Bull. Nov. Comp. Center, Num. Anal., 6 (1994), 37–42

Locally two-sided approximate solutions in parabolic problems*

K.A. Chertova

By means of the fictitous regions method [1] locally two-sided estimations the initial boundary value problem with the second order parabolic equation of the first type are obtained.

In order to find approximate solutions of mathematical physics boundary value problems it is useful to obtain locally two-sided approximate solutions because with their help the approximation order can be increased. By means of the fictitous regions method (FRM) locally two-sided approximate solutions for boundary value problems with the second order elliptic operator are examined in ([2], [3]).

In the present work one type of the auxiliary problem for the second order parabolic equation is determined by means of the FRM. It was proved that two continuations with different sign coefficients of the auxiliary problem are possible. It was shown that the main members of errors in the expansions of the solutions with different signs are also of different signs. And therefore the semi-sum of solutions approximates an exact solution with the order higher than that of each one-sided approximate solution.

Thus, let $D \subset R^n$ be a finite one-connected domain with the boundary $\gamma \in C^2$, $T > 0$, and let $D_T = \{(x,t): \ x \in D, \ t \in (0,T)\}$ be a cylinder with the side surface $\gamma_T = \{(x,t): \ x \in \gamma, \ t \in (0,T)\}$.

Let us consider the initial boundary value problem for the second order parabolic operator of the first kind

$$\begin{aligned} &Lu \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}\Big) + a(x,t)u = f(x,t), \quad (x,t) \in D_T, \\ &u(x,0) = \varphi(x), \quad x \in D, \qquad\qquad (1) \\ &u(x,t) = 0, \quad (x,t) \in \gamma_T. \end{aligned}$$

*Supported by the Russian Foundation of Fundamental Research under Grant 93–012–494.

The coefficients $a_{i,j}(x,t)$ and $a(x,t)$ satisfy the following conditions:

$$a(x,t) \geq 0, a_{i,j}(x,t) = a_{j,i}(x,t),$$
$$\nu_0 \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \leq \nu_1 \sum_{i=1}^{n} \xi_i^2, \quad \nu_0 > 0,$$

for all $(x,t) \in D_T$ and $(\xi_1, \dots, \xi_n)$.

We suppose that

$$a_{i,j}(x,t) \in C(\bar{D}_T), \quad \partial a_{i,j}/\partial x_k(x,t) \in C(\bar{D}_T), \quad a(x,t) \in C(\bar{D}_T),$$
$$f(x,t) \in L_2(D_T), \quad \varphi(x) \in \overset{\circ}{W}{}_2^1(D).$$

With such suppositions problem (1) has the only solution which belongs to the space $W_{2,0}^{2,1}(D_T)$. In this case the following estimate takes place ([4], p. 388):

$$\|u\|_{W_{2,0}^{2,1}(D_T)} \leq C_1\big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\overset{\circ}{W}{}_2^1(D)}\big). \tag{2}$$

By means of the fictitous region D_1 we shall supplement D to some other region D_0 with the boundary Γ, which does not have common points with γ. Let us designate $D_T^i = \{(x,t): \ x \in D_i, \ t \in (0,T)\}$, $i = 0,1$, $\Gamma_T = \{(x,t): \ x \in \Gamma, \ t \in (0,T)\}$.

The auxiliary problem of the FRM is determined in the following way:

$$\begin{array}{ll} L_\varepsilon u_\varepsilon = f, \quad (x,t) \in D_T, & \dfrac{\partial u_\varepsilon}{\partial t} - \triangle u_\varepsilon = 0, \quad (x,t) \in D_T^1, \\ u_\varepsilon(x,0) = \varphi, \quad x \in D, & u_\varepsilon(x,0) = 0, \quad x \in D_1. \\ u_\varepsilon(x,t) = 0, \quad (x,t) \in \Gamma_T; & \end{array} \tag{3}$$

On the surface γ_T, where coefficients are discontinuous, the agreement conditions are assumed

$$u_\varepsilon\Big|_{\gamma_T}^{+} = u_\varepsilon\Big|_{\gamma_T}^{-}, \qquad \frac{\partial u_\varepsilon}{\partial N}\Big|_{\gamma_T}^{+} = \frac{Q}{\varepsilon}\frac{\partial u_\varepsilon}{\partial n}\Big|_{\gamma_T}^{-}. \tag{4}$$

Here $\varepsilon > 0$, $\triangle$ is the Laplace operator,

$$\frac{\partial u_\varepsilon}{\partial N} = \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial u_\varepsilon}{\partial x_j}\cos(\bar{n}, \bar{x}_i), \quad \frac{\partial u_\varepsilon}{\partial n} = \sum_{i=1}^{n} \frac{\partial u_\varepsilon}{\partial x_i}\cos(\bar{n}, \bar{x}_i),$$

$\bar{n}$ is the vector of the external normal to γ, signs "plus" or "minus" mean that the corresponding value is a limit value while a point (x,t) approaches γ_T inside or outside the domain D_T, Q is a parameter which can equal either 1 or -1.

Let us show that there exists such a parameter ε_0 that for all $\varepsilon < \varepsilon_0$ the auxiliary problem (3), (4) is uniquely solvable at any sign of Q. In this case:

$$\|u - \frac{1}{2}(u_\varepsilon^+ + u_\varepsilon^-)\|_{W_2^{2,1}(D_T)} \le C\varepsilon^2, \tag{5}$$

where u_ε^+ and u_ε^- are the solutions of the auxiliary problem (3), (4) at $Q = 1$ and $Q = -1$, respectively.

Let us seek for the solution of problem (3), (4) as power series $S_1 = \sum_{k=0}^{\infty} \varepsilon^k v_k$ in the domain D_T, $S_2 = \sum_{k=1}^{\infty} \varepsilon^k w_k$ in the domain D_T^1. Substituting formally S_1, S_2 in (3), (4) and equating the expressions at corresponding power of ε, one can obtain a sequence of problems for the determination of v_k and w_k:

$$\begin{aligned}
&Lv_0 = f, \quad (x,t) \in D_T, && \frac{\partial w_1}{\partial t} - \Delta w_1 = 0, \quad (x,t) \in D_T^1,\\
&v_0(x,0) = \varphi(x), \quad x \in D, && w_1(x,0) = 0, \quad x \in D_1,\\
&v_0 = 0, \quad (x,t) \in \gamma_T; && \frac{\partial w_1}{\partial n} = Q\frac{\partial v_0}{\partial N}, \quad (x,t) \in \gamma_T,\\
& && w_1 = 0, \quad (x,t) \in \Gamma_T;
\end{aligned} \tag{6}$$

for $k \ge 1$:

$$\begin{aligned}
&Lv_k = 0, \quad (x,t) \in D_T, && \frac{\partial w_{k+1}}{\partial t} - \Delta w_{k+1} = 0, \quad (x,t) \in D_T^1,\\
&v_k(x,0) = 0, \quad x \in D, && w_{k+1}(x,0) = 0, \quad x \in D_1,\\
&v_k = w_k, \quad (x,t) \in \gamma_T; && \frac{\partial w_{k+1}}{\partial n} = Q\frac{\partial v_k}{\partial N}, \quad (x,t) \in \gamma_T,\\
& && w_{k+1} = 0, \quad (x,t) \in \Gamma_T.
\end{aligned}$$

With such suppositions all the problems in (6) are uniquely solvable and $v_k \in W_2^{2,1}(D_T)$, $k = 0, 1, \ldots$, $w_k \in W_2^{2,1}(D_T^1)$, $k = 1, 2, \ldots$.

Theorem 1. *There exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ the series S_1 and S_2 are absolutely convergent in the spaces $W_2^{2,1}(D_T)$ and $W_2^{2,1}(D_T^1)$ correspondingly, and the following equalities take place*

$$u_\varepsilon = S_1, \quad (x,t) \in D_T, \qquad u_\varepsilon = S_2, \quad (x,t) \in D_T^1, \tag{7}$$

where u_ε is the solution to problem (3), (4).

Proof. Using *a priori* estimates for the solutions, traces propeties and agreement conditions on γ_T, we have

$$\begin{aligned}\|w_k\|_{W_2^{2,1}(D_T^1)} &\leq C_2\Big\|\frac{\partial w_k}{\partial n}\Big\|_{W_2^{\frac{1}{2},1}(\gamma_T)} = C_2\Big\|\frac{\partial v_{k-1}}{\partial N}\Big\|_{W_2^{\frac{1}{2},1}(\gamma_T)} \\ &\leq C_2C_3\|v_{k-1}\|_{W_2^{2,1}(D_T)},\end{aligned} \tag{8}$$

where the constants C_2 and C_3 depend only on the regions D, D_1 and the coefficients of the operator L and are ε-independent.

The estimate (8) shows that the convergence of the series S_1 in the norm of the space $W_2^{2,1}(D_T)$ is followed by the convergence of the series S_2 in the norm of the space $W_2^{2,1}(D_T^1)$. Let us prove the convergence of the series S_1.

$$\|v_k\|_{W_2^{2,1}(D_T)} \leq C_4\|v_k\|_{W_2^{\frac{3}{2},1}(\gamma_T)} = C_4\|w_k\|_{W_2^{\frac{3}{2},1}(\gamma_T)} \leq C_4C_5\|w_k\|_{W_2^{2,1}(D_T^1)}.$$

Taking into account (2) and (8), one can obtain

$$\begin{aligned}\|v_k\|_{W_2^{2,1}(D_T)} &\leq C_6\|v_{k-1}\|_{W_2^{2,1}(D_T)}, \quad k \geq 1, \\ \|v_0\|_{W_2^{2,1}(D_T)} &\leq C_1\big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\overset{\circ}{W}{}_2^1(D)}\big),\end{aligned} \tag{9}$$

where $C_6 = C_2C_3C_4C_5$.

Thus, if $\varepsilon < \varepsilon_0 = C_6^{-1}$, the series S_1 is absolutely convergent in the norm of $W_2^{2,1}(D_T)$, and therefore, the series S_2 is also absolutely convergent in the norm of $W_2^{2,1}(D_T^1)$. From the completeness of the space $W_2^{2,1}$ it follows that $S_1 \in W_2^{2,1}(D_T)$ and $S_2 \in W_2^{2,1}(D_T^1)$.

Multiplying equations (6) for v_k and w_k by ε^k and summing with respect to k, one can obtain

$$\begin{aligned} &LS_1 = f, \quad (x,t) \in D_T, && \frac{\partial S_2}{\partial t} - \triangle S_2 = 0, \quad (x,t) \in D_T^1, \\ &S_1(x,0) = \varphi(x), \quad x \in D, && S_2(x,0) = 0, \quad x \in D_1, \\ &S_1 = S_2, \quad (x,t) \in \gamma_T; && \frac{\partial S_2}{\partial n} = Q\varepsilon\frac{\partial S_1}{\partial N}, \quad (x,t) \in \gamma_T, \\ & && S_2(x,t) = 0, \quad (x,t) \in \Gamma_T. \end{aligned} \tag{10}$$

Thus, $u_\varepsilon = S_1$ in D_T, $u_\varepsilon = S_2$ in D_T^1 for all $0 < \varepsilon < \varepsilon_0$. □

It follows from Theorem 1 that the auxiliary problem (3), (4) is uniquely solvable for all $0 < \varepsilon < \varepsilon_0$ and the following approximation estimate takes place

$$\begin{aligned} \|u - u_\varepsilon^+\|_{W_2^{2,1}(D_T)} &\leq C_7'\varepsilon\big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\mathring{W}_2^1(D)}\big), \\ \|u - u_\varepsilon^-\|_{W_2^{2,1}(D_T)} &\leq C_7''\varepsilon\big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\mathring{W}_2^1(D)}\big), \end{aligned} \tag{11}$$

where $u_\varepsilon^+ = u_\varepsilon$ at $Q = 1$, $u_\varepsilon^- = u_\varepsilon$ at $Q = -1$. For the proof it is sufficient to compare u with the expantion u_ε in the series S_1 and to take into consideration that the first member of the expansion coincides with u.

Theorem 2. *For all* $0 < \varepsilon < \varepsilon_0$ *the following estimate takes place*

$$\|u - \frac{1}{2}(u_\varepsilon^+ + u_\varepsilon^-)\|_{W_2^{2,1}(D_T)} \leq C_8\varepsilon^2\big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\mathring{W}_2^1(D)}\big), \tag{12}$$

where u *is the solution of* (1), u_ε^+, u_ε^- *are the solutions to the auxiliary problem* (3), (4) *at* $Q = 1$ *and* $Q = -1$ *respectively.*

Proof. It is obvious from Theorem 1 that the following expansion is true:

$$u_\varepsilon^+ = \sum_{k=0}^{\infty} \varepsilon^k v_k^+, \ \ (x,t) \in D_T, \quad u_\varepsilon^+ = \sum_{k=1}^{\infty} \varepsilon^k w_k^+, \ \ (x,t) \in D_T^1. \tag{13}$$

where v_k^+, w_k^+ are the solutions of (6) at $Q = 1$.

Similarly, it is evident from Theorem 1 that the expansion for u_ε^- is true

$$u_\varepsilon^- = \sum_{k=0}^{\infty} \varepsilon^k v_k^-, \ \ (x,t) \in D_T, \quad u_\varepsilon^- = \sum_{k=1}^{\infty} \varepsilon^k w_k^-, \ \ (x,t) \in D_T^1, \tag{14}$$

where v_k^-, w_k^- are the solutions of (6) at $Q = -1$.

One can see that $v_0^+ \equiv v_0^- \equiv u$, where u is the solution to the problem (1).

Let us consider $\tilde{w}_1 = w_1^+ + w_1^-$. The function $\tilde{w}_1$ is the solution to the problem

$$\begin{aligned} &\frac{\partial \tilde{w}_1}{\partial t} - \Delta \tilde{w}_1 = 0, \quad (x,t) \in D_T^1, && \frac{\partial \tilde{w}_1}{\partial n} = 0, \quad (x,t) \in \gamma_T, \\ &\tilde{w}_1(x,0) = 0, \quad x \in D_1, && \tilde{w}_1 = 0, \quad (x,t) \in \Gamma_T, \end{aligned}$$

whence it follows that $\tilde{w}_1 = 0$ or $w_1^+ = -w_1^-$.

Then, designate $\tilde{v}_1 = v_1^+ + v_1^-$. The function $\tilde{v}_1$ is the solution to the problem

$$L\tilde{v}_1 = 0, \quad (x,t) \in D_T,$$
$$\tilde{v}_1(x,0) = 0, \quad x \in D,$$
$$\tilde{v}_1(x,t) = 0, \quad (x,t) \in \gamma_T,$$

whence it appears that $\tilde{v}_1 = 0$. It means that $v_1^+ = -v_1^-$.

Considering $\tilde{w}_2 = w_2^+ - w_2^-$, $\tilde{v}_2 = v_2^+ - v_2^-$ one can obtain $w_2^+ = w_2^-$, $v_2^+ = v_2^-$. Using this line of reasoning, one can derive the correlation

$$\begin{aligned} v_k^+ &= v_k^-, && \text{when } k \text{ is even,} \\ v_k^+ &= -v_k^-, && \text{when } k \text{ is odd.} \end{aligned} \tag{15}$$

Taking into account (15), one can rewrite the expansions (13) and (14) in the domain D_T

$$\begin{aligned} u_\varepsilon^+ &= u + \varepsilon v_1^+ + \varepsilon^2 v_2^+ + \ldots, \\ u_\varepsilon^- &= u - \varepsilon v_1^+ + \varepsilon^2 v_2^+ - \ldots. \end{aligned} \tag{16}$$

Taking advantage of the representations (16) and the estimate (9), one can find that for all $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} \Big\| u - \frac{1}{2}(u_\varepsilon^+ + u_\varepsilon^-) \Big\|_{W_2^{2,1}(D_T)} &\le \varepsilon^2 \big\| v_2^+ + \varepsilon^2 v_4^+ + \ldots \big\|_{W_2^{2,1}(D_T)} \\ &\le C_8' \varepsilon^2 \| v_0^+ \|_{W_2^{2,1}(D_T)} \\ &\le C_8 \varepsilon^2 \big(\|f\|_{L_2(D_T)} + \|\varphi\|_{\mathring{W}_2^1(D)} \big). \qquad \square \end{aligned}$$

One can see from the proof of Theorem 2 that main members of errors for u_ε^+, u_ε^- in the expansions (16) are of different signs. From the supposition of the function sufficient smoothness it follows that the approximate solutions obtained by means of the fictitous regions method are locally two-sided. It allows to increase the approximation order: the semi-sum of the solutions u_ε^+, u_ε^- has a higher order than each of u_ε^+, u_ε^-.

References

[1] A. N. Konovalov, *Fictitous regions method for problems of two-phase incompressible fluid with capillary forces*, Sb."Chislennye metody mekhaniki sploshnoy sredy", Vol. 3, No. 3, 1972 (in Russian).

[2] G.V. Konuh, *On two-sided methods for solving partial differential equations of the second order*, Thesis of candidate's degree of phys.- math. scienses, Novosibirsk, 1988 (in Russian).

[3] G.V. Konuh, *The accuracy increasing in the fictitous regions method*, Modelirovanie v mekhanike: Sb. nauchnyh trudov, Novosibirsk, Vol. 1, No. 1, 1987 (in Russian).

[4] O.A. Ladygenskaya, *Linear and quasi-linear equations of parabolic type*, Nauka, Moscow, 1967 (in Russian).