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Polar and azimuthal geometrical divergences in two-dimensional media with a blockwise constant gradient

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Introduction

Beginning with paper [1], the ray method has long been used to solve numerically various problems of mathematical seismics. Many papers (see, e.g., [2–5]) were devoted to questions of calculation of geometrical divergence in combined two- and three-dimensional media with curvilinear interfaces. For layered and blockwise gradient media (in this paper, no distinction is made between these media), explicit expressions for ray, eikonal, and geometrical divergence (see [6–8]) have been widely used in computer realizations of the ray method. For instance, explicit expressions used in the calculation of geometrical divergence for a three-dimensional blockwise linear medium written in a special curvilinear system of coordinates and not yet adapted to the two-dimensional case are presented in [8].

In this paper, the authors present their long experience of work with computational schemes of the ray method for the calculation of geometrical divergence in two-dimensional blockwise linear media. The authors believe that the following two considerations justify the appearance of this paper:

1. The algorithms obtained for the calculation of geometrical divergence $l_{\|}$ in the plane of a ray and geometrical divergence $l_{\perp}$ in the direction perpendicular to the plane of the ray are new and, what is more important, they are represented in a form that can be easily used for programming.

2. Results 1 and 2 formulated in Section 5 for divergence $l_{\perp}$ in an arbitrary (not necessarily gradient) block medium with piecewise smooth velocity $V(P)$ are interesting from the point of view of theory. For instance, a method of derivation of explicit solutions for a class of direct and inverse problems of ray acoustics proposed in [9, 10] is based, instead of variable t, on the following variable (denoted by the same symbol):

$$t = \int\limits_{\gamma(y,z)} \frac{|dx|}{\lambda(x)}$$

[10, p. 194]. In our notation, it is written in the form

$$t = \int_{L(M_0,M)} V(P)\, ds_P.$$

Here $L(M_0, M)$ is the ray connecting points M_0 and M. It turns out (and this follows from Result 2) that the variable t has a simple geometrical meaning: it coincides with the divergence $l_\perp$ along the ray $L(M_0, M)$ up to a factor $1/V(M_0)$.

1. Model of medium

Assume that the properties of an elastic medium do not depend on the coordinate y in a rectangular Cartesian system of coordinates O, x, y, z. The propagation velocity $V \equiv V(x, z)$ of a longitudinal or a transverse elastic wave as a function of x and z is considered to be specified in a rectangle $0 \leq x \leq L$, $0 \leq z \leq H$. It is assumed to be a piecewise smooth function which has at piecewise smooth "interfaces" either a discontinuity or a discontinuity of its first-order partial derivatives. A region of smoothness of the function V will be called, as usual, a block. It is assumed that the number of blocks is finite and that the boundary of each block consists of a finite number of smooth sections of interfaces and straight-line segments. The straight-line segments bound the global domain of the function V. Every smooth section of the boundary of a block is either specified by an equation of the form $z = f(x)$ or $x = f(z)$, or is a vertical or horizontal section at the boundary of the rectangular domain in which the propagation process is being studied.

A variant of the geometry of an interface in a block medium described above is presented in Figure 1.

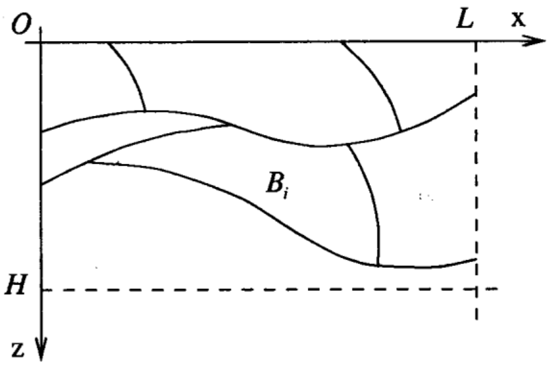

Figure 1. An example of block structure of a medium

Unless otherwise specified, the propagation velocity V_i of a perturbation inside each block B_i, $i = 1, 2, \ldots, N$, is a linear function

$$V_i(x, z) = a_i x + b_i z + c_i. \tag{1}$$

2. Problem statement

Formulas to calculate the geometrical divergence of central ray fields along rays located in the plane Oxz are derived for the model described above. The purpose of this section is to define the terminology used here.

Let us consider an arbitrary ray L going out of the point M_0. At the points M_i, $i = 1, \dots, n-1$, it meets successively with interfaces at which reflection, refraction, or exchange may take place. Then it comes to the point M_n.

We call this ray regular if:

1) The projection M_i' of each of the points M_i, $i = 1, \dots, n-1$, on the plane Oxz has a circular ε_i-vicinity, in which the interface is a smooth arc with ends located on the circle;

2) the ray glancing angle at each of the points M_i is not zero.

We say that a ray is locally equivalent to the initial ray if it:

(a) begins at M_0;

(b) passes successively the same $n-1$ interfaces in spherical ε_i-vicinities of the points M_i, $i = 1, \dots, n-1$, as the ray L with the same (refraction-reflection) type of passage and exchange;

(c) ends at a point M of the block with the end point M_n of the ray L, and

(d) satisfies the conditions of regularity.

In general, we say that the two rays starting from a point and having end points in a common block are equivalent, if they can be made the first and the last members of a finite succession of rays, in which rays with neighboring numbers are locally equivalent.

Note that equivalent rays are also called rays of a common code.

Now assume that a ray L is in the plane Oxz. We call it a basic, or initial, ray, because we shall calculate all the necessary values along it.

Let us introduce ray parameters φ, θ, t of the points M of all kinds of rays equivalent to the initial ray: φ and θ are the azimuthal and polar angles of a ray starting at a point M_0. These parameters are referred to a system of coordinates obtained from the initial system of coordinates by a translation of the origin to the point M_0; t is the time of propagation of a perturbation along a ray with angular parameters φ and θ from the point M_0 to a point M. Let us denote the ray parameters of the final point M_n of the ray L by φ_0, θ_0, and t_0. Since L lies in the plane Oxz, $\varphi_0 = 0$ or π. For the sake of definiteness, we assume that $\varphi_0 = 0$ from here on.

For each admissible value of (φ, θ, t) by $\vec{r}(t, \varphi, \theta)$ we denote the radius vector $\overrightarrow{OM}$ of the point M with the given values of ray parameters. Thus, the vector function

$$(\varphi, \theta, t) \to \vec{r}(t, \varphi, \theta) \equiv \begin{pmatrix} x(t, \varphi, \theta) \\ y(t, \varphi, \theta) \\ z(t, \varphi, \theta) \end{pmatrix}, \tag{2}$$

is defined. Here φ, θ, and t are ray parameters of all kinds of points lying on rays equivalent (or locally equivalent) to the basic ray.

When studying the differential properties of function (2) in the vicinity of $(\varphi_0, \theta_0, t_0)$, it is convenient to consider that the corresponding point M_n (it is the final point of the initial ray) does not lie on the boundary of a block. This requirement does not cause loss of generality, because otherwise the velocity in this block can be considered somewhat prolonged outside the boundary of the block near M_n by expression (1) that specifies it. Then the function (2) under study is a restriction of the similar function determined for the block thus modified.

We assume that all above conditions are satisfied and that the boundaries of the interface near points M_i, $i = 1, \ldots, n-1$, are sufficiently smooth curves. Then we can assert that, for all φ, θ, t that are close enough to their values at the points of the basic ray not coinciding with M_i, $i = 1, \ldots, n-1$, function (2) is defined and repeatedly differentiable. Then the quantities

$$\vec{l}_\perp \equiv \vec{l}_\perp(M_0, M) \stackrel{\text{df}}{=} \left(\frac{1}{\sin\theta} \frac{\partial \vec{r}(t, \varphi, \theta)}{\partial \varphi} \right) \Bigg|_{\theta=\theta_0}^{\varphi=\varphi_0\,(=0)}, \tag{3}$$

$$\vec{l}_\| \equiv \vec{l}_\|(M_0, M) \stackrel{\text{df}}{=} \frac{\partial \vec{r}(t, \varphi, \theta)}{\partial \theta} \Bigg|_{\theta=\theta_0}^{\varphi=\varphi_0\,(=0)}, \tag{4}$$

are defined. Here $M \neq M_i$, $i = 1, \ldots, n-1$, and the vector $\vec{r}(t, \varphi, \theta)$ is defined above. We call (3) azimuthal divergence, and (4) polar geometrical divergence at the point M along the ray L. From here on, we write $\vec{l}_{\perp,\|}(M)$ instead of $\vec{l}_{\perp,\|}(M_0, M)$.

The purpose of the present paper is to obtain explicit expressions for $l_\|$ and $l_\perp$ for the model of medium described in Section 1.

3. Equation for variations

It follows from definitions (3) and (4) that vectors $\vec{l}_\perp$ and $\vec{l}_\|$ are tangential to the surface $t =$ const or, equivalently, they are orthogonal to the ray L at the point M. In this case, $\vec{l}_\|$ lies in the plane of the ray, and $\vec{l}_\perp$ is orthogonal to this plane, because function (2) is even with respect to φ. Let

$\vec{\tau}$ denote the unit vector tangential to the ray L at the point M. The vector is oriented in the direction where t increases. Let $\vec{\nu}$ denote the unit vector orthogonal to $\vec{\tau}$ lying in a plane of the ray and such that the pair $(\vec{\tau}, \vec{\nu})$ has the same orientation as the pair $(\vec{k}, \vec{i})$ of the unit vectors of the z and x axes, respectively. The unit vector of the y axis is denoted, as usual, by $\vec{j}$. Therefore, assuming that

$$l_\perp = (\vec{l}_\perp \cdot \vec{j}), \quad l_\| = (\vec{l}_\| \cdot \vec{\nu}), \tag{5}$$

we can write

$$\vec{l}_\perp = l_\perp \vec{j}, \quad \vec{l}_\| = l_\| \vec{\nu}. \tag{6}$$

Here we show that the quantities $l_\perp$ and $l_\|$ satisfy the same differential equation which follows from the equations of variations for the ray equation along its sections between the points M_i and M_{i+1}, $i = 0, \ldots, n-1$. This justifies the use of the same symbol $u(t, \varphi_0, \theta_0)$ to denote

$$\sin\theta_0 l_\perp := u(t, \varphi_0, \theta_0) \quad (\varphi_0 = 0), \tag{7}$$

$$l_\| := u(t, \varphi_0, \theta_0) \quad (\varphi_0 = 0). \tag{8}$$

Let us derive the equation we promised.

We choose the following ray equation parameterized with respect to time t:

$$\frac{d}{dt}\left(\frac{\dot{\vec{r}}_t}{V^2}\right) = -\frac{\nabla V}{V}. \tag{9}$$

Here the symbols $d/dt(\ldots)$ and $(\dot{\ldots})_t$ denote the same, i.e., (partial) derivatives of $(\ldots)$ of the variables t, φ, and θ with respect to t. Using the equality

$$\frac{\dot{\vec{r}}_t}{V^2} = \vec{\tau}\frac{1}{V}, \tag{10}$$

we rewrite (9) in the form

$$\frac{1}{V}\frac{d\vec{\tau}}{dt} + \vec{\tau}\frac{d}{dt}\left(\frac{1}{V}\right) = -\frac{\nabla V}{V}.$$

Hence, taking into account that

$$\frac{d}{dt}\left(\frac{1}{V}\right) = -\frac{1}{V}\frac{d}{dt}\ln V,$$

we obtain

$$\frac{d\vec{\tau}}{dt} - \vec{\tau}\frac{d}{dt}\ln V = -\nabla V. \tag{11}$$

Now let us differentiate (11) with respect to a parameter α, where $\alpha := \varphi$ or $\alpha := \theta$, at $\alpha = \alpha_0$ ($\alpha_0 = \varphi_0$ or θ_0, respectively). Changing the order of differentiation with respect to α and t (which is possible because function (2) is smooth), we write the result in the following form:

$$\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial\alpha}\right) - \left(\frac{\partial\vec{r}}{\partial\alpha}\right)\frac{d}{dt}\ln V = \vec{r}\frac{d}{dt}\left(\frac{\partial}{\partial\alpha}\ln V\right) - \frac{\partial}{\partial\alpha}(\nabla V) \equiv \vec{F}_1 + \vec{F}_2. \quad (12)$$

Case $\alpha = \varphi$. Since the function V and, hence, ∇V and $\ln V$ do not depend on y,

$$\frac{\partial}{\partial\alpha}V = \frac{\partial}{\partial\alpha}(\ln V) = 0, \quad \frac{\partial}{\partial\alpha}(\nabla V) = 0$$

and, consequently,

$$\vec{F}_1 + \vec{F}_2 = 0, \quad (13)$$

$$\frac{\partial\vec{r}}{\partial\alpha} \overset{(10)}{:=} \frac{\partial}{\partial\alpha}\left(\dot{\vec{r}}_t\frac{1}{V}\right) = \frac{1}{V}\frac{\partial}{\partial\alpha}(\dot{\vec{r}}_t). \quad (14)$$

Changing the order of differentiation in the right-hand side of (14) and taking into account formulas (3), (6), and (7), we obtain

$$\frac{\partial\vec{r}}{\partial\alpha} = \frac{1}{V}\frac{d}{dt}(u\vec{j}) = \left(\frac{\dot{u}_t}{V}\right)\vec{j}. \quad (15)$$

Finally, from (12), (13), and (15) we obtain the following equation:

$$\frac{d}{dt}\left(\frac{\dot{u}_t}{V}\right) - \left(\frac{\dot{u}_t}{V}\right)\frac{d}{dt}\ln V = 0. \quad (16)$$

The first integral of this equation for each of the intervals $[t_i, t_{i+1}]$ is evidently

$$\frac{\dot{u}_t}{V^2} = \text{const} \quad \text{or} \quad \frac{\dot{u}_s}{V} = \text{const}. \quad (17)$$

Here $\dot{u}_s$ is the derivative of $u(t, \varphi_0, \theta_0)$ with respect to the arc length s of the ray L.

Conclusion 1. Relations (17) were obtained without the use of the property of linearity of function V in the block under consideration.

Case $\alpha = \theta$. Since the function V in the block under consideration is linear, $\nabla V = \text{const}$. Therefore, $\vec{F}_2 = 0$ (in (12)). Let us multiply the both sides of (12) scalarly by $\vec{\nu}$. Then, taking into account that $(\vec{\nu}\cdot\vec{F}_1) = 0$, we obtain

$$\left(\vec{\nu}\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial\alpha}\right)\right) - \left(\vec{\nu}\cdot\frac{\partial\vec{r}}{\partial\alpha}\right)\frac{d}{dt}\ln V = 0. \quad (18)$$

From the relation of orthogonality

$$\left(\frac{d\vec{\nu}}{dt}\cdot\frac{\partial\vec{r}}{\partial\alpha}\right) = 0$$

the first term in (18) can be written in the following form:

$$\left(\vec{\nu}\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial\alpha}\right)\right)=\frac{d}{dt}\left(\vec{\nu}\cdot\frac{\partial\vec{r}}{\partial\alpha}\right). \tag{19}$$

If we use the relations of orthogonality

$$\text{(a)}\quad (\vec{\nu}\cdot\vec{r}_t)=0 \qquad \text{and} \qquad \text{(b)}\quad \left(\vec{\nu}\cdot\frac{d\vec{\nu}}{dt}\right)=0$$

and the equality

$$\text{(c)}\quad \frac{\partial\vec{r}}{\partial\theta}=u\vec{\nu},$$

obtained from (4), (6), and (8), we transform the expression $\left(\vec{\nu}\cdot\dfrac{\partial\vec{r}}{\partial\alpha}\right)$ as follows:

$$\begin{aligned}\left(\vec{\nu}\cdot\frac{\partial\vec{r}}{\partial\alpha}\right) &\overset{(10)}{:=} \left(\vec{\nu}\cdot\frac{\partial}{\partial\theta}\left(\frac{\dot{\vec{r}}_t}{V}\right)\right)=(\vec{\nu}\cdot\vec{r}_t)\frac{\partial}{\partial\theta}\left(\frac{1}{V}\right)+\frac{1}{V}\left(\vec{\nu}\cdot\frac{\partial}{\partial\theta}\vec{r}_t\right)\\ &\overset{(a)}{=} \frac{1}{V}\left(\vec{\nu}\cdot\frac{\partial}{\partial\theta}\vec{r}_t\right)\overset{(b)}{=}\frac{1}{V}\left(\vec{\nu}\cdot\frac{d}{dt}(u\vec{\nu})\right)\\ &= \frac{\dot{u}_t}{V}+\frac{u}{V}\left(\vec{\nu}\cdot\frac{d\vec{\nu}}{dt}\right)\overset{(c)}{=}\frac{\dot{u}_t}{V}.\end{aligned} \tag{20}$$

Thus, in accordance with (18)–(20), as in the case $\alpha := \varphi$, for $(\dot{u}_t/V)$ equation (16) (and, hence, equalities (17)) is valid.

To conclude, returning from u to $l_\perp$, $l_{\|}$, we formulate some results that follow from the fact that (16) has the first integral (17):

For any two points M and M' of the ray L lying between M_i and M_{i+1}, $i=0,\dots,n-1$, the following equalities hold:

$$\frac{(\dot{l}_\perp)_s(M)}{V(M)}=\frac{(\dot{l}_\perp)_s(M')}{V(M')}, \tag{21}$$

$$\frac{(\dot{l}_{\|})_s(M)}{V(M)}=\frac{(\dot{l}_{\|})_s(M')}{V(M')}. \tag{22}$$

Equality (21) need not be valid only for a linear function V. From (21) and (22), integrating over the arc length s of the ray L between the points M and M', we derive that

$$l_\perp(M)=l_\perp(M')+(\dot{l}_\perp)_s(M')\,l(M',M), \tag{23}$$

$$l_{\|}(M)=l_{\|}(M')+(\dot{l}_{\|})_s(M')\,l(M',M). \tag{24}$$

Here

$$l(M',M):=\frac{1}{V(M')}\int\limits_{S(M')}^{S(M)}V\Big(\vec{r}(s,\varphi_0,\theta_0)\Big)\,ds=\frac{1}{V(M')}\int\limits_{M'}^{M}V(P)\,ds_P. \tag{25}$$

Let us apply (24) for a ray that lies on L but starts not at M_0 but at M'. Then $l_{\|}(M') = 0$, $(\dot{l}_{\|})_s(M') = 1$. We see that

$$l(M', M) := l_{\|}(M', M). \tag{26}$$

That is, $l(M', M)$ has the meaning of polar geometrical divergence at the point M along a ray starting at M' (lying on L).

Since $l_{\|}(M', M)$ has an explicit expression for media with linear velocity, we obtained explicit expressions (21)–(26) which relate the values of quantities $l_{\perp,\|}$, $(\dot{l}_{\perp,\|})_s$ at two arbitrary points lying in a smooth section of the ray L.

4. Formulas for recalculation at interfaces

To obtain final formulas for geometrical divergence, we should find relations between the values of quantities $l_{\perp,\|}(M)$, $(\dot{l}_{\perp,\|})_s(M)$ as the point M passes through the points M_i, $i = 1, \dots, n-1$, that lie on interfaces.

Let M_* denote any point M_i, and let S denote an interface as a surface in the space $Oxyz$ on which M_* lies (the same letter S, if necessary, will denote the trace of the interface on the plane Oxz). We write the boundary of S in the vicinity of M_* in implicit form:

$$F(x, y, z) = 0. \tag{27}$$

Here $F(x, y, z)$ is a sufficiently smooth function that does not depend on y and such that $\nabla F(x, y, z) \neq 0$ for all $(x, y, z) \in S$ in the vicinity of M_*.

All values referring to a ray incident on S at a point of incidence M will have superscript "–", and values of an outgoing (reflected or refracted) ray will have superscript "+". Using these symbols, we write the first basic relation on S that expresses the condition of continuity of a ray during its passage through S:

$$\vec{r}^{\,-}(t, \varphi, \theta) = \vec{r}^{\,+}(t, \varphi, \theta). \tag{28}$$

Here

$$t = t^{\pm}(\varphi, \theta) \tag{29}$$

is the equation of the surface S in the space of ray parameters t, φ, θ. That is, $t^{\pm}(\varphi, \theta)$ is a solution for t of equation

$$F\Big(\vec{r}(t, \varphi, \theta)\Big) = 0 \tag{30}$$

in the vicinity of ray parameters φ_0, θ_0, and t_* of the point M_*.

We write the second basic relation on S which is an expression of the Snellius law of ray refraction and reflection. For this, we consider the unit vectors $\vec{n}^+$ and $\vec{n}^-$ normal to S at the point M with ray parameters t, φ, and θ having acute angles with the unit vectors $\vec{\tau}^+$ and $\vec{\tau}^-$ tangent to the ray at the point M. It is clear that $\vec{n}^+ = \vec{n}^-$ under refraction and $\vec{n}^+ = -\vec{n}^-$ under reflection. We write the Snellius law in the following form:

$$\left[\vec{\tau}^+(t,\varphi_0,\theta)\times\vec{n}^+(M)\right] = (\vec{n}^+\cdot\vec{n}^-)\frac{V^+(M)}{V^-(M)}\left[\vec{\tau}^-(t,\varphi_0,\theta)\times\vec{n}^-(M)\right]. \tag{31}$$

Our derivation of formulas of recalculation is based on the differentiation of relations (28), (30), and (31) with respect to φ and θ under the assumption (29). Prior to performing these operations, we note that, as in Section 2, some stages of the derivation will be the same for each of the parameters φ and θ. Therefore, we give them, as in Section 2, a common name "α", $\alpha := \varphi$, or $\alpha := \theta$, and assume that

$$\frac{\partial \vec{r}^{\,\pm}}{\partial \alpha}(t,\varphi,\theta) = \vec{u}^{\pm}(t,\varphi,\theta). \tag{32}$$

Then

$$\vec{u}(t,\varphi_0,\theta) = \begin{cases} u^{\pm}(t,\varphi_0,\theta)\vec{j} = (\sin\theta)l_{\perp}^{\pm}\vec{j} & \text{at } \alpha := \varphi; \\ u^{\pm}(t,\varphi_0,\theta)\vec{\nu} = l_{\parallel}^{\pm}\vec{\nu} & \text{at } \alpha := \theta. \end{cases} \tag{33}$$

Recall that here () $\vec{j}$ is the unit vector of the Oy axis, $\vec{\nu}^{\pm}$ are the unit vectors orthogonal to $\vec{\tau}^{\pm}$ (respectively), oriented so that the frames $(M,\vec{\nu}^{\pm},\vec{\tau}^{\pm})$ are of the same type as the frame $(O,\vec{k},\vec{i})$.

First we differentiate (30) with respect to α under condition (29). Then, using (32) and the equalities

$$\vec{n}^{\pm} = \pm\frac{\nabla F}{|\nabla F|}, \qquad \dot{\vec{r}}_t = \dot{\vec{r}}_s \cdot V, \tag{34}$$

we obtain

$$(t^{\pm})'_{\alpha} = -\frac{(\vec{n}^{\pm}\cdot\vec{u}^{\pm})}{V^{\pm}(\vec{n}^{\pm}\cdot\vec{\tau}^{\pm})}. \tag{35}$$

Now let us differentiate relation (28) with respect to α under condition (29) and make use of (32), (34), and (35). We have

$$\vec{u}^+ - \frac{(\vec{n}^+\cdot\vec{u}^+)}{(\vec{n}^+\cdot\vec{\tau}^+)}\vec{\tau}^+ = \vec{u}^- - \frac{(\vec{n}^-\cdot\vec{u}^-)}{(\vec{n}^-\cdot\vec{\tau}^-)}\vec{\tau}^-. \tag{36}$$

From (36), we derive formulas for recalculation of the quantities $l_{\perp}$ and $l_{\parallel}$.

Recalculation of $l_{\perp}$. In this case, we take $\alpha := \varphi$ in (32), and $\theta := \theta_0$ in (33). Then, in accordance with (33), $\vec{u}^{\pm}$ are parallel to $\vec{j}$. Since $\vec{n}^{\pm}$ lie in the plane Oxz, $(\vec{n}^{\pm}\cdot\vec{u}^{\pm}) = 0$. Therefore, from (36) and (33) we obtain

$$l_{\perp}^{+}(M_*) = l_{\perp}^{-}(M_*). \tag{37}$$

Thus, $M \to l_{\perp}(M)$ is a continuous function along the ray L.

Recalculation of $l_{\|}$. Here we set $\alpha := \theta$ in (32) and $\theta = \theta_0$ in (33). From (33) and (36) we obtain

$$l_{\|}^{+}\Big(\vec{\nu} - \frac{(\vec{n}\cdot\vec{\nu})}{(\vec{n}\cdot\vec{\tau})}\vec{\tau}\Big)^{+} = l_{\|}^{-}\Big(\vec{\nu} - \frac{(\vec{n}\cdot\vec{\nu})}{(\vec{n}\cdot\vec{\tau})}\vec{\tau}\Big)^{-}.$$

We verify directly that the vectors

$$\vec{n}_{\perp}^{\pm} := \Big[(\vec{n}\cdot\vec{\tau})\vec{\nu} - (\vec{n}\cdot\vec{\nu})\vec{\tau}\Big]^{\pm} \tag{38}$$

are orthogonal, respectively, to the vectors $\vec{n}^{\pm}$, where

$$\vec{n}_{\perp}^{+} = \begin{cases} \vec{n}_{\perp}^{-} & \text{under refraction,} \\ -\vec{n}_{\perp}^{-} & \text{under reflection.} \end{cases}$$

As a result, we obtain

$$\frac{l_{\|}^{+}}{(\vec{n}^{+}\cdot\vec{\tau}^{+})}\vec{n}_{\perp}^{+} = \frac{l_{\|}^{-}}{(\vec{n}^{-}\cdot\vec{\tau}^{-})}\vec{n}_{\perp}^{-}, \tag{39}$$

or, if we introduce angles of the incidence $\beta^{\pm}$,

$$\frac{l_{\|}^{+}}{\cos\beta^{+}} = (\vec{n}^{+}\cdot\vec{n}^{-})\frac{l_{\|}^{-}}{\cos\beta^{-}}. \tag{40}$$

Equality (39) has a simple geometrical interpretation given in Figure 2 for the case of reflection.

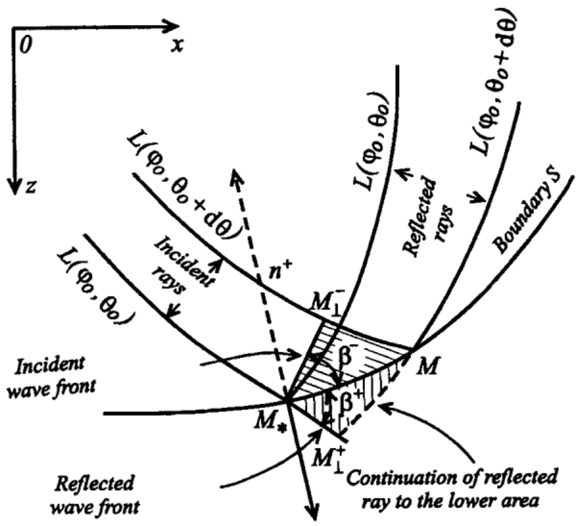

Figure 2. Geometrical interpretation of the recalculation of $l_{\|}$

In Figure 2, $\smile M_* M_\perp^-$ is the trace of the incident wave front; the dashed line $\smile M M_\perp^+$ is a continuation of the ray reflected at the point M to the lower medium; $\smile M_* M_\perp^+$ is a continuation of the trace of the reflected wave front. From the curvilinear rectangular triangles $\triangle M_* M_\perp^+ M$ and $\triangle M_* M_\perp^- M$ with a common "hypotenuse" $\smile M_* M$ with an accuracy up to $o(d\theta)$ we obtain the equality

$$\frac{|M_* M_\perp^+|}{\cos\beta^+} = \frac{|M_* M_\perp^-|}{\cos\beta^-} \quad (= |M_* M|),$$

in which $|M_* M_\perp^\pm| = |l_\parallel^\pm| d\theta + o(d\theta)$.

Now let us derive formulas of recalculation for $(\dot{l}_{\perp,\parallel})_s$. For this purpose, we differentiate relation (31) with respect to α under condition (29). That is, we operate on the both sides of (31) by the differential operator

$$\Lambda_\alpha(\ldots) := (\dot{\ldots})_t\, t'_\alpha + \frac{\partial}{\partial\alpha}(\ldots). \tag{41}$$

Here t'_α is calculated by using any of formulas (35). For the present, we write the result of the differentiation in a "compact" form, and relate it to the point M_*:

$$\begin{aligned}(\vec{n}^+ \cdot \vec{n}^-)\Big[[(\Lambda_\alpha\vec{\tau}) \times \vec{n}] + [\vec{\tau} \times (\Lambda_\alpha\vec{n})]\Big]^+ \\ = \gamma\Big[[(\Lambda_\alpha\vec{\tau}) \times \vec{n}] + [\vec{\tau} \times (\Lambda_\alpha\vec{n})]\Big]^- + (\Lambda_\alpha\gamma)^-[\vec{\tau} \times \vec{n}]^-,\end{aligned} \tag{42}$$

$$\gamma \equiv \gamma(M) = \frac{V^+(M)}{V^-(M)}. \tag{43}$$

Recalculation of $(\dot{l}_\perp)_s$. Let us assume that $\alpha = \varphi$ in (42). Then $\Lambda_\alpha(\ldots) = \frac{\partial}{\partial\varphi}(\ldots)$, $\Lambda_\alpha\vec{n} = 0$, $\Lambda_\alpha\gamma = 0$. These equalities are related to the symmetry of the medium and the ray field about the plane Oxz. Equality (42) takes a very simple form:

$$\left[\frac{\partial\vec{\tau}}{\partial\varphi} \times \vec{n}\right]^+ = \gamma\left[\frac{\partial\vec{\tau}}{\partial\varphi} \times \vec{n}\right]^-. \tag{44}$$

However,

$$\frac{\partial\vec{\tau}}{\partial\varphi} \overset{(15)}{=} \left(\frac{\dot{u}_t}{V}\right)\vec{j} \overset{(7),(33)}{=} \sin\theta_0\,\frac{(\dot{l}_\perp)_t}{V}\,\vec{j} = \sin\theta_0\,(\dot{l}_\perp)_s\,\vec{j}. \tag{45}$$

Therefore, (44) and (43) can be written in the form

$$\frac{(\dot{l}_\perp)_s^+(M_*)}{V^+(M_*)} = \frac{(\dot{l}_\perp)_s^-(M_*)}{V^-(M_*)}. \tag{46}$$

Conclusion 2. It is likely that the following alternative form of relation (46) is in a better agreement with (21):

$$\frac{(\dot{l}_\perp)_s(M_*^+)}{V(M_*^+)} = \frac{(\dot{l}_\perp)_s(M_*^-)}{V(M_*^-)}. \tag{47}$$

Here the symbols $(\ldots)(M_*^+)$, $(\ldots)(M_*^-)$ denote limiting values of the quantity $(\ldots)$ at the point M_* for the incident ray and outgoing ray, respectively.

Recalculation of $(\dot{l}_\parallel)_s$. Let us assume that $\alpha = \theta$ in (42). For simplicity of representation, we omit the signs "±" below, and give them only in the final result. To expand equality (42), we need the following two rules:

1. The rule of operating by the operator Λ_θ on the superposition $\Phi(\vec{r}(t, \varphi_0, \theta))$. Here Φ is either a scalar function or a vector function that is sufficiently smooth. This rule is represented by the formula

$$\Lambda_\theta \Phi = d\Phi \langle \Lambda_\theta \vec{r} \rangle. \tag{48}$$

Here the right-hand side is the value of the differential $d\Phi$ for the vector $\Lambda_\theta \vec{r}$.

2. The rule of operating by the operator Λ_θ on $\vec{r}(t, \varphi_0, \theta)$. We have actually obtained this rule. Recall that the operating by the operator Λ_θ on a function of t, φ, θ is the differentiation of this function with respect to θ under condition (29). We applied this very operation to the functions $\vec{r}^{\,\pm}(t, \varphi, \theta)$ prior to formula (36), and formula (39) resulted from it. Hence, we obtain the following formula (as before, we omit "±"):

$$\Lambda_\theta \vec{r}(t, \varphi_0, \theta) = \frac{l_\parallel}{(\vec{n} \cdot \vec{\tau})} \vec{n}_\perp. \tag{49}$$

Applying rules (48) and (49) to $\Lambda_\theta \gamma$ and $\Lambda_\theta \vec{n}$, we find

$$\Lambda_\theta \gamma \equiv \Lambda_\alpha \left| \frac{V^+(\vec{r}(t, \varphi_0, \theta))}{V^-(\vec{r}(t, \varphi_0, \theta))} \right| \overset{(48)}{=} d\gamma \langle \Lambda_\alpha \vec{r} \rangle \overset{(49)}{=} \frac{\partial \gamma}{\partial \vec{n}_\perp} \frac{l_\parallel}{(\vec{n} \cdot \vec{\tau})}, \tag{50}$$

$$\Lambda_\theta \vec{n} \equiv \vec{n}\Big(\vec{r}(t, \varphi_0, \theta)\Big) \overset{(48),(49)}{=} \frac{d\vec{n}\langle \vec{n}_\perp \rangle}{(\vec{n} \cdot \vec{\tau})} l_\parallel = \frac{\partial \vec{n}}{\partial \vec{n}_\perp} \frac{l_\parallel}{(\vec{n} \cdot \vec{\tau})}. \tag{51}$$

Now let us find $\Lambda_\theta \vec{\tau}$. For this, we transform expression (41) by substituting t'_α from (35) into it and taking into account that $\vec{u} = l_\parallel \vec{\nu}$ at $\alpha := \theta$ (see (33)). We obtain

$$\Lambda_\theta(\ldots) = -l_\parallel \frac{(\vec{n} \cdot \vec{\nu})}{(\vec{n} \cdot \vec{\tau})} (\ldots)_s + \frac{\partial}{\partial \theta}(\ldots).$$

Then

$$\Lambda_\theta \vec{\tau} = \frac{\partial \vec{\tau}}{\partial \theta} - l_\parallel \frac{(\vec{n} \cdot \vec{\nu})}{(\vec{n} \cdot \vec{\tau})} \frac{\partial \vec{\tau}}{\partial s}.$$

Using (20) and (33) we derive that

$$\frac{\partial \vec{\tau}}{\partial \theta} = \left(\vec{\nu} \cdot \frac{\partial \vec{\tau}}{\partial \theta}\right)\vec{\nu} \overset{(20)}{=} \frac{\dot{u}_t}{V}\vec{\nu} = \dot{u}_s\vec{\nu} \overset{(33)}{=} (\dot{l}_{\|})_s\vec{\nu}. \tag{52}$$

Therefore

$$\Lambda_\theta \vec{\tau} = (\dot{l}_{\|})_s\vec{\nu} - l_{\|}\,\frac{(\vec{n}\cdot\vec{\nu})}{(\vec{n}\cdot\vec{\tau})}\,\frac{\partial \vec{\tau}}{\partial s}. \tag{53}$$

Let us introduce $k_{\rm r}$, $k_{\rm b}$ as the curvatures of the ray and the boundary, respectively, at a point $M \equiv M(t, \varphi_0, \theta)$ by choosing their signs in accordance with the equalities

$$\frac{\partial \vec{\tau}}{\partial s} = k_{\rm r}\vec{\nu}, \tag{54}$$

$$\frac{\partial \vec{n}}{\partial \vec{n}_\perp} = k_{\rm b}\,(\vec{n}\cdot\vec{n}^-)\,\vec{n}_\perp. \tag{55}$$

Note that due to the factor $(\vec{n}\cdot\vec{n}^-)$ introduced in (55), the equality $k_{\rm b}^+ = k_{\rm b}^- = k_{\rm b}$ holds. Here it turns out that $k_{\rm b} \geq 0$ if the direction of convexity of the boundary (along some of the axes) is an acute angle with the vector $\vec{n}^-$, and $k_{\rm b} \leq 0$ otherwise.

We use the notation

$$\frac{l_{\|}}{(\vec{n}\cdot\vec{\tau})} =: \Delta$$

(Δ is the divergence along S, in contrast to $l_{\|}$, where it is the divergence along $t = \text{const}$ or across the ray). Then

$$\begin{aligned}
\Lambda_\theta \vec{\tau} &= \Big[(\dot{l}_{\|})_s - \Delta(\vec{n}\cdot\vec{\nu})\cdot k_{\rm r}\Big]\vec{n} && \text{– see (53) and (54);}\\
\Lambda_\theta \vec{n} &= \Big[\Delta(\vec{n}\cdot\vec{n}^-)\cdot k_{\rm b}\Big]\vec{n}_\perp && \text{– see (51) and (55);}\\
(\Lambda_\theta \gamma)^- &= \frac{\partial \gamma}{\partial \vec{n}_\perp}\,\Delta^- && \text{– see (50).}
\end{aligned}$$

Now, let us express $[\vec{\nu}\times\vec{n}]$, $[\vec{\tau}\times\vec{n}_\perp]$, $[\vec{\tau}\times\vec{n}]$ in terms of $[\vec{\nu}\times\vec{\tau}]$. We have

$$\begin{aligned}
[\vec{\nu}\times\vec{n}] &= \Big[\vec{\nu}\times((\vec{n}\cdot\vec{\nu})\vec{\nu} + (\vec{n}\cdot\vec{\tau})\vec{\tau})\Big] &&= (\vec{n}\cdot\vec{\tau})[\vec{\nu}\times\vec{\tau}],\\
[\vec{\tau}\times\vec{n}_\perp] &= \Big[\vec{\tau}\times((\vec{n}_\perp\cdot\vec{\tau})\vec{\tau} + (\vec{n}_\perp\cdot\vec{\nu})\vec{\nu})\Big] &&= (\vec{n}_\perp\cdot\vec{\nu})[\vec{\tau}\times\vec{\nu}],\\
[\vec{\tau}\times\vec{n}] &= \Big[\vec{\tau}\times((\vec{n}\cdot\vec{\tau})\vec{\tau} + (\vec{n}\cdot\vec{\nu})\vec{\nu})\Big] &&= (\vec{n}\cdot\vec{\nu})[\vec{\tau}\times\vec{\nu}],
\end{aligned}$$

so that

$$\begin{aligned}
[\vec{\nu}\times\vec{n}] &= (\cos\beta)[\vec{\nu}\times\vec{\tau}],\\
[\vec{\tau}\times\vec{n}_\perp] &= -(\cos\beta)[\vec{\nu}\times\vec{\tau}],\\
[\vec{\tau}\times\vec{n}] &= -\operatorname{sgn}(\vec{n}\cdot\vec{\nu})(\sin\beta)[\vec{\nu}\times\vec{\tau}].
\end{aligned}$$

We find from these formulas that

$$\Big[[(\Lambda_\theta\vec{\tau}) \times \vec{n}] + [\vec{\tau} \times (\Lambda_\theta\vec{n})]\Big]^{\pm}$$
$$= \Big[(\dot{l}_{||})_s - \Delta(k_{\rm r}(\vec{n}\cdot\vec{\nu}) - k_{\rm b}(\vec{n}\cdot\vec{n}^-))\Big]^{\pm} \cos\beta^{\pm}[\vec{\nu} \times \vec{\tau}]^{\pm},$$
$$(\Lambda_\theta\gamma)^-[\vec{\tau} \times \vec{n}] = -\frac{\partial\gamma}{\partial\vec{n}_\perp^-}\,\Delta^-(\vec{n}\cdot\vec{\nu})[\vec{\nu} \times \vec{\tau}]. \qquad (56)$$

The equality $[\vec{\nu}^+ \times \vec{\tau}^+] = [\vec{\nu}^- \times \vec{\tau}^-]$ holds due to the above construction. Note that $\operatorname{sgn}(\vec{n}\cdot\vec{\nu})$ controls the positiveness of negativeness of the angle of incidence β if we consider it oriented between $\vec{\tau}$ and $\vec{n}$: if $(\vec{n}\cdot\vec{\nu}) > 0$, then $\arg(\vec{\tau},\vec{n}) > 0$; if $(\vec{n}\cdot\vec{\nu}) < 0$, then $\arg(\vec{\tau},\vec{n}) < 0$.

And, finally, by introducing the angle of incidence β^- and the angle of reflection/refraction β^+ and by using expressions (56) in (42) (written for $\alpha := \theta$), we obtain a formula for recalculation of $(l_{||})_s$ in the following form (all values are related to the point M_*)

$$(\vec{n}^+\cdot\vec{n}^-)\Big[(\dot{l}_{||})_s - \frac{l_{||}}{\cos\beta^+}\Big(k_{\rm r}(\vec{n}\cdot\vec{\nu}) + k_{\rm b}(\vec{n}\cdot\vec{n}^-)\Big)\Big]^+\cos\beta^+ \qquad (57)$$
$$= \gamma\Big[(\dot{l}_{||})_s - \frac{l_{||}}{\cos\beta^-}\Big(k_{\rm r}(\vec{n}\cdot\vec{\nu}) + k_{\rm b}(\vec{n}\cdot\vec{n}^-)\Big)\Big]^-\cos\beta^- - \frac{\partial\gamma}{\partial\vec{n}_\perp^-}\frac{l_{||}^-(\vec{n}^-\cdot\vec{\nu}^-)}{\cos\beta^-};$$
$$(\vec{n}\cdot\vec{\nu}) = \operatorname{sgn}(\vec{n}\cdot\vec{\nu})\sin\beta.$$

5. Azimuthal divergence. Summary of results

Formulas (46) and (47) make it possible to formulate the following

Result 1. In a medium with a two-dimensional piecewise smooth velocity for each regular ray L lying in the transverse plane, the following identity is valid:

$$\frac{(\dot{l}_\perp)_s(M)}{V(M)} = \text{const}, \qquad M \in L. \qquad (58)$$

Here $l_\perp(M)$ is the azimuthal divergence at a point M along L, const, and the constant depends only on L. At the points $M = M_i$, $i = 1,\dots,n-1$, the ray meets with the interfaces: here the identity (58) is understood as an equality of the limiting values at the incident and refracted/reflected parts of the ray L. Then, in accordance with (37), $l_\perp(M)$ is a continuous function of a point $M \in L$. From (58), we have

$$(\dot{l}_\perp)_s(M) = \frac{(\dot{l}_\perp)_s(M_0)}{V(M_0)}, \qquad (59)$$

where $(\dot{l}_\perp)_s(M_0) = 1$.

Therefore, integrating (59) over the length of the arc s along L from M_0 to M_n, we obtain the following

Result 2. In a medium with a two-dimensional piecewise smooth velocity, the azimuthal geometrical divergence $l_\perp$ along any regular ray L is given by the formula

$$l_\perp(M_n) \equiv l_\perp(M_0, M_n) = \frac{1}{V(M_0)} \int\limits_{M_0}^{M_n} V(M) ds_M. \tag{60}$$

Assume now that in each block B_i, where a section $L(M_i, M_{i+1})$ of the ray L between the points M_i and M_{i+1}, the velocity V_i has a constant direction $\vec{q}_i$ ($|\vec{q}_i| = 1$) of the gradient (the modulus of the gradient does not need to be constant). Then the Snellius law is satisfied in each block B_i (cf. (58)):

$$\frac{[\dot{\vec{r}}_s(M) \times \vec{q}_i]_y}{V(M)} = \text{const}_i, \quad M \in L(M_i, M_{i+1}).$$

Hence, we can write

$$V(M) = \frac{V(M_i^+)}{[\vec{r}(M_i^+) \times \vec{q}_i]_y} \left[\dot{\vec{r}}_s(M) \times \vec{q}_i\right]_y.$$

Integrating this equality along L from M_i to M_{i+1}, we find that

$$\begin{aligned} \int\limits_{M_i}^{M_{i+1}} V(M) ds_M &= \frac{V^+(M_i)}{[\vec{r}^+(M_i) \times \vec{q}_i]_y} \left[\left(\int\limits_{s(M_i)}^{s(M_{i+1})} \dot{\vec{r}}_s ds\right) \times \vec{q}_i\right]_y \\ &= V^+(M_i) \frac{|\overrightarrow{M_i M_{i+1}} \times \vec{q}_i|}{|\vec{r}^+(M_i) \times \vec{q}_i|}. \end{aligned} \tag{61}$$

Let us denote (this notation does not contradict (25))

$$l(M_i, M_{i+1}) = \frac{|\overrightarrow{M_i M_{i+1}} \times \vec{q}_i|}{|\vec{r}^+(M_i) \times \vec{q}_i|}. \tag{62}$$

We obtain from (60)–(62)

Result 3. In a two-dimensional block medium with a constant direction $\vec{q}_i$ of the velocity gradient V_i in each block B_i, the azimuthal geometrical divergence along a regular ray $L(M_0 M_n)$ is given by the formula

$$\begin{aligned} l_\perp(M_n) \equiv l_\perp(M_0, M_n) = \frac{1}{V(M_0)} \Big[V(M_0) l(M_0, M_1) + \\ V^+(M_1) l(M_1, M_2) + \ldots + V^+(M_{n-1}) l(M_{n-1}, M_n) \Big]. \end{aligned} \tag{63}$$

Here $V^+(M_i)$, $i = 1,\dots,n-1$, are the limiting vales of the velocity V at the points M_i from the refracted/reflected part of the ray L. The quantities $l(M_i, M_{i+1})$, $i = 0,\dots,n-1$, are calculated by using (62). They are the azimuthal geometrical divergence along sections $L(M_i, M_{i+1})$ of the ray L with a polar axis at the source M_i directed along the vector $\vec{q}_i$.

Below we show how to construct, using dividers and ruler, a section equal to $l(M_i, M_{i+1})$, if the points M_i, M_{i+1} and the vectors $\vec{\tau}^+(M_i) \equiv \vec{\tau}_i$, $\vec{q}_i$ are given (Figure 3).

Let us draw a straight line parallel to the vector $\vec{q}_i$ through the point M_{i+1}. Let us draw another straight line parallel to the vector $\vec{\tau}_i$ through the point M_i. Let M_i^{i+1} be the point of intersection of these straight lines. Then

$$l(M_i, M_{i+1}) = |M_i M_i^{i+1}|.$$

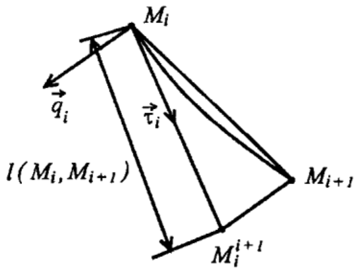

Figure 3. Construction of the geometrical divergence $l(M_i, M_{i+1})$

We adapt expression (63) for the case when all $b_i = 0$ in formula (1) , i.e., when $V_i = a_i x + c_i$. Then $\vec{q}_i = \operatorname{sgn} a_i \vec{i}$ ($\vec{i}$ is the unit vector of the Ox-axis), and (62) is written in the form

$$l(M_i, M_{i+1}) = \frac{|\overrightarrow{M_i M_{i+1}} \times \vec{i}|}{|\vec{\tau}^+(M_i) \times \vec{i}|}.$$

But $|\overrightarrow{M_i M_{i+1}} \times \vec{i}| = |z_{i+1} - z_i|$, where $z_i = z(M_i)$, $i = 0,\dots,n-1$, and

$$|\vec{\tau}^+(M_i) \times \vec{i}| = \sin \psi^+(M_i).$$

Here $\psi^+(M_i)$ is the angle of the vector $\vec{\tau}^+(M_i)$ with the axis Ox. Thus, we have obtained

Result 4. For the model of medium described in Section 1 when all $b_i = 0$ in (1), the quantity $l_\perp(M_n)$ determined from (3) at $M := M_n$ is calculated by the formula

$$l_\perp(M_n) = \frac{1}{V(M_0)} \sum_{i=0}^{n-1} V^+(M_i) \frac{|z_{i+1} - z_i|}{\sin \psi^+(M_i)}. \tag{64}$$

Here $|z_{i+1} - z_i|$ is the shift of the L ray section along the Oz axis between the points M_i and M_{i+1}, $\psi^+(M_i)$ is the angle of the Ox axis with the tangential vector M_i to the part of the ray L going from the point M_i; $V^+(M_i)$ is the limiting velocity value at the point M_i from the part of the ray L that goes from M_i, $i = 0,\dots,n-1$.

6. Polar geometrical divergence. Summary of results

Note that for polar divergence there is no single expression similar to (63). Calculation of $l_{||}$ at the final point M_n of a ray L is made in n steps by performing at every i-th step ($i = 0, \ldots, n-1$) the following operations of (a) recalculation and (b) translation:

(a) at $i = 0$, the initial conditions $l_{||}^{+}(M_i) := 0$ and $(\dot{l}_{||})_s^{+}(M_i) := 1$ for $l_{||}$ at the source are set; at $i \neq 0$, the recalculation of $l_{||}^{-}(M_i)$ and $(\dot{l}_{||})_s^{-}(M_i)$ to $l_{||}^{+}(M_i)$ and $(\dot{l}_{||})_s^{+}(M_i)$ is done by formulas (40) and (57);

(b) the "translation" of $l_{||}$ and $\dot{l}_{||}$ along $L(M_iM_{i+1})$ from their values $l_{||}^{+}(M_i)$ and $\dot{l}_{||}^{+}(M_i)$ at the point M_i to the values $l_{||}^{-}(M_{i+1})$ and $\dot{l}_{||}^{-}(M_{i+1})$ at the point M_{i+1} is provided by formulas (22), (24), and (25); in this case, $l_{||}^{-}(M_n) \equiv l_{||}(M_n)$, and therefore we get the result at the step $i = n - 1$.

The above algorithm is only a preliminary formulation of finding $l_{||}(M_n)$, because the recalculation and translation formulas are still not written in a form that was announced. Besides, recalculation formulas should be specialized for linear velocities, because the expressions presented were obtained without the assumption of linearity. Here we are going to give expressions for the derivative $\partial\gamma/\partial n_{\perp}^{-}$ (see (57)) and for the ray curvature.

Let us find an expression for $(\partial\gamma/\partial n_{\perp}^{-})(M_i)$. Recall that

$$V_i(x, z) = a_i x + b_i z + c_i, \quad i = 0, \ldots, n-1, \tag{65}$$

is the velocity along the section of the ray L between the points M_i and M_{i+1} (i.e., the velocity in block B_i). Then the function γ (see (43)) at the point M_i is given by the formula

$$\gamma(M) \overset{(43)}{=} \frac{V^{+}(M)}{V^{-}(M)} = \frac{V_i(M)}{V_{i-1}(M)} = \frac{a_i x + b_i z + c_i}{a_{i-1}x + b_{i-1}z + c_{i-1}}. \tag{66}$$

Hence, we find that

$$\nabla\gamma(M_i) = \frac{1}{V_{i-1}(M_i)}\Big[(a_i - \gamma_i a_{i-1})\vec{i} + (b_i - \gamma_i b_{i-1})\vec{k}\Big], \tag{67}$$

where

$$\gamma_i \equiv \gamma(M_i) = \frac{a_i x_i + b_i z_i + c_i}{a_{i-1}x_i + b_{i-1}z_i + c_{i-1}}. \tag{68}$$

It can be easily derived from (38) that

$$\vec{n}_{\perp}^{-}(M_i) \equiv \vec{n}_{i\perp}^{-} = n_z^{-}(M_i)\cdot\vec{i} - n_x^{-}(M_i)\cdot\vec{k} \equiv n_{iz}^{-}\vec{i} - n_{ix}^{-}\vec{k}. \quad (69)$$

Thus, in accordance with (66) and (68), we have

$$\begin{aligned}\frac{\partial\gamma}{\partial\vec{n}_{\perp}^{-}}(M_i) &\stackrel{\text{df}}{=} (\nabla\gamma\cdot\vec{n}_{\perp}^{-}) \\ &= \frac{1}{V_{i-1}(M_i)}\Big[(a_i - \gamma_i a_{i-1})n_{iz}^{-} - (b_i - \gamma_i b_{i-1})n_{ix}^{-}\Big]. \end{aligned} \quad (70)$$

Now let us find an expression for the ray curvature k_r given by formula (54). It is known that in the case of linear velocity $V(x,z) = ax+bz+c$ each ray is a circumference. Its center O_*, for a ray lying in the plane Oxz, is at the intersection of the straight line $V(x,z)=0$ with a straight line going through a point $M_* = (x_*, z_*)$ of the ray normal to the tangential vector $\vec{\tau}$ to the ray at this point (Figure 4).

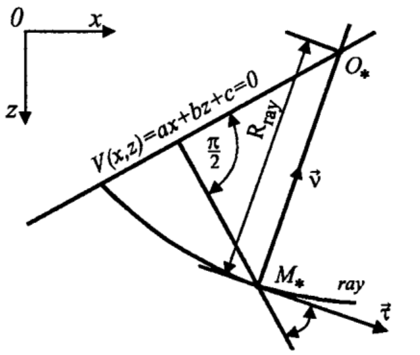

Figure 4. Construction of a ray passing through M_* in the direction $\vec{\tau}$ in a medium with linear velocity

We write the equation $V(x,z)=0$ in the form

$$a(x-x_*) + b(z-z_*) = -V(x_*,z_*) \equiv -V(M_*).$$

The equation of a straight line passing through M_* orthogonally to $\vec{\tau}$ is

$$\tau_x(x-x_*) + \tau_z(z-z_*) = 0.$$

Solving this system for $x-x_*$ and $z-z_*$ (the coordinates of the vector $\overrightarrow{M_*O_*}$), we find that

$$\begin{aligned}(M_*O_*)_x &\equiv (x-x_*) = -\frac{V(M_*)\tau_z}{a\tau_z - b\tau_x}, \\ (M_*O_*)_z &\equiv (z-z_*) = \frac{V(M_*)\tau_x}{a\tau_z - b\tau_x}.\end{aligned}$$

Hence, taking into account that $\vec{\nu} \stackrel{\text{df}}{=} \tau_z\vec{i} + (-\tau_x)\vec{k}$, $\nabla V = a\vec{i} + b\vec{k}$, we can write

$$\frac{\partial \vec{\tau}}{\partial s}(M_*) = -\frac{(\nabla V \cdot \vec{\nu})}{V(M_*)}\vec{\nu}.$$

Comparing it with (54), we obtain

$$k_{\rm r} = -\frac{(\nabla V \cdot \vec{\nu})}{V(M_*)}, \quad \vec{\nu} = \tau_z \vec{i} + (-\tau_x)\vec{k}. \tag{71}$$

Let us introduce the following notation:

$$k_i = -\frac{(\nabla V_i \cdot \vec{\nu}_i^+)}{V_i(M_i)} \quad \left(= -\frac{(\nabla V_i \cdot \vec{\nu}_{i+1}^-)}{V_i(M_{i+1})}\right). \tag{72}$$

Here

$$\vec{\nu}_i^\pm = \vec{\nu}^\pm(M_i) = \tau_z^\pm(M_i)\vec{i} - \tau_x^\pm(M_i)\vec{k} \equiv \tau_{iz}^\pm \vec{i} - \tau_{ix}^\pm \vec{k}. \tag{73}$$

Then the quantities $k_{\rm r}^\pm(M_i)$ in the recalculation formula (57) at $M_* = M_i$, $i = 0, \ldots, n-1$, are given by the equalities

$$\begin{aligned} k_{\rm r}^+(M_i) &= k_i, \quad i = 0, \ldots, n-1; \\ k_{\rm r}^-(M_i) &= k_{i-1}, \quad i = 1, \ldots, n. \end{aligned} \tag{74}$$

Now we have done all the necessary work to write formulas for calculation of $l_{||}(M_n)$. We try to make these formulas convenient for programming. Since the final result is the quantity $l_{||}(M_n)$, we simplify the notation for the intermediate quantities $l_{||}^\pm(M_i)$, $(\dot{l}_{||}^\pm)_s(M_i)$, and $l(M_i, M_{i+1})$ by using the symbols $l_i^\pm$, $\dot{l}_i^\pm$, and l_i^{i+1}, respectively. The meaning of other notation will be explained when necessary in the process of derivation of the formulas independently of the preceding text. The numbers of formulas from which some equalities were obtained will be indicated over the equality signs or within the brackets given in the right-hand sides. Thus, we have obtained

Result 5: An algorithm for calculation of polar geometrical divergence $l_{||}$. We calculate $l_{||}(M_n)$, which is the polar geometrical divergence along a ray L lying in the plane Oxz with the start at $M_0 = (x_0, z_0)$ for the two-dimensional block linear velocity model described in Section 1. The ray L reaches the final point $M_n = (x_n, z_n)$ through $(n-1)$ interfaces S_i at the points $M_i = (x_i, z_i)$, $i = 1, \ldots, n-1$ at a speed in each block B_i, where the ray L lies between the points M_i and M_{i+1}. For this speed we use the formula

$$V_i(x, z) = a_i x + b_i z + c_i.$$

The interfaces S_i in the vicinity of the points M_i are specified by equations of the form $z = f_i(x)$ or $x = f_i(z)$. The calculation is based on finding recurrently quantities l_i^+, $\dot{l}_i^+$, $i = 0, \ldots, n-1$, l_i^-, $\dot{l}_i^-$, $i = 1, \ldots, n$, where $l_n^- = l_{||}(M_n)$. The rules used are as follows:

1) Initial conditions: $l_0^+ = 0$, $\dot{l}_0^+ = 1$;

2) Translation formulas:

$$l_{i+1}^- = l_i^+ + \dot{l}_i^+ l_i^{i+1} \quad \text{(obtained from (24))},$$
$$\dot{l}_{i+1}^- = \frac{V_i(M_{i+1})}{V_i(M_i)} \dot{l}_i^+ \quad \text{(obtained from (22))},$$

$i = 0, \ldots, n-1$. Here

$$l_i^{i+1} \overset{(25)}{=} l(M_i, M_{i+1}) \overset{(62)}{=} \frac{\begin{vmatrix} (x_{i+1} - x_i) & a_i \\ (z_{i+1} - z_i) & b_i \end{vmatrix}}{\begin{vmatrix} \tau_{ix}^+ & a_i \\ \tau_{iz}^+ & b_i \end{vmatrix}},$$

τ_{ix}^+, τ_{iz}^+ are the components of the unit vector $\vec{\tau}_i^+ = \vec{\tau}^+(M_i)$. It is tangent to the reflected/refracted part of the ray L at the point M_i;

3) Recalculation formulas:

$$l_i^+ = C_i l_i^- \quad \text{(from (40))},$$
$$\dot{l}_i^+ = A_i \dot{l}_i^- + B_i l_i^- \quad \text{(from (57) and (40))},$$

$i = 1, \ldots, n-1$. Here

$$C_i = (\vec{n}_i^+ \cdot \vec{n}_i^-) \frac{\cos\beta_i^+}{\cos\beta_i^-}, \quad \cos\beta_i^\pm = |(\vec{n}_i \cdot \vec{\tau}_i^\pm)|, \quad \vec{\tau}_i^\pm = \tau_{ix}^\pm \vec{i} + \tau_{iz}^\pm \vec{k} \equiv \vec{\tau}^\pm(M_i),$$

are the unit vectors tangent to the incident part of the ray L at the point M_i for the sign "$-$" and to the reflected-refracted part for the sign "$+$"; $\vec{n}_i$ is the unit vector normal to S_i at the point M_i. It is determined by the following equalities:

$$\vec{n}_i = \begin{cases} \dfrac{f_i'(x_i)}{\sqrt{1 + (f_i'(x_i))^2}} \vec{i} - \dfrac{1}{\sqrt{1 + (f_i'(x_i))^2}} \vec{k}, & \text{if } S_i \text{ is specified by equation } z = f_i(x); \\ \dfrac{1}{\sqrt{1 + (f_i'(z_i))^2}} \vec{i} - \dfrac{f_i'(z_i)}{\sqrt{1 + (f_i'(z_i))^2}} \vec{k}, & \text{if } S_i \text{ is specified by equation } x = f_i(z); \end{cases}$$

$$\vec{n}_i^\pm = \operatorname{sgn}(\vec{n}_i \cdot \vec{\tau}_i^\pm)\vec{n}_i; \quad A_i = \frac{\gamma_i}{C_i}; \quad \gamma_i = \frac{V_i(M_i)}{V_{i-1}(M_i)};$$

$$B_i = \frac{(\vec{n}^+ \cdot \vec{n}^-)}{\cos\beta_i^-} \left[K_i^+ - (\vec{n}^+ \cdot \vec{n}^-) A_i K_i^- - \frac{(\vec{n}_i^- \cdot \vec{\nu}_i^-)}{\cos\beta_i^+} Q_i \right];$$

$$\vec{\nu}_i^{\pm} = \tau_{iz}^{\pm}\vec{i} + (-\tau_{ix}^{\pm})\vec{k}, \quad (\vec{\nu}_i^{\pm} \perp \vec{\tau}_i^{\pm});$$
$$K_i^{\pm} = k_{\mathrm{r}i}^{\pm}(\vec{n}_i^{\pm} \cdot \vec{\nu}_i^{\pm}) + k_{\mathrm{b}i}(\vec{n}_i^{\pm} \cdot \vec{n}_i^{-});$$
$$k_{\mathrm{r}i}^{+} \overset{(74)}{=} k_i; \quad k_{\mathrm{r}i}^{-} \overset{(74)}{=} k_{i-1}; \quad k_i \overset{(72)}{=} -\frac{a_i\nu_{ix}^{+} + b_i\nu_{iz}^{+}}{V_i(M_i)};$$
$$Q_i \equiv \left(\frac{\partial\gamma}{\partial n_{\perp}^{-}}(M_i)\right) \overset{(70)}{=} \frac{(a_i - \gamma_i a_{i-1})n_{iz}^{-} - (b_i - \gamma_i b_{i-1})n_{ix}^{-}}{V_{i-1}(M_i)};$$

$k_{\mathrm{b}i}$ is the curvature of the interface S_i at the point M_i determined by using the rule (see (55)):

$$k_{\mathrm{b}i} = \begin{cases} (-\operatorname{sgn} n_{iz}^{-})\chi_i & \text{if } S_i \text{ is specified by equation } z = f_i(x), \\ (-\operatorname{sgn} n_{ix}^{-})\chi_i & \text{if } S_i \text{ is specified by equation } x = f_i(z); \end{cases}$$
$$\chi_i = \begin{cases} \dfrac{f_i''(x_i)}{\left(1 + f_i'^2(x_i)\right)^{3/2}} & \text{if } S_i \text{ is specified by equation } z = f_i(x), \\ \dfrac{f_i''(z_i)}{\left(1 + f_i'^2(z_i)\right)^{3/2}} & \text{if } S_i \text{ is specified by equation } x = f_i(z). \end{cases}$$

Here the calculation algorithm for $l_{\|}(M_n)$ terminates.

Conclusion 3 (On correctness of the calculation formulas). In the calculation formulas for $l_{\|}$ and $l_{\perp}$ presented, the expressions we used for $l_i^{i+1} = l(M_i, M_{i+1})$ (see (28), (62)) represent an indeterminate form of $0/0$ when the ray goes from the point M_i in the direction $\vec{\tau}_i^{+}$. This direction is parallel to the gradient direction of the velocity $V_i(x,z) = a_i x + b_i z + c_i$, i.e., the determinant $\begin{vmatrix} \tau_{ix}^{+} & a_i \\ \tau_{iz}^{+} & b_i \end{vmatrix}$ is equal or close to zero, or when the difference $z_{i+1} - z_i$ in (64) is equal or close to zero. Here one of the formulas known in the literature for the evaluation of indeterminate forms should be used to calculate l_i^{i+1}. This is one of such formulas, which we give in vector form:

$$l_i^{i+1} = |M_i M_{i+1}|\left(\cos(\widehat{\vec{\tau}_i^{+}, \overrightarrow{M_i M_{i+1}}}) + \frac{|M_i M_{i+1}|}{2H_i}(\vec{q}_i \cdot \vec{\tau}_i^{+})\right).$$

Here H_i is the distance from the point M_i to the straight line $V_i(x,z) = 0$,

$$H_i = \frac{V_i(M_i)}{|\nabla V_i|}, \quad \vec{q}_i = \frac{\nabla V_i}{|\nabla V_i|}. \tag{75}$$

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References

[1] Alekseev A.S., Gelchinsky B.Ya. On the ray method to calculate wave fields in the case of inhomogeneous media with curvilinear interfaces // Questions of the Dynamic Theory of Seismic Wave Propagation. – Leningrad, 1956. – Vol. III. – P. 107–160.

[2] Belonosova A.V., Tadzhimukhamedova S.S., Alekseev A.S. On calculating hodographs and geometrical divergence of rays in inhomogeneous media // Some Methods and Algorithms for Interpretation of Geophysical Data. – Moscow, 1967. – P. 124–136.

[3] Popov M.M. On one method to calculate geometrical divergence in an inhomogeneous medium with interfaces // Dokl. Akad. Nauk SSSR. – 1977. – Vol. 237, № 5. – P. 1059–1062.

[4] Popov M.M., Psencik I. Computation of ray amplitudes in inhomogeneous media with curved interfaces // Studia Geoph. et Geod. – 1978. – Vol. 22. – P. 248.

[5] Belonosova A.V., Tsetsokho V.A. On the calculation of geometrical divergence in inhomogeneous media with smooth interfaces // Numerical Methods in Interpretation of Geophysical Observations. – Novosibirsk, 1980. – P. 14–24.

[6] Gelchinsky B.Ya. A formula for geometrical divergence // Questions of the Dynamic Theory of Seismic Wave Propagation. – Leningrad, 1961. – Issue 5. – P. 47–53.

[7] Goldin S.V. et al. Curvature of the seismic wave front in layered-gradient media // Geologiya i Geofizika. – 1978. – № 4. – P. 118–124.

[8] Matveyeva N.N., Antonova L.N. A method and a program for the calculation of the kinematics and dynamics of volume waves in three-dimensional inhomogeneous block media // Programs for Interpretation of Seismic Observations. – Leningrad, 1977. – Issue 2. – P. 173–221.

[9] Anikonov Yu.E. Some special solutions of an inverse kinematic problem // Mathematical Problems of Geophysics. – Novosibirsk, 1973. – Issue 4. – P. 30–60.

[10] Anikonov Yu.E., Pivovarova N.B. Formulas of ray, time, and amplitude for one simple class of velocities // Mathematical Problems of Geophysics. – Novosibirsk, 1973. – Issue 4. – P. 193–198.