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"Walk on spheres" algorithms for solving Helmholtz equation in the n-dimensional Euclidean space

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In this paper the algorithms of Monte Carlo methods for solving the n-dimensional Helmholtz equation are investigated. The dependence of the computational efficiency of the algorithms on n is studied.

1. Main problem. "Walk on spheres" process

Let us consider a n-dimensional Dirichlet problem for a variable-parameter Helmholtz equation

$$\Delta u + cu = -g, \quad u|_\Gamma = \psi, \tag{1.1}$$

as a main problem in a domain $D \in R^n$ with the boundary Γ, where $c(r) < c^*$, and $-c^*$ is the first eigenvalue of the Laplasian operator for the domain D, $r = (x_1, \ldots, x_n) \in D$.

The conditions for the functions c, g, ψ and Γ to be regular are assumed to be fulfilled. These conditions guarantee the existence and uniqueness of the solution of problem (1.1), as well as its probabilistic representation and integral representation with making use of Green's function for a ball (see, e.g., [1–3]).

The well-known rule of taking a derivative of a resolvent with respect to the parameter c shows that the derivative

$$v_p = u^{(p)} = \frac{\partial^p u}{\partial c^p}$$

is a solution of the following Rikye problem:

$$\begin{aligned} &(\Delta + c)^{p+1} v_p = p!(-1)^{p+1} g, \\ &(\Delta + c)^k v_p\Big|_\Gamma = \frac{(-1)^k p!}{(p-k)!}\psi^{(p-k)}, \quad k = 0, \ldots, p. \end{aligned} \tag{1.2}$$

This relationship can also be easily obtained by direct differentiation of problem (1.1) with respect to c. Hence, the solution of problem (1.2) can be estimated by direct differentiation of the estimates of problem (1.1).

The estimates to be considered below are associated with the so-called "walk on spheres" process within the domain D. To describe the process, let us introduce the following notation: $\overline{D}$ is a closure of domain D; $d(P)$ is a distance from the point P to the boundary Γ; Γ_ε is an ε-neighbourhood of the boundary Γ, i.e.,

$$\Gamma_\varepsilon = \{P \in \overline{D}: \ d(P) < \varepsilon\};$$

$S(P)$ is the largest sphere of those centred at the point P and entirely lying in $\overline{D}$, i.e.,

$$S(P) = \{Q \in \overline{D}: \ |Q - P| = d(P)\}.$$

In the "walk on spheres" process, every next point P_{k+1} is chosen uniformly over the surface of the sphere $S(P_k)$; the process terminates when a point find itself inside Γ_ε.

Let us denote by $S_0(P,\varepsilon)$ the surface of that part of the sphere $S(P)$, that belongs to the set Γ_ε. Construct a sphere S_ε with radius ε centred at the point of contact of the boundary Γ with the sphere $S(P)$. Then the area of the part of the sphere $S(P)$, entirely lies inside S_ε equals

$$\frac{(\sqrt{\pi})^{n-1}}{\Gamma((n-1)/2)(n-1)}\varepsilon^{n-1}.$$

This yields the following lower bound for the probability of the next point ending up inside Γ_ε:

$$\frac{S_0(P,\varepsilon)}{S(P)} \geq \frac{\Gamma(n/2)}{\pi\Gamma((n-1)/2)(n-1)} \cdot \frac{\varepsilon^{n-1}}{d_{max}^{n-1}} = \nu(\varepsilon), \tag{1.3}$$

where d_{max} is the exact upper boundary of radii of the spheres entirely lying in D. Evidently, the mean number $EN = q(P,\varepsilon)$ of the transitions in the "walk on spheres" chain, which governs the average time it takes for a computer to model the chain, does not exceed $\nu^{-1}(\varepsilon)$.

Moreover, for a wide class of boundaries Γ, the following logarithmic estimate is true [1–3]:

$$q(P,\varepsilon) \leq C|\ln\varepsilon|. \tag{1.4}$$

This estimate is obviously valid for convex domains. Muller [4] has shown that the mean number of the transitions in the "walk on spheres" chain depends linearly on n for large values of n

$$q(P,\varepsilon) \sim n. \tag{1.5}$$

2. The use of probabilistic representation

For $c = const < c^*$, the probabilistic representation of the solution of problem (1.1) has the form [1]

$$u(r_0) = E\int_0^{\tau} e^{ct} g(\xi(t))dt + E[e^{c\tau}\psi(\xi(\tau))],$$

where $\xi(t)$ is the diffusion process originating at the point r_0 and corresponding to the Laplasian operator, τ is the instant the process leaves the domain D for the first time. Proceeding from the process being strictly Markovian, we have

$$u(r_0) = E\sum_{i=0}^{\infty}\int_{\tau_i}^{\tau_{i+1}} e^{ct} g(\xi(t))dt + E\Big[\psi(\xi(\tau))\prod_{i=0}^{\infty} e^{c(\tau_{i+1}-\tau_i)}\Big]$$

$$= \sum_{i=0}^{\infty} E\Big[e^{c\tau_i}\int_0^{\tau_{i+1}-\tau_i} e^{ct} g(t+\tau_i)dt\Big] + E\Big[\psi(\xi(\tau))\prod_{i=0}^{\infty} e^{c(\tau_{i+1}-\tau_i)}\Big],$$

where τ_i is the instant the process $\xi(t)$ for the first time arrives at the surface of the i-th sphere of the corresponding "walk on spheres" $\{r_n\}$, $n = 0, \ldots, N$, and $r_N \in \Gamma_\varepsilon$.

On the other hand, the solution of problem (1.1) at the centre of the ball $D(r)$ can be represented as follows:

$$u(r) = \frac{s(c,d)}{\omega_n d^{n-1}}\int_{S(r)} u(r(s))ds + \int_{D(r)} G_r(r')g(r')dr',$$

where $G_r(r')$ is the Green's function for a ball for the Helmholtz' equation, ω_n is the area of the n-dimensional unit sphere

$$\omega_n = 2(\sqrt{\pi})^n/\Gamma(n/2),$$

$$s(c,d) = \begin{cases} \dfrac{(d\sqrt{c}/2)^{(n-2)/2}}{\Gamma(n/2)J_{(n-2)/2}(d\sqrt{c})}, & c > 0, \\ \dfrac{(d\sqrt{c}/2)^{(n-2)/2}}{\Gamma(n/2)I_{(n-2)/2}(d\sqrt{c})}, & c \le 0. \end{cases}$$

Here $d\sqrt{c} < \alpha_n$, where α_n is the first positive root of the function $J_{(n-2)/2}(x)$. The latter requirement is evidently met, if $c < c^*$. Indeed,

the quantity c^* decreases with expanding the domain, and for a ball D^* of radius d_{max} the relationship $d_{max}\sqrt{c^*(D^*)} = \alpha_n$ is valid.

Let us denote by P the point of the boundary Γ closest to r_n. Using the repeated averaging procedure, we can easily obtain [1] that

$$u(r_0) \approx u_\varepsilon(r_0) = E\eta_\varepsilon^{(0)},$$

where

$$\eta_\varepsilon^{(0)} = \sum_{i=0}^{N}\Big[\prod_{j=0}^{i-1} s(c,d_j)\Big]\int\limits_{D(r_i)} G(\rho,c,d_i)g(\rho)d\rho + \Big[\prod_{j=0}^{N-1} s(c,d_j)\Big]\psi(P). \quad (2.1)$$

Here $d_j = d(r_j)$, $D(r_i)$ is the ball of radius d_i centred at the point r_i. If the first derivatives of the solution are bounded in $\overline{D}$, then

$$|u(r) - u_\varepsilon(r)| \le C\varepsilon, \quad r \in D.$$

Futher, each of the integrals appearing in (2.1) can be estimated by its own random node ρ_i [5]. It is reasonable to choose random nodes in $D(r_i)$ with the density proportional to $G(\rho,c,d_i)$. In this case, instead of $\eta_\varepsilon^{(0)}$ we obtain the estimate

$$\eta_\varepsilon = \sum_{i=0}^{N}\Big[\prod_{j=0}^{i-1} s(c,d_j)\Big]g(\rho_i)F_{r_i} + \Big[\prod_{j=0}^{N-1} s(c,d_j)\Big]\psi(P),$$

where

$$F_r = \int\limits_{D(r)} G(\rho,c,d(r))d\rho \le C_1 d^2(r). \quad (2.2)$$

The repeated averaging shows that

$$E\eta_\varepsilon = E\eta_\varepsilon^{(0)} = u_\varepsilon(r_0).$$

Now let us consider the question of uniform in ε boundedness of the variance $D\eta_\varepsilon$.

First, assume that $g \equiv 0$. In this case

$$\eta_\varepsilon \le \eta = C_2\Big[\prod_{j=0}^{\infty} s(c,d_j)\Big],$$

and $E\eta < +\infty$ for $c < c^*$. Hence,

$$E\eta_\varepsilon^2 \le C_2^2 E\Big[\prod_{j=0}^{\infty} s^2(c,d_j)\Big].$$

Now let us show that for $c < c^*$ the following inequality holds:

$$s^2(c/2,d) \le s(c,d). \tag{2.3}$$

Evidently, that $s(c,d) = v(r)$, where $v(r)$ is the solution of the problem

$$\Delta v + cv = 0, \quad v|_{S(r)} = 1,$$

where $S(r)$ is the sphere with radius d centred at the point r. On the other hand, $v(r) = Ee^{c\tau}$, where τ is the instant the diffusion process originating at the point r leaves the ball $D(r)$. Hence,

$$s^2(c/2,d) = (Ee^{c\tau/2})^2 \le Ee^{c\tau} = s(c,d),$$

and the uniform in ε bondedness of the variance $D\eta_\varepsilon$ has been proved for $c < c^*/2$ at $g \equiv 0$.

Now let us turn to problem (1.1) for an inhomogeneous Helmholtz' equation. Evidently, it will suffice to prove uniform boundedness of the variance for $g \equiv const$, and the statement is true, if for $\varepsilon = 0$ the variance of the quantity

$$\eta_n = \sum_{i=0}^{n}\Big[\prod_{j=0}^{i-1} s(c,d_j)\Big]d_i^2$$

is bounded uniformly in n.

Let us consider the relationship

$$\prod_{j=0}^{n} s(c,d_j) = 1 + \sum_{i=0}^{n}\Big[\prod_{j=0}^{i-1} s(c,d_j)\Big][-1 + s(c,d_i)] = \zeta_n.$$

As $d_{max}\sqrt{c} < \alpha_n$, then

$$s(c,d_i) - 1 \ge C_3 d_i^2, \quad \eta_n < C_3^{-1}\zeta_n.$$

From the above we have

$$\lim_{n\to\infty} D\zeta_n < +\infty,$$

and the quantity $D\eta_n$ possesses the same property.

Now let us estimate the number $R_{\varepsilon,n}$ of arithmetic operations, required to obtain a prescribed error ε in estimating the solution. Let us note that

$$R_{\varepsilon,n} \sim L_\varepsilon Q_{\varepsilon,n} C_n,$$

where L_ε is the number of the trajectories to be modelled. It should be $C\varepsilon^{-2}$ for the probabilistic error to be of the order of ε. $Q_{\varepsilon,n}$ is the number of spheres, when the trajectories are terminated in Γ_ε, it is equal to $n|ln\varepsilon|$ by the order of magnitude. C_n is the number of arithmetic operations per one sphere in an n-dimensional space. For large values of n it depends linearly on n. Hence,

$$R_{\varepsilon,n} \sim n^2 |\ln \varepsilon| / \varepsilon^2.$$

Let us dwell on the results of methodical calculations. The Monte Carlo method was used to solve the Dirichlet problem in a cube $0 \le x_i \le 1, \quad i = 1, \ldots, n$ for the equation

$$\Delta u + cu = 0, \quad u|_\Gamma = \cos(x_1\sqrt{c/n}) \times \ldots \times \cos(x_n\sqrt{c/n}). \tag{2.4}$$

The exact solution has the form

$$u(r) = \cos(x_1\sqrt{c/n}) \times \ldots \times \cos(x_n\sqrt{c/n}).$$

It is known that for a n-dimensional unit cube $c^* = n\pi^2$. The solution was estimated for different values of c, for $n = 4$ at the point with the coordinates $x_1 = \ldots = x_n = 0.9$, with $\varepsilon = 0.0001$, $N = 40000$, $c^* \approx 39.5$ (Table 1), and for $n = 10$ at the point $x_1 = \ldots = x_n = 0.1$, with $\varepsilon = 0.01$, $N = 10000$, $c^* \approx 98.7$ (Table 2). Here y_n is the estimate of solution, σ_n is the estimate of the corresponding mean-square error and σ^2 is the estimate of the quantity $D\eta_\varepsilon$.

Table 1. Results of calculation for problem (2.4) with $n = 4$, $N = 40000$

c	$u(r,c)$	$y_n \pm \sigma_n$	σ^2
15	0.00086	0.00096 ± 0.00020	0.00168
20	0.03338	0.03347 ± 0.00025	0.00253
25	0.15571	0.15490 ± 0.00120	0.05722
30	0.36931	0.36357 ± 0.00631	1.59144
35	0.61982	0.59343 ± 0.02730	29.8515
40	0.83753	0.73218 ± 0.10720	459.666
45	0.97017	0.97484 ± 0.39973	6391.475

Table 2. Results of calculation with $n = 10$, $N = 10000$

c	$u(r,c)$	$y_n \pm \sigma_n$	σ^2
30	0.860	0.859 ± 0.0005	0.003
50	0.777	0.776 ± 0.0007	0.006
100	0.601	0.600 ± 0.0010	0.014
120	0.542	0.541 ± 0.0013	0.016

The analysis of results shows that insignificant deviations from the exact solution have been obtained for $c > c^*/2$ as well.

The following problem was also solved in the unit cube:

$$\Delta u + cu = 0, \quad u|_\Gamma = 1. \tag{2.5}$$

The solution was estimated at the point $x_1 = \ldots = x_n = 0.5$, for $n = 4$, $\varepsilon = 0.001$, $N = 40000$ (Table 3), and for $n = 10$, $\varepsilon = 0.01$, $N = 10000$ (Table 4). There exists no exact solution of this problem, but the results agree satisfactorily with the solution of problem (2.5) for the ball of radius d, where d is chosen in such a way that the quantity c^* for the ball coincides with the quantity c^* for the cube, i.e., $d = \alpha_n/\pi\sqrt{n}$. In Tables 3, 4 $s(c,d)$ is the exact solution of problem (2.5) at the centre of the ball.

Table 3. Results of calculation for problem (2.5) with $n = 4$, $N = 40000$

c	$y_n \pm \sigma_n$	σ^2	$s(c,d)$
18	2.7665 ± 0.0116	5.3890	2.7276
19	2.9703 ± 0.0143	8.1388	2.9268
20	3.1966 ± 0.0176	12.3927	3.1484
22	3.7313 ± 0.0217	29.5330	3.6743

Table 4. Results of calculation with $n = 10$, $N = 10000$

c	$y_n \pm \sigma_n$	σ^2	$s(c,d)$
33	2.9101 ± 0.0173	2.9919	2.8824
34	2.9937 ± 0.0194	3.7476	2.9876
50	5.6103 ± 0.1018	103.5363	5.5542
55	7.0724 ± 0.1318	173.8112	6.8959

It is interesting to note that for the same values of y_n the quantity σ_n decreases with growing n. The results are listed in Table 5. The time it takes for the computer to model problem (2.5) for $n_1 = 4$ with $\varepsilon = 0.001$, $N = 10000$ is $t_1 = 5\text{min.}20\text{sec.}$, and for $n_2 = 10$ is $t_2 = 34\text{min.}21\text{sec.}$, i.e., $t_1/t_2 \approx (n_1/n_2)^2 = 0.16$. The result agrees with the estimate $R_{\varepsilon,n} \sim n^2$.

Table 5. Results of calculation with $N = 10000$, $\varepsilon = 0.01$

n	c	$y_n \pm \sigma_n$	n	c	$y_n \pm \sigma_n$
2	11.0	2.9191 ± 0.0331	6	25.0	2.9281 ± 0.0270
3	15.6	2.9559 ± 0.0287	8	29.9	2.9934 ± 0.0207
4	19.3	2.9381 ± 0.0260	10	34.0	2.9937 ± 0.0194

3. Algorithms for variable $c(r)$

Let us rewrite (1.1) in the form

$$\Delta u = -cu - g, \quad u|_\Gamma = \psi. \tag{3.1}$$

Using the integral representation of the solution of Poisson's equation at the ball center [6], we can write the following integral equation for the function $u(r)$:

$$u = Ku + h. \tag{3.2}$$

In $D \setminus \Gamma_\varepsilon$ this equation has the form

$$u(r) = \frac{1}{\omega_n d^{n-1}(r)} \int\limits_{S(r)} u(r(s))ds + \int\limits_{D(r)} c(r')G_r(r')u(r')dr'$$
$$+ \int\limits_{D(r)} G_r(r')g(r')dr', \qquad r \in D \setminus \Gamma_\varepsilon,$$

while in Γ_ε it transforms into the identity

$$u(r) \equiv u(r), \quad r \in \Gamma_\varepsilon,$$

i.e., $k(r,r') \equiv 0$ and $h(r) \equiv u(r)$ for $r \in \Gamma_\varepsilon$. Here $k(r,r')$ is the kernel of the integral operator K.

$$G_r(r') = \frac{1}{(n-2)\omega_n}\left(\frac{1}{|r-r'|^{n-2}} - \frac{1}{d(r)^{n-2}}\right), \quad |r-r'| \le d(r),$$

is the Green's function for a ball at the centre of the ball $D(r)$. For $r \in D \setminus \Gamma_\varepsilon$ we write the integral operator K in the form

$$[1 - c_0 d^2(r)/2n] \int\limits_{S(r)} \frac{[1 - c_0 d^2(r)/2n]^{-1}}{\omega_n d^{n-1}(r)} u(r(s))ds$$
$$+ \frac{c_0 d^2(r)}{2n} \int\limits_{D(r)} \frac{c(r')}{c_0} 2n d^{-2}(r) G_r(r')u(r')dr', \tag{3.3}$$

with c_0 satisfying the inequality $c_0 d^2_{max}/2n < 1$.

The standard Monte Carlo estimate that implements a probabilistic representation of the Neumann series has the form [1]

$$\xi = \sum_{n=0}^{N} Q_n h(r_n), \quad u(r_0) = E\xi.$$

According to (3.2), the following "walk on spheres" and balls are accomplished: with a probability of $1 - c_0 d^2(r_i)/2n$, a new point r_{i+1} is chosen uniformly over the sphere $S(r_i)$, while the weight Q_i is multiplied by the quantity

$$q(r_i, c_0) = [1 - c_0 d^2(r_i)/2n]^{-1}$$

with an opposite probability $c_0 d^2(r_i)/2n$, a point r_{i+1} is chosen within the ball with respect to the density $2n d^{-2}(r_i) G_{r_i}(r')$, while the weight is multiplied by $c(r_{i+1})/c_0$. When the chain find itself inside Γ_ε it is terminated, and the solution estimate multiplied by the weight is added to the sum in the counter. The resulting rundom weights Q_n has the form

$$Q_n = \Big[\prod_{i=1}^{m_n} \frac{c(r_{k_i})}{c_0}\Big]\Big[\prod_{i=1}^{n-m_n} q(r_{t_i-1}, c_0)\Big],$$

and the function $h(r)$ is calculated exactly or in a randomized manner. Here $\{r_{t_i}\}$, $i = 1, \ldots, n - m_n$ are the points chosen on the spheres, while $\{r_{k_i}\}$, $i = 1, \ldots, m_n$ are the points chosen inside the balls.

For the new "walk on spheres" and balls the average modelling costs $S_{\varepsilon,n}$ also admit an estimate [5]

$$S_{\varepsilon,n} \leq \text{const}|\ln \varepsilon|n^2.$$

Below we shall justify an asymptotical unbiasedness of the quantity ξ_ε considered as an estimate of the solution of the initial problem.

Lemma 1. *Provided*

$$c_0 \geq |c(r)| \quad \textit{and} \quad c_0 < 2nc^*/\alpha_n^2, \tag{3.4}$$

the Neumann series for equation (3.2) *converges.*

Proof. Let us consider c_0 satisfying the inequality

$$\frac{c_0}{c^*} < \min_{x\in[0,\alpha_n]} \left\{\frac{2n}{x^2}\left(1 - \frac{\Gamma(n/2)J_{(n-2)/2}(x)}{(x/2)^{(n-2)/2}}\right)\right\}, \tag{3.5}$$

where $x = d\sqrt{c^*}$, than for all $d_i \leq d_{max}$

$$\frac{c_0}{c^*} < \frac{2n}{d_i^2 c^*}\left(1 - \frac{\Gamma(n/2)J_{(n-2)/2}(d_i\sqrt{c^*})}{(d_i\sqrt{c^*}/2)^{(n-2)/2}}\right),$$

that entails the inequality

$$\frac{1}{1 - c_0 d_i^2/2n} < \frac{(d_i\sqrt{c^*}/2)^{(n-2)/2}}{\Gamma(n/2)J_{(n-2)/2}(d_i\sqrt{c^*})}. \tag{3.6}$$

The minimized function monotone decreases in $[0, \alpha_n]$, hence, inequality (3.5) is equivalent to $c_0 < 2nc^*/\alpha_n^2$. With allowance made for inequality $|c(r)|/c_0 < 1$, relationship (3.6) implies that quantity ξ_ε can be termwise majorized by the standard estimate η (see Section 2) on "walk on spheres" $\{r_i\}$ (not necessarily maximal) with a certain $c < c^*$ for $\varepsilon = 0$. The Neumann series corresponding to estimate η converges, as η can be obtained by the partial averaging of the probabilistic representation of the solution. □

Let us note that for large n quantity α_n depends linearly on n, hence, $2n/\alpha_n^2 \to 0$ for $n \to \infty$. For $n = 3$ the Neumann series converges with $c_0 < 0.6079c^*$, for $n = 4$ with $c_0 < 0.5454c^*$, for $n = 10$ with $c_0 < 0.3481c^*$.

In [5] it is proved that under the hypotheses of Lemma 1 there exists a unique bounded solution to equation (3.2) that can be represented by the corresponding Neumann series and coincides with the solution of problem (3.1).

Now let us proceed with constructing and studying the actual estimate ξ, which is obtained by the replacement of $u(r)$ in Γ_ε with the value of the boundary function ψ at the point $P(r) \in \Gamma$ closest to r. In this case

$$h(r) = \psi(P(r)), \quad r \in \Gamma_\varepsilon.$$

If the first derivatives of the solution of problem (1.1) are bounded in D, then, provided the conditions of Lemma 1 are satisfied, $E\xi_\varepsilon = u_\varepsilon$ exists [5] and

$$|u(r) - u_\varepsilon(r)| \le \text{const} \cdot \varepsilon, \quad r \in D, \quad \varepsilon > 0.$$

Lemma 2. *If the conditions*

$$c_0 \ge |c(r)|, \quad \frac{c_0}{c^*} < \min_{x \in [0, \alpha_n]} \left\{ \frac{2n}{x^2} \left(1 - \frac{\Gamma(n/2) J_{(n-2)/2}(x)}{(x/2)^{(n-2)/2}} \right)^{1/2} \right\} \tag{3.7}$$

are satisfied and $g \equiv 0$, *then* $D\xi_\varepsilon < c_d < +\infty$ *for all* $\varepsilon > 0$.

Proof. Relationship (3.7) implies that

$$(1 - c_0 d_i^2/2n)^{-2} < \frac{(d_i\sqrt{c^*}/2)^{(n-2)/2}}{\Gamma(n/2) J_{(n-2)/2}(d_i\sqrt{c^*})}.$$

This means that the quantity ξ_ε^2 here can be majorized in the same way as ξ in the proof of Lemma 1. □

Direct calculations show that the variance is bounded for $n = 3$ with $c_0 < 0.488c^*$, for $n = 4$ with $c_0 < 0.465c^*$, for $n = 10$ with $c_0 < 0.336c^*$.

Now let us consider the variance for an inhomogeneous equation. With the aim to extend the condition of the finiteness of the variance, instead of the estimate ξ_ε we shall consider the estimate

$$\xi_{\varepsilon,1} = \sum_{n=0}^{N-m_N} Q_n h_1(r_n) = \sum_{i=1}^{N-m_n} Q_{t_i} h(r_{t_i-1}) + Q_N \psi(P(r_N)).$$

The repeated averaging yields $E\xi_{\varepsilon,1} = E\xi$.

Under the hypotheses of Lemma 2 we have

$$D\xi_{\varepsilon,1} < C_{d,1} < +\infty \quad \forall \varepsilon > 0.$$

Now let us construct estimates of the derivatives

$$u^{(k)}(r,c) = \frac{\partial^k u(r,c)}{\partial c^k},$$

while assuming $c = \text{const} < c_0$. Evidently, it is sufficient to consider the case when the functions g and ψ are independed of c. Let us differentiate the estimate $\xi_{\varepsilon,1}$ k-times with respect to c to get

$$\xi_{\varepsilon,1}^{(k)} = \frac{\partial^k \xi_{\varepsilon,1}}{\partial c^k} = \sum_{i=1}^{N-m_N} \frac{\mu_i!}{(\mu_i - k)!} c^{-k} Q_{t_i} h(r_{t_i-1}) + \frac{m_N!}{(m_N - k)!} c^{-k} Q_N \psi(P(r_N)).$$

Here μ_i is the number of steps into the balls among the first t_i steps, i.e., $\mu_i = m_{t_i}$. If $\mu_i < k$, then the corresponding term equals zero. Therefore, by analogy with Lemmas 1,2 we can state that $E\xi_{\varepsilon,1}^{(k)}$ exists and the quantity $D\xi_{\varepsilon,1}^{(k)}$ is bounded uniformly in ε.

The new algorithms have been used for calculating a number of versions of problem (2.5) at the point $x_1 = \ldots = x_n = 0.5$ for $n = 4$ with $\varepsilon = 0.001$, $N = 40000$ and for $n = 10$ with $\varepsilon = 0.01$, $N = 10000$ (Table 6).

Table 6. Results of calculations by the new method for problem (2.5)

n	c	c_0	$(y_n \pm \sigma_n) \cdot 10^3$	σ^2	$u'_c \cdot 10^3$	$u''_c \cdot 10^3$	$u'''_c \cdot 10^3$
4	18	18.1	2761 ± 11.8	5.568	189.5 ± 2.4	18.19 ± 0.56	2.22 ± 0.14
10	33	33.1	2939 ± 18.1	3.299	107.2 ± 2.1	4.83 ± 0.25	0.26 ± 0.03

The results presented are in good agreement with the results obtained by the standard "walk on spheres" method. The costs of the new method are practically the same as those of the standard one.

On the basis of these estimates of the solution and on the basis of the first five derivatives using the Taylor series in the point $c = 18$ for $n = 4$ and $c = 33$ for $n = 10$, the values of the solution have been estimated for other values of c with the aim to compare with the corresponding results from Section 2. The mean-square error was estimated by the formula

$$\sigma\Big(\sum_{i=1}^{n} \xi_i\Big) \le \sum_{i=1}^{n} \sigma(\xi_i) = \tilde{\sigma}.$$

The results are included in Tables 7, 8.

Table 7. Estimates with the Taylor series at the point $c = 14$ for $n = 4$

c	$y_n \pm \sigma_n$	Estimates from Section 2
18	2.761 ± 0.0118	2.767 ± 0.0116
19	2.960 ± 0.0145	2.970 ± 0.0143
20	3.180 ± 0.0180	3.197 ± 0.0176
22	3.692 ± 0.0280	3.731 ± 0.0217

Table 8. Estimates with the Taylor series at the point $c = 33$ for $n = 10$

c	$y_n \pm \sigma_n$	Estimates from Section 2
33	2.939 ± 0.0182	2.910 ± 0.0173
34	3.049 ± 0.0204	2.994 ± 0.0194
50	5.737 ± 0.1271	5.610 ± 0.1018
55	7.117 ± 0.2153	7.072 ± 0.1318

References

[1] B.S. Elepov, A.A. Kronberg, G.A. Mikhailov, and K.K. Sabelfeld, Solution of Boundary-Value Problems by the Monte Carlo Method, Nauka, Moscow, 1980 (in Russian).

[2] S.M. Ermakov, V.V. Nekrutkin, and A.S. Sipin, Random Processes for Solving Classical Equation of Mathematical Phisics, Nauka, Moscow, 1984 (in Russian).

[3] K.K. Sabelfeld, Monte Carlo Methods in Boundary-Value Problems, Nauka, Novosibirsk, 1989 (in Russian).

[4] M.E. Muller, Some continuons Monte Carlo methods for the Dirichlet problem, Ann. Math. Stat., Vol. 27, No. 3, 1956, 569–589.

[5] G.A. Mikhailov, "Walk on spheres" algorithms for solving Helmholtz equation, Russ. J. Numer. Anal. Math. Modelling, Vol. 7, No. 6, 1992, 515–536.

[6] R. Courant, Partial Differential Equations, Mir, Moscow, 1964.