.

By this method finite difference approximations were constructed for advective–diffusive operators at the stage of convection-diffusion and in the problem of dynamic adjustment of meteofields, where on the basis of the continuity equation a second-order equation for pressure is obtained.

A three-dimensional nonhydrostatic model of local atmospheric motions above a complicated relief numerically was considered. The model was given by the system of equations [2, 3]:

$$
\begin{aligned}
\frac{\partial u'}{\partial t} + \bar{u}\,\mathrm{grad}\,u' &= -\frac{\partial \pi'}{\partial x} + lv' + \Delta u', \\
\frac{\partial v'}{\partial t} + \bar{u}\,\mathrm{grad}\,v' &= -\frac{\partial \pi'}{\partial y} - lu' + \Delta v', \\
\frac{\partial w'}{\partial t} + \bar{u}\,\mathrm{grad}\,w' &= -\frac{\partial \pi'}{\partial z} + \lambda\theta' + \Delta w', \\
\frac{\partial \theta'}{\partial t} + \bar{u}\,\mathrm{grad}\,\theta' &= -Sw' + \Delta\theta', \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0.
\end{aligned}
\tag{1}
$$

*Supported by the Russian Foundation for Basic Research under Grant 00-15-98543.

Here t is time, x, y, z are the Cartesian coordinates, u, v, w are the components of the wind velocity vector in the directions x, y, z, respectively, $\bar{u} = (u, v, w)$, π is the Exner function, θ, l, S, λ are the potential temperature, the Coriolis parameter, the parameter of stratification, the parameter of buoyancy, respectively.

The letters with primes denote deviations from basic fields as in [2].

The operator

$$\Delta = \frac{\partial}{\partial x}\mu_x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\mu_y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\nu\frac{\partial}{\partial z},$$

where μ_x, μ_y, ν are coefficients of turbulent diffusion in the direction of the coordinates x, y, z respectively. The equations are integrated in the area $D_t = D \times [0, T]$, where $D = \{0 \le x \le X,\ 0 \le y \le Y,\ \delta(x, y) \le z \le H\}$ is the range of space variables and $0 \le t \le T$ is the interval of time, $\delta(x, y)$ is the function of the relief.

Boundary conditions are specified by the following formulas:

$$\begin{array}{lll}
\dfrac{\partial u'}{\partial x} = \dfrac{\partial v'}{\partial x} = \dfrac{\partial w'}{\partial x} = 0, & \dfrac{\partial \theta'}{\partial x} = 0, & x = 0, X, \\
\dfrac{\partial u'}{\partial y} = \dfrac{\partial v'}{\partial y} = \dfrac{\partial w'}{\partial y} = 0, & \dfrac{\partial \theta'}{\partial y} = 0, & y = 0, Y, \\
u' = v' = w' = 0, & \theta' = 0, & z = H, \\
u = v = w = 0, & \theta' = \theta_0(x, y, t), & z = \delta(x, y).
\end{array} \qquad (2)$$

Equations (1) together with boundary conditions (2) describe the motion of air mass above the relief. In this case, this set of equations can be considered as a separate model of the atmosphere above a geometrically complicated area. At the same time, this model is also one of the blocks of a more complicated model.

Methods of construction of effective algorithms of realization of numerical models of dynamics of the atmosphere on the base of methods of decomposition in a combination with variational principle are developed by V.V. Penenko in works [1, 2]. For the solution of problem (1), (2) a methodological approach explained in the papers was used.

We chose the following method of construction of schemes of splitting [1]. At first the initial system of differential equations is reduced to a series of more simple problems. Then for each of these problems an integral identity is created and is approximated.

At the first stage of splitting the set of equations is

$$L\varphi = \Delta\varphi - \bar{u}\,\mathrm{grad}\,\varphi = \frac{\partial \varphi}{\partial t}, \qquad \varphi = (u', v', w', \theta'). \qquad (3)$$

For construction of discrete approximations an integral identity is applied to system (3). The identity is obtained by the integration of the equations which are multiplied by rather smooth functions u^*, v^*, w^*, θ^*.

$$\int_{D_t}\left(L\varphi-\frac{\partial\varphi}{\partial t}\right)\varphi^*\,dD\,dt=0, \tag{4}$$

$$L\varphi=-\bar{u}\,\mathrm{grad}\,\varphi+\frac{\partial}{\partial x}\mu_x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\mu_y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\nu\frac{\partial}{\partial z},$$

$$\varphi=(u',v',w',\theta'),\quad \varphi^*=(u^*,v^*,w^*,\theta^*)^T.$$

The ideas of the method of fictitious areas [2, 3, 5] were used in the construction of numerical models. Let us introduce a grid area ${D_t}^h$ as the direct product of the one-dimensional grids along each of the coordinates x, y, z, t: ${D_t}^h=D^h\times[0,T]^h$, $D^h=\omega_x\times\omega_y\times\omega_z$. The grid D^h covers the complement $\tilde{D}$ of the area D up to the parallelepiped

$$\tilde{D}=\{(x,y,z):0\le x\le X,0\le y\le Y,0\le z\le H\}.$$

The one-dimensional grids are defined as follows:

$$\omega_x=\{x_i:\ x_0=0,\ x_{i+1}=x_i+\Delta x_i,\ i=0,\dots,N-1,\ x_N=X\},$$
$$\omega_y=\{y_m:\ y_0=0,\ y_{m+1}=y_m+\Delta y_m,\ m=0,\dots,M-1,\ y_M=Y\},$$
$$\omega_z=\{z_k:\ z_0=0,\ z_{k+1}=z_k+\Delta z_k,\ k=0,\dots,K-1,\ z_K=H\},$$
$$[0,T]^h=\{t_j:\ t_0=0,\ t_{j+1}=t_j+\Delta t,\ j=0,\dots,J-1,\ t_J=T\}.$$

The intersections of coordinate lines of the grid D^h (i.e., of the straight lines $x=x_i$, $y=y_m$, $i=0,\dots,N$, $m=0,\dots,M$) with the surface $z=\delta(x,y)$ are assumed to coincide with the nodes of the grid ω_z. In other words, the points (x_i,y_m,z_{im}) belong to the grid area D^h, where $z_{im}=\delta(x_i,y_m)$, $i=0,\dots,N$, $m=0,\dots,M$. More general cases will not be considered. For the vector functions φ, φ^* and parameters of the model the following statement is true [6]: for the components of these vector functions there are finitary extensions with the same class of a smoothness in $\tilde{D}$ as the components have. Thus, we can formally determine the integrand expressions in the volumetric integrals on the whole parallelepiped $\tilde{D}$. The integrand without the term $\frac{\partial\varphi}{\partial t}$ is multiplied by the indicatrix χ of the area D. Now the integral is considered on the area $\tilde{D}\times[0,T]$. In other words, we have determined the problem by the expression $\frac{\partial\varphi}{\partial t}=0$ in the area $\tilde{D}\setminus D$. Thus, the obtained integral identity still corresponds to the initial problem. At the same time, the modified integral identity allows practically completely to reproduce the scheme of construction of finite difference approximations which is applied in the case of a rectangular area.

The space approximation of equations (2) was carried out by a method which is described in work [4]. Specifically, solutions of local adjoint problems are used as weight functions. For equations with variable coefficients an appropriate approximation in each mesh of grid area is chosen. This approach allows the building of monotonic difference schemes of second order of space approximation. As a result, the differential-difference equations of the following form were obtained:

$$\begin{aligned}
&R_{kmi}\left(\frac{\partial\varphi}{\partial t}\right)_{imk} + \\
&\Big[-(uA_1B_1)_{i+1/2}(\varphi_{i+1mk}-\varphi_{imk}) + (uA_1)_{i-1/2}(\varphi_{imk}-\varphi_{i-1mk})\Big]\delta y_m\delta z_k + \\
&\Big[-(vA_2B_2)_{m+1/2}(\varphi_{im+1k}-\varphi_{imk}) + (vA_2)_{m-1/2}(\varphi_{imk}-\varphi_{im-1k})\Big]\delta x_i\delta z_k + \\
&\Big[-(wA_3B_3)_{k+1/2}(\varphi_{imk+1}-\varphi_{imk}) + (wA_3)_{k-1/2}(\varphi_{imk}-\varphi_{imk-1})\Big]\delta x_i\delta y_m \\
&\quad = 0,
\end{aligned} \tag{5}$$

$$(u_sA_sB_s) \geq 0, \quad (u_sA_s) \geq 0, \quad R_{kmi} \geq 0, \qquad u_s = u,\ v,\ w,$$
$$\delta r_n = (r_{n+1} - r_{n-1})/2, \quad r = x,\ y,\ z, \quad n = i,\ m,\ k.$$

The properties of coefficients in (5) ensure monotonicity and stability of the difference scheme. Therefore, time approximations for this equation should be selected so that the property of monotonicity be conserved. It can be done easily with the help of implicit schemes.

To solve the problem of convection-diffusion two methods of time approximation of the equations were used. In the first case, the method of weak approximation with fractional steps was used [1]. In the second case, an implicit scheme was chosen. The problems were solved by space splitting. In the first case, on each fractional step the scheme of Crank–Nicolson type was formed. Thus two numerical schemes for the solution of the set of equations of convection-diffusion are constructed. The schemes are absolutely stable with second order of approximation in space. The first scheme is conditionally monotonic; the second scheme is monotonic. Because of the linearization of the model both schemes are of the first order of time approximation.

At the adjustment stage the following set of equations is solved:

$$\begin{aligned}
&\frac{u^{j+1}+u^j}{\Delta t} + \frac{\partial\pi^{j+1}}{\partial x} - lv^{j+1} = 0, \qquad \frac{v^{j+1}+v^j}{\Delta t} + \frac{\partial\pi^{j+1}}{\partial y} + lu^{j+1} = 0, \\
&\frac{w^{j+1}+w^j}{\Delta t} + \frac{\partial\pi^{j+1}}{\partial z} - \lambda\theta^{j+1} = 0, \qquad \frac{\theta^{j+1}+\theta^j}{\Delta t} + Sw^{j+1} = 0, \\
&\frac{\partial u^{j+1}}{\partial x} + \frac{\partial v^{j+1}}{\partial y} + \frac{\partial w^{j+1}}{\partial z} = 0.
\end{aligned} \tag{6}$$

For simplicity the primes at all unknown quantities are omitted. u^{j+1}, v^{j+1}, w^{j+1} are expressed through functions from the previous time step and are substituted in the discrete analog of the continuity equation. The equation concerning function π is obtained:

$$u_1\frac{\partial\pi^{j+1}}{\partial x}+v_1\frac{\partial\pi^{j+1}}{\partial y}+w_1\frac{\partial\pi^{j+1}}{\partial z}-\frac{\partial}{\partial x}\mu_{1x}\frac{\partial\pi^{j+1}}{\partial x}-\frac{\partial}{\partial y}\mu_{1y}\frac{\partial\pi^{j+1}}{\partial y}-\nu_1\frac{\partial}{\partial z}\frac{\partial\pi^{j+1}}{\partial z}=F. \qquad (7)$$

Equation (7) has the same spatial structure as equations (3). Therefore, approximation in space for equation (7) is constructed by the same scheme. For the solution of the obtained system of difference equations the method of minimal discrepancies is used [5].

For testing of the obtained difference schemes some numerical experiments were carried out. Some problems of forming of the atmospheric circulation above an inhomogeneous relief were considered. All experiments were carried out with the following input parameters $X = 29$ km, $Y = 29$ km, $H = 2.9$ km, $\Delta x = \Delta y = 1000$ m, $\Delta z = 200$ m, $\Delta t = 30$ s, $\mu_x = \mu_y = 1000$ m^2/s, $\nu = 10$ m^2/s. In Figures, the two-dimensional vertical sections of the fields of wind velocity and temperature are represented in the plane (x, z) at $y = 14$ km.

In Figures 1 and 2, the evolution of circulation under the influence of a temperature inhomogeneity of the underlying surface is given and the surface is represented by a pyramid D_0 in the center of the area,

$$\begin{aligned} D_0 = \; & \{11 \le x \le 17 \text{ (km)},\quad 11 \le y \le 17 \text{ (km)},\quad 0 \le z \le 200 \text{ (m)}\} \cup \\ & \{13 \le x \le 15 \text{ (km)},\quad 13 \le y \le 15 \text{ (km)},\quad 200 \le z \le 400 \text{ (m)}\}. \end{aligned}$$

The deviation of temperature for the pyramid is 2 degrees. The calculations are given at $t = 5$ min, in Figure 1 for the monotonic scheme and in Figure 2 for the conditionally monotonic one.

The results of calculations represented in Figures 1 and 2, are characteristic for meteoprocesses formed in day time, when the slopes are heated up. There is a vertical rise of the air above top of the mountain, and downward streams are observed above the slopes.

In Figures 3 and 4, the calculation results of forming of the atmospheric process in a long and deep hollow D_0 are given, where

$$\begin{aligned} D_0 &= D_1 \cup D_2 \cup D_3 \\ &= \{0 \le x \le 9 \text{ (km)},\quad 0 \le y \le 29 \text{ (km)},\quad 0 \le z \le 200 \text{ (m)}\} \cup \\ &\quad\ \{9 \le x \le 19 \text{ (km)},\quad 0 \le y \le 29 \text{ (km)},\quad z = 0 \text{ m}\} \cup \\ &\quad\ \{19 \le x \le 29 \text{ (km)},\quad 0 \le y \le 29 \text{ (km)},\quad 0 \le z \le 200 \text{ (m)}\}. \end{aligned}$$

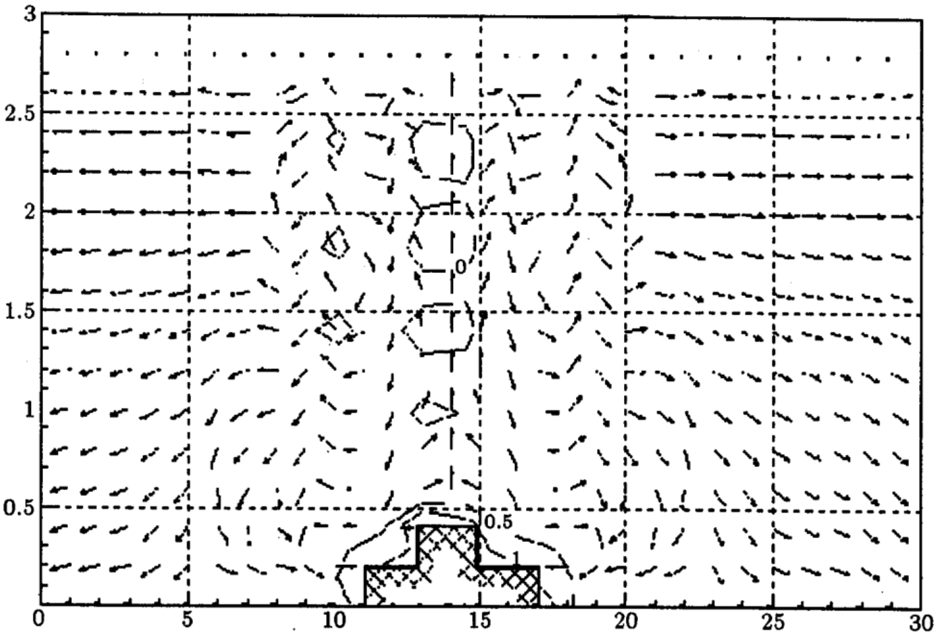

Figure 1

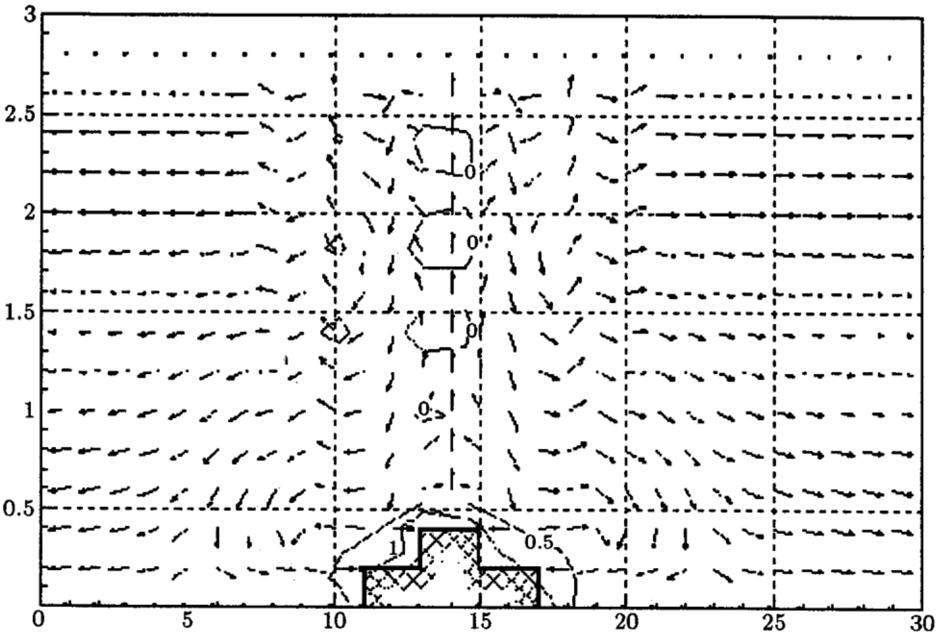

Figure 2

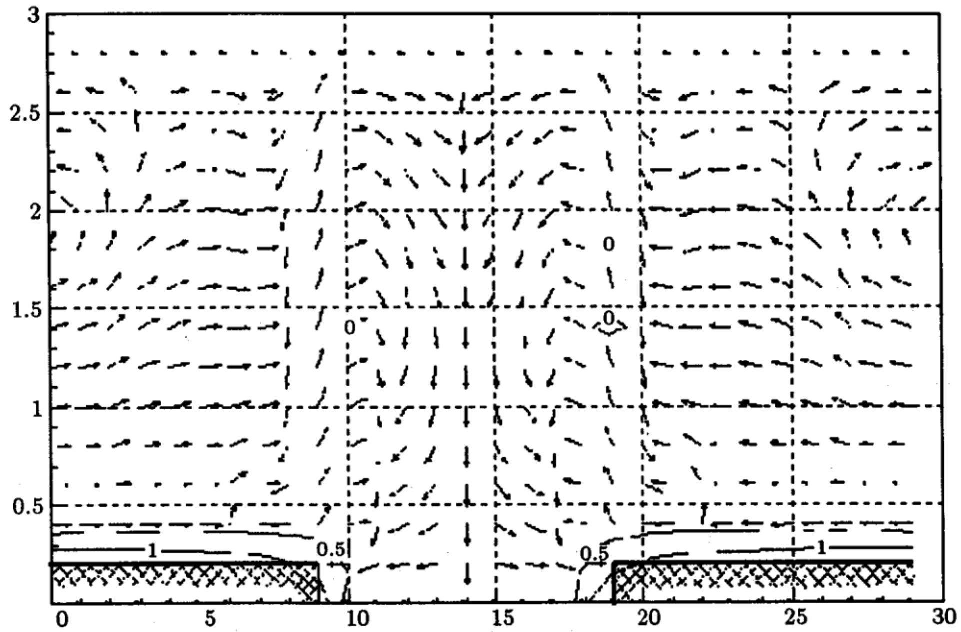

Figure 3

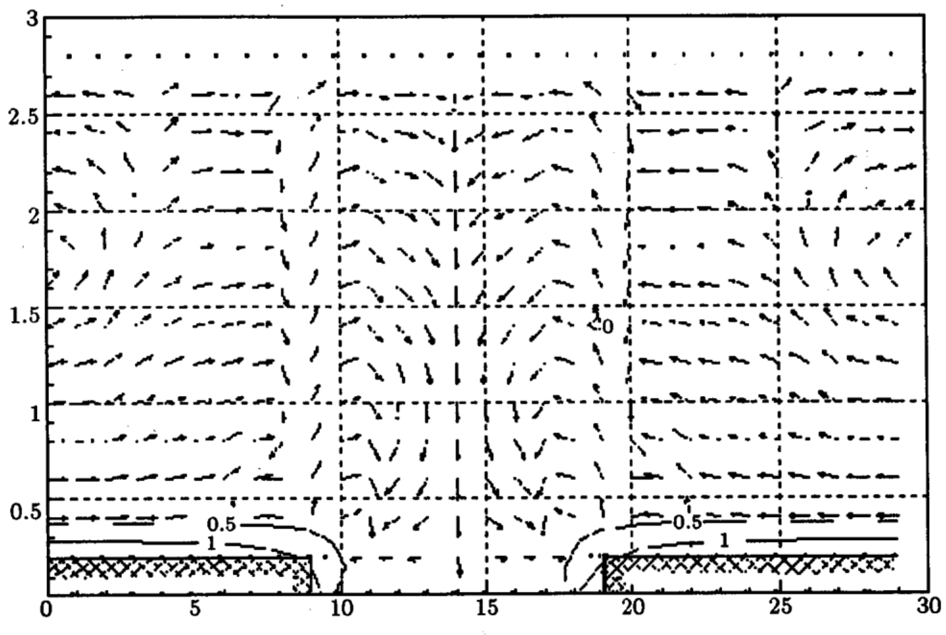

Figure 4

In the areas D_1 and D_2, the perturbation of temperature is 2 degrees. The calculations are given at $t = 5$ min, in Figure 3 for the monotonic scheme and in Figure 4 for the conditionally monotonic one. The following evolution of the circulation is observed: there are ascending streams above the heated up surfaces and there is a descending stream to the bottom of the hollow. Also a weak leaking of the air from the bottom of the hollow up to the heated surface is seen.

On the whole, all figures show the probable direction of development of circulations. But further research of the behaviour of the constructed schemes is necessary.

References

[1] Penenko V.V. Methods of Numerical Modeling of Atmosphere Processes. – Leningrad: Hydrometeoizdat, 1981 (in Russian).

[2] Penenko V.V., Aloyan A.E. Models and methods for environment preservation problems. – Novosibirsk: Nauka, 1985 (in Russian).

[3] Mathematical methods mining industry control and management / Kuzin, Mazalov, Kerdman, Faleychik. – Irkutsk University, 1991 (in Russian).

[4] Penenko V.V. Numerical schemes for advective–diffusion equations with local adjoint problems. – Novosibirsk, 1993. – (Preprint / RAN. Sib. Div. Comp. Cent.; 984) (in Russian).

[5] Marchuk G.I. Methods of computational mathematics. – Moscow: Nauka, 1977 (in Russian).

[6] Mikhailov V.P. Differential equations with partial derivatives. – Moscow: Nauka, 1976 (in Russian).