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Operator alternating-triangular method for the three-dimentional static problem in elasticity theory*

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To solve the tree-dimensional static problems in elasticity theory in the displacements a new class of effective iteration methods – factorized operator-triangular methods was studied. The additive expansion of the diagonal operator leads to the analogous expansion of the initial matrix operator in the sum of triangular-matrix operators. The degree of convergence of the factorized operator-triangular method is higher than of factorized operator-diagonal method, when, as in the two-dimensional case, the corresponding iterative parameters were taken.

When solving numerically the three-dimentional static problem in elasticity theory in displacements, we will proceed, as in [1–3], from a factorized representation of the Lamè operator $\boldsymbol{A}$:

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{x} \in \boldsymbol{G}, \quad \boldsymbol{A} = \boldsymbol{R}^*\boldsymbol{K}\boldsymbol{R}, \\ &\boldsymbol{u} = \boldsymbol{0},\ \boldsymbol{x} \in \boldsymbol{\Gamma}. \end{aligned} \tag{1}$$

In (1) $\boldsymbol{u} = (u_1, u_2, u_3)^T$ is the column vector of elastic displacements, $\overline{\boldsymbol{G}} = \boldsymbol{G} \cup \boldsymbol{\Gamma} = \{\boldsymbol{x} = (x_1, x_2, x_3)^T,\ 0 \le x_i \le 1\}$, the vector $\boldsymbol{f} = (f_1, f_2, f_3)^T$ gives the field of mass forces, and T is the transposition.

Let $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})^T$ be the vector of elastic deformations, $\varepsilon_{ik} = \varepsilon_{ki}$, $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^T$ be the vector of elastic stresses, and $\sigma_{ik} = \sigma_{ki}$. For a physically linear medium, we have

$$\boldsymbol{\sigma} = \boldsymbol{K}\boldsymbol{\varepsilon}, \quad \boldsymbol{K} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}. \tag{2}$$

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Here $\lambda > 0$, $\mu > 0$ are the constants characterizing the properties of the elastic medium. Their positiveness provides invertibility of the Hooke law (2), so that $\boldsymbol{\varepsilon} = \boldsymbol{K}^{-1}\boldsymbol{\sigma}$.

As it is known, for a geometrically linear medium,

$$\boldsymbol{\varepsilon} = \boldsymbol{R}\boldsymbol{u}, \quad \boldsymbol{R} = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{pmatrix}^T, \tag{3}$$

and the operator $\boldsymbol{R}$:

$$\boldsymbol{H}^*(\boldsymbol{u}) \to \boldsymbol{H}(\boldsymbol{\varepsilon}) = \boldsymbol{H}(\boldsymbol{K}^{-1}\boldsymbol{\sigma})$$

is thereby defined.

The set of vectors $\boldsymbol{u} \in \boldsymbol{H}^*(\boldsymbol{u})$ whose components possess a desired smoothness and satisfy the homogeneous boundary value problem from (1) is assumed to be the domain of definition $\boldsymbol{U}(\boldsymbol{R})$ of the operator $\boldsymbol{R}$. Next,

$$[\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}]_{H^*} = \sum_{i=1}^{3}(u_i^{(1)}, u_i^{(2)}) = \sum_{i=1}^{3}\int_{\boldsymbol{G}} u_i^{(1)} \cdot u_i^{(2)} d\boldsymbol{x}.$$

As for the Hilbert spaces $\boldsymbol{H}(\boldsymbol{\varepsilon})$ and $\boldsymbol{H}(\boldsymbol{\sigma})$,

$$\begin{aligned}[\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}]_H &= (\boldsymbol{\sigma}^{(1)}, \boldsymbol{K}^{-1}\boldsymbol{\sigma}^{(2)}) = (\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}^{(2)}) \\ &= \sum_{i,k=1}^{3}(\sigma_{ik}^{(1)}, \varepsilon_{ik}^{(2)}) = \sum_{i,k=1}^{3}\int_{\boldsymbol{G}} \sigma_{ik}^{(1)} \cdot \varepsilon_{ik}^{(2)} d\boldsymbol{x}, \quad k \geq i,\end{aligned}$$

and the space $\boldsymbol{H}(\boldsymbol{\varepsilon})$ is an image of $\boldsymbol{H}(\boldsymbol{\sigma})$ for mapping $\boldsymbol{K}^{-1}$.

By definition,

$$[\boldsymbol{R}\boldsymbol{u}, \boldsymbol{\sigma}]_H = [\boldsymbol{u}, \boldsymbol{R}^*\boldsymbol{\sigma}]_{H^*}.$$

If $\boldsymbol{u} \in \boldsymbol{U}(\boldsymbol{R})$, then $\ker \boldsymbol{R} = \{0\}$, in addition $\boldsymbol{K} = \boldsymbol{K}^T > 0$. Therefore, $\boldsymbol{A} = \boldsymbol{A}^* > 0$ in (1).

Let us introduce in $\overline{\boldsymbol{G}}$ a uniform grid

$$\overline{\boldsymbol{G}}_h = \{\boldsymbol{x}_{ijk} = (x_{1i}, x_{2j}, x_{3k})^T,\ x_{1i} = ih,\ x_{2j} = jh,\ x_{3k} = kh,\ 0 \leq i,j,k \leq N\}.$$

Notice that a concrete choice of $\boldsymbol{G}$, $\boldsymbol{G}_h$ is not essential and is only used for calculating the constans of energy equivalence.

Of fundamental significance is the fact that if a certain grid approximation $\boldsymbol{R}_h$ of the operator $\boldsymbol{R}$ for which $\ker \boldsymbol{R}_h = \{0\}$ is chosen, then

$$A_h = R_h^* K R_h = A_h^* > 0\,. \tag{4}$$

We will set the operator R_h as follows:

$$R_h = \begin{pmatrix} (\cdot)_{x_1} & 0 & 0 & (\cdot)_{x_2} & (\cdot)_{x_3} & 0 \\ 0 & (\cdot)_{x_2} & 0 & (\cdot)_{x_1} & 0 & (\cdot)_{x_3} \\ 0 & 0 & (\cdot)_{x_3} & 0 & (\cdot)_{x_1} & (\cdot)_{x_2} \end{pmatrix}^T . \tag{5}$$

The designations in (5) and below we do not explain are standard in the theory of difference schemes (see [4–6]). Let $u_h = y$. For $y \in U(R_h)$ we assume that $y = 0$ if $x_{ijk} \in \Gamma_h$. For such vectors, $\ker R_h = \{0\}$. If one takes into account that, by definition,

$$[\sigma_h, R_h y]_{H_h} = [R_h^* \sigma_h, y]_{H_h^*},$$

for R_h^* we have

$$R_h^* = -\begin{pmatrix} (\cdot)_{\bar{x}_1} & 0 & 0 & (\cdot)_{\bar{x}_2} & (\cdot)_{\bar{x}_3} & 0 \\ 0 & (\cdot)_{\bar{x}_2} & 0 & (\cdot)_{\bar{x}_1} & 0 & (\cdot)_{\bar{x}_3} \\ 0 & 0 & (\cdot)_{\bar{x}_3} & 0 & (\cdot)_{\bar{x}_1} & (\cdot)_{\bar{x}_2} \end{pmatrix} . \tag{6}$$

Now (4)–(6) determine the grid Lamè operator

$$A_h = \begin{pmatrix} A_{11} & A_{21}^* & A_{31}^* \\ A_{21} & A_{22} & A_{32}^* \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_h^* > 0. \tag{7}$$

Here

$$\begin{aligned} A_{11} &= -(\lambda + 2\mu)(\cdot)_{x_1\bar{x}_1} - \mu(\cdot)_{x_2\bar{x}_2} - \mu(\cdot)_{x_3\bar{x}_3}, \\ A_{22} &= -\mu(\cdot)_{x_1\bar{x}_1} - (\lambda + 2\mu)(\cdot)_{x_2\bar{x}_2} - \mu(\cdot)_{x_3\bar{x}_3}, \\ A_{33} &= -\mu(\cdot)_{x_1\bar{x}_1} - \mu(\cdot)_{x_2\bar{x}_2} - (\lambda + 2\mu)(\cdot)_{x_3\bar{x}_3}, \\ A_{ij} &= -\lambda(\cdot)_{x_j\bar{x}_i} - \mu(\cdot)_{x_i\bar{x}_j}, \quad i > j\,. \end{aligned}$$

Thereby the grid problem

$$A_h y = f_h, \quad y = 0, \quad x_{ijk} \in \Gamma_h, \tag{8}$$

is also determined.

To solve the problem (8) numerically, we will consider a two-layer stationary iterative method

$$B\frac{y^{m+1} - y^m}{\tau} + A_h y^m = f_h, \quad y^m = 0, \quad x_{ijk} \in \Gamma_h. \tag{9}$$

Let $\Lambda = \operatorname{diag} A_h = \operatorname{diag}\{A_{11}, A_{22}, A_{33}\}$. Then

$$\Lambda = \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{22} & 0 \\ 0 & 0 & \Lambda_{33} \end{pmatrix} = \begin{pmatrix} P+P^* & 0 & 0 \\ 0 & Q+Q^* & 0 \\ 0 & 0 & M+M^* \end{pmatrix}$$
$$= \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & M \end{pmatrix} + \begin{pmatrix} P^* & 0 & 0 \\ 0 & Q^* & 0 \\ 0 & 0 & M^* \end{pmatrix} = \Lambda_1 + \Lambda_2,\ \Lambda_2 = \Lambda_1^*. \tag{10}$$

The operator diagonal decomposition (10) produces the operator triangular decomposition A_h:

$$A_h = \begin{pmatrix} P & 0 & 0 \\ A_{21} & Q & 0 \\ A_{31} & A_{32} & M \end{pmatrix} + \begin{pmatrix} P^* & A_{21}^* & A_{31}^* \\ 0 & Q^* & A_{32}^* \\ 0 & 0 & M^* \end{pmatrix} = A_1 + A_2,\ A_2 = A_1^*. \tag{11}$$

Consider now two iterative methods of the type (9):

$$(E+\omega\Lambda_1)(E+\omega\Lambda_1^*)\frac{y^{m+1}-y^m}{\tau} + A_h y^m = f_h, \tag{12}$$
$$(E+\omega A_1)(E+\omega A_1^*)\frac{y^{m+1}-y^m}{\tau} + A_h y^m = f_h. \tag{13}$$

The iterative methods (12), (13) are of insignificant difference in the number of operations needed to perform one iteration. In [3], the degree of convergence of the iterative method (13) is shown to be higher for the two-dimensional case than that of the iterative method (12). As will be shown below, this result is also true for the three-dimensional case.

Lemma 1. *If $y \in U(R_h)$, then the following operator inequalities are valid:*

$$\frac{\lambda\mu}{\lambda^2+3\lambda\mu+\mu^2}\Lambda \le A_h \le \frac{(\lambda+2\mu)(\mu+2\lambda)}{\lambda^2+3\lambda\mu+\mu^2}\Lambda, \tag{14}$$
$$\Lambda \ge \frac{4}{h^2}(\lambda+4\mu)\sin^2\frac{\pi h}{2}E,\quad \Lambda_1\Lambda_1^* \le \frac{(\lambda+4\mu)}{h^2}\Lambda. \tag{15}$$

Lemma 2. *If $y \in U(R_h)$, then the following operator inequalities:*

$$(A_h y, y) \ge \mu\|y\|_1^2,\quad A_h \ge \frac{8\mu\sin^2\frac{\pi h}{2}}{h^2}E,\quad A_1A_2 \le \mu\frac{N_0}{h^2}A_h, \tag{16}$$
$$N_0 = 4+3\gamma^2+\varepsilon,\ \varepsilon = \frac{\beta^2+\beta\sqrt{\beta^2+2(1+\gamma^2)}}{1+\gamma^2},\ \beta = \frac{\lambda+\mu}{\mu},\ \gamma = \frac{\lambda+2\mu}{\mu}. \tag{17}$$

are true.

We will comment on some important points associated with the proofs of Lemmas 1 and 2. The validity of the first inequality in (15) and the first two inequalities in (16) is established in a standard manner (see, e.g., [6]). Turn to the second inequality in (15). Let $\boldsymbol{y} = (u, v, w)^T$. Then, by definition

$$\begin{aligned} \boldsymbol{P}u &= -\frac{1}{h}\big((\lambda+2\mu)u_{x_1} + \mu u_{x_2} + u_{x_3}\big), \\ \boldsymbol{P}^*u &= \frac{1}{h}\big((\lambda+2\mu)u_{\bar{x}_1} + \mu u_{\bar{x}_2} + \mu u_{\bar{x}_3}\big), \\ \boldsymbol{Q}v &= -\frac{1}{h}\big(\mu v_{x_1} + (\lambda+2\mu)v_{x_2} + \mu v_{x_3}\big), \\ \boldsymbol{Q}^*v &= \frac{1}{h}\big(\mu v_{\bar{x}_1} + (\lambda+2\mu)v_{\bar{x}_2} + \mu v_{\bar{x}_3}\big), \\ \boldsymbol{M}w &= -\frac{1}{h}\big(\mu w_{x_1} + \mu w_{x_2} + (\lambda+2\mu)w_{x_3}\big), \\ \boldsymbol{M}^*w &= \frac{1}{h}\big(\mu w_{\bar{x}_1} + \mu w_{\bar{x}_2} + (\lambda+2\mu)w_{\bar{x}_3}\big). \end{aligned}$$

Therefore,

$$(\boldsymbol{A}_1\boldsymbol{A}_1^*\boldsymbol{y}, \boldsymbol{y}) = \|\boldsymbol{P}^*u\|^2 + \|\boldsymbol{Q}^*v\|^2 + \|\boldsymbol{M}^*w\|^2. \tag{18}$$

Next,

$$\begin{aligned} \|P^*u\|^2 &= \frac{1}{h^2}\|(\lambda+2\mu)u_{\bar{x}_1} + \mu u_{\bar{x}_2} + \mu u_{\bar{x}_3}\|^2 \\ &\le \frac{\lambda+4\mu}{h^2}\big[(\lambda+2\mu)\|u_{\bar{x}_1}\|^2 + \mu\|u_{\bar{x}_2}\|^2 + \mu\|u_{\bar{x}_3}\|^2\big]. \end{aligned} \tag{19}$$

Similar estimates also take place for $\|\boldsymbol{Q}^*v\|^2$, $\|\boldsymbol{M}^*w\|^2$. Thus,

$$(\boldsymbol{A}_1\boldsymbol{A}_1^*\boldsymbol{y}, \boldsymbol{y}) \le \frac{\lambda+4\mu}{h^2}\big[(\lambda+2\mu)S_1 + \mu S_2\big],$$

where

$$\begin{aligned} S_1 &= \|u_{x_1}\|^2 + \|v_{x_2}\|^2 + \|w_{x_3}\|^2, \\ S_2 &= \|u_{x_2}\|^2 + \|u_{x_3}\|^2 + \|v_{x_1}\|^2 + \|v_{x_3}\|^2 + \|w_{x_1}\|^2 + \|w_{x_2}\|^2. \end{aligned}$$

It remains to note that

$$(\boldsymbol{A}\boldsymbol{y}, \boldsymbol{y}) = (\lambda+2\mu)S_1 + \mu S_2.$$

For the operator inequality (14), we will prove, for example, the left-hand inequality. We have

$$(\boldsymbol{A}_h \boldsymbol{y}, \boldsymbol{y}) = (\lambda + 2\mu)S_1 + \mu S_2 + S_3,$$

where

$$S_3 = -2\big[(\boldsymbol{A}_{21}u, v) + (\boldsymbol{A}_{31}u, w) + (\boldsymbol{A}_{32}u, w)\big].$$

Then

$$S_3 \geq -(\lambda + \mu)(\varepsilon + \varepsilon^{-1})S_1, \quad S_3 \geq -(\lambda + \mu)S_2. \tag{20}$$

Now it may be noted that

$$S_3 = \nu_1 S_3 + (1 - \nu_1)S_3, \quad 0 < \nu_1 < 1;$$

and the point is the determination of the constant c_1 in the inequality $\boldsymbol{A}_h > c_1 \boldsymbol{A}$. We have

$$\begin{aligned}(\boldsymbol{A}_h \boldsymbol{y}, \boldsymbol{y}) &\geq (\lambda + 2\mu)S_1 + \mu S_2 - \nu_1(\lambda + \mu)(\varepsilon + \varepsilon^{-1})S_1 - (1 - \nu_1)(\lambda + \mu)S_2 \\ &\geq c_1[(\lambda + 2\mu)S_1 + \mu S_2].\end{aligned}$$

For the unknown constant c_1, we get

$$\begin{aligned}(\lambda + 2\mu) - \nu_1(\lambda + \mu)(\varepsilon + \varepsilon^{-1}) &= (\lambda + 2\mu)c_1, \\ \mu - (1 - \nu_1)(\lambda + \mu) &= \mu c_1,\end{aligned} \tag{21}$$

whence

$$c_1 = \frac{\lambda\mu}{\lambda^2 + 3\lambda\mu + \mu^2}, \quad \varepsilon + \frac{1}{\varepsilon} = \frac{\lambda + \mu}{\lambda}, \quad 0 < \nu_1 = \frac{\lambda}{\lambda + \mu} < 1.$$

The validity of the right-hand inequality in (14) is established in a similar way. The accuracy (unimprovability) of the estimate (14) is evidenced by the fact that

$$\frac{\lambda\mu}{\lambda^2 + 3\lambda\mu + \mu^2} + \frac{(\lambda + 2\mu)(\mu + 2\lambda)}{\lambda^2 + 3\lambda\mu + \mu^2} = 2.$$

Finally, as for the operator inequality (16)

$$\sum = (\boldsymbol{A}_1 \boldsymbol{A}_1^* \boldsymbol{y}, \boldsymbol{y}) = \sum\nolimits_1 + \sum\nolimits_2. \tag{22}$$

The scalar products of the type $(\boldsymbol{P}^* u, \boldsymbol{P}^* u)$ and $(\boldsymbol{P}^* u, \boldsymbol{A}_{21}^* v)$ are incorporated into the term $\sum_1$, while the remaining ones are included in the term $\sum_2$. All the scalar products entering (22) will be estimated through $\|u\|_1^2$, $\|v\|_1^2$, $\|w\|_1^2$. For $\sum_1$, we have the following estimate:

$$\sum_1 \leq \frac{2\mu^2 + (\lambda + 2\mu)^2}{h^2} \|y\|_1^2.$$

The term $\sum_2$, will be estimated using the Cauchy-Bunjakovsky inequality and the ε-inequality. When choosing ε, we proceed from the equality of the coefficient at $\|u\|_1^2$, $\|v\|_1^2$ and $\|w\|_1^2$ in $\sum$. Then

$$\begin{aligned}(A_1 A_1^* y, y) &\leq \frac{\mu^2}{h^2}\left[2 + 2\gamma^2 + \varepsilon + 2 + \left(\frac{\lambda + 2\mu}{\mu}\right)^2\right]\|y\|_1^2 \\ &= \frac{\mu^2}{h^2}(4 + 3\gamma^2 + \varepsilon)\|y\|_1^2 \leq \frac{\mu}{h^2}(4 + 3\gamma^2 + \varepsilon)(A_h y, y).\end{aligned}$$

Using Lemmas 1 and 2, it is possible to find all the needed constans in order to estimate the degree of divergence of the iterative methods (12) and (13). When choosing the parameters w and τ, the norms of the transition operator are assumed to be minimal (see [4]).

Theorem 1. *If y and y^m are the solutions to the problems* (8) *and* (12) *respectively, then*

$$\|y^m - y\|_{A_h} \leq \rho_1^m \|y^0 - y\|_{A_h},$$

$$\rho_1 = \frac{1 - \eta_1}{1 + \eta_1}, \quad \eta_1 = \frac{2\lambda\mu}{\lambda^2 + 3\lambda\mu + \mu^2} \cdot \frac{\sin\frac{\pi h}{2}}{1 + \sin\frac{\pi h}{2}}.$$

Theorem 2. *If y and y^m are the solutions to the problems* (8) *and* (13) *respectively, then*

$$\|y^m - y\|_{A_h} \leq \rho_2^m \|y^0 - y\|_{A_h},$$

$$\rho_1 > \rho_2 = \frac{1 - \eta_2}{1 + \eta_2}, \quad \eta_1 < \eta_2 = \frac{2\sqrt{\xi_2}}{1 + \sqrt{\xi_2}}, \quad \xi_2 = \frac{2\sin^2\frac{\pi h}{2}}{N_0}.$$

Remark 1. The inequality $\eta_1 > \eta_2$ from Theorem 2 is checked by using simple, but cumbersome munipulations. Their essence is reduced to the following. Let

$$\eta_1 = \frac{2t}{(t+2)(2t+1)} \cdot \frac{\sin\frac{\pi h}{2}}{1 + \sin\frac{\pi h}{2}}.$$

Then

$$\eta_2 = \frac{2\sqrt{2}\sin\frac{\pi h}{2}}{\sqrt{N_0} + \sqrt{2}\sin\frac{\pi h}{2}}$$

and we should convince ourselves in the validity of the inequality

$$N_0 < \frac{2(t+2)^2(2t+1)^2}{t^2}.$$

Remark 2. The inequality $\rho_2 < \rho_1$ also holds true when (12) and (13) are applied in the nonstationary version, with a stable Chebyshev set of iterative parameters τ_{m+1}.

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