Bull. Nov. Comp. Center, Math. Model. in Geoph., 1 (1994), 87–98

Full waveform inversion of VSP data (normally-incident plane wave)

V.A. Tcheverda and T.A. Voronina

The problem to recover structure of vertically-inhomogeneous medium by VSP data with unknown source-function is considered on the basis of optimization approach. It is proved that any stationary point of the cost function provides identity of a simulated wave field with a recorded one.

A numerical algorithm to reconstruct velocity of wave propagation is proposed, justified and realized. Results of numerical simulations are presented.

1. Introduction

We are going to consider the problem of recovering a wavepropagation velocity in a vertically inhomogeneous medium (acoustic half-space) given VSP data. We do not assume that any information about source excitation is available. To invert VSP data and to reconstruct the velocity distribution simultaneously with the input impulse we apply an optimization approach. We would like to note here that this approach was proposed and justified in [1] and modified to process VSP data in [2], where numerical algorithm was proposed and synthesized and field data processed. In some way our article continues these ones and is devoted to the problem of local minima of a cost function. We also propose and test an algorithm which does not need any *a priori* information about a source function. Moreover, we illustrate by means of numerical experiments that bad fitting of the reconstructed source function with the real one does not lead to the same for, the wave propagation velocity.

As far as to the reconstruction of the trend under lack of information for low time frequencies, we do not deal with it at all as it was shown in [3] and [4] that it is unresolved for VSP data.

2. Statement of the problem

Let us suppose plane wave with a source function $f(t)$ ($f(t) = 0$, $t < 0$, $f(t) = 0$, $t > T$) incidents normally on the half space $z \geq 0$ fulfilled with a vertically inhomogeneous medium with a wave propagation velocity $c(z) \in C^2(0,\infty)$. Then the wave field within this half space is described by the boundary-valued problem

$$\frac{d^2u}{dz^2} + \omega^2 n^2(z)u = 0, \quad n(z) = c_1(z); \tag{1}$$

$$\left.\frac{du}{dz}\right|_{z=0} = F(\omega), \quad u = F(\omega)S(\omega)\exp\left(i\omega n(H)z\right), \quad z > H, \tag{2}$$

and we are going to reconstruct functions $F(\omega)$ and $n^2(z)$ by the data

$$u_j(\omega) = u(z_j;\omega); \quad z_j \in [0,H]; \quad j = 1,\dots,N; \quad \omega_1 \leq \omega \leq \omega_2. \tag{3}$$

We will suppose that

$$n^2(z) \in M = \left\{n^2(z) \in C^2(0,\infty) : n^2(z) \equiv n_H^2,\ z \geq H\right\},$$

and introduce the scalar product in M

$$\left(n_1^2, n_2^2\right)_M = \int_0^\infty \left(n_1^2(z) - n_{1\infty}\right)\left(n_2^2(z) - n_{2\infty}^2\right)dz.$$

Then, $L_2(\omega_1,\omega_2) \times M$, with a scalar product

$$\left(\{F_1, n_1^2\}, \{F_2, n_2^2\}\right) = (F_1, F_2)_{L_2} + (n_1^2, n_2^2)_M,$$

produces a space of models. Operator $B_j[F, n^2](\omega)$, which transforms point (F, n^2) from this space to the solution of a boundary value problem (1), (2) in the point z_j acts from $L_2(\omega_1,\omega_2) \times M$ to $L_2(\omega_1,\omega_2)$ and has Freschet's derivatives with respect to $F_1 = \operatorname{Re} F$, $F_2 = \operatorname{Im} F$ and n^2.

If one is going to solve inverse problem (1)–(3) minimizing the cost function

$$\Phi\left[F; n^2(z)\right] = \sum_{j=1}^{N} \int_{\omega_1}^{\omega_2} \left|u_j(\omega) - B_j[F; n^2](\omega)\right|^2 d\omega, \tag{4}$$

the key question is if this cost function has the only one minimum or a lot of ones. If it has the only one stationary point – global minimum – it would be reasonable to expect that at least for rather well initial approximation iteration process will converge to real source function and slowness.

3. Analysis of the cost function

After simple, but bulky calculations one comes to the following expressions for the gradients of cost function (4):

$$(\nabla_{F_1}\Phi[F,n^2])(\omega) = -2\,\mathrm{Re}\sum_{j=1}^{N}\left[u_j(\omega)-F(\omega)G(z_j;0;\omega)\right]\bar{G}(z_j;\omega); \tag{5}$$

$$(\nabla_{F_2}\Phi[F,n^2])(\omega) = -2\,\mathrm{Im}\sum_{j=1}^{N}\left[u_j(\omega)-F(\omega)G(z_j;0;\omega)\right]\bar{G}(z_j;\omega); \tag{6}$$

$$(\nabla_{n^2}\Phi[F,n^2])(z) = -2\,\mathrm{Re}\sum_{j=1}^{N}\int_{\omega_1}^{\omega 2}(\omega^2+\varepsilon^2)\bar{F}(\omega)\left[u_j(\omega)-F(\omega)G(z_j;0;\omega)\right] \times \bar{G}(z;0;\omega)\bar{G}(z;z_j;\omega)d\omega. \tag{7}$$

If these gradients are equal to zero in a point $(F^S, n_S^2) \in L_2(\omega_1,\omega_2)\times M$, then by (5) and (6) we have:

$$\sum_{j=1}^{N}\left[u_j(\omega)-F^S(\omega)G(z_j;0;\omega)\right]\bar{G}(z_j;0;\omega)=0,$$

hence,

$$F^S(\omega)=\frac{\sum_{j=1}^{N}\bar{G}(z_j;0;\omega)u_j(\omega)}{\sum_{j=1}^{N}|G(z_j;0;\omega)|^2}, \tag{8}$$

where $G(\xi;\eta;\omega)$ satisfies the boundary value problem

$$\frac{d^2G}{d\xi^2}+\omega^2G=\delta(\xi-\eta), \quad \left.\frac{dG}{d\xi}\right|_{\xi=0}=0,$$

$$G(\xi;\eta;\omega)=S(\omega;\eta)\exp\left(in(H)\omega\xi\right), \quad \xi>\max(H,\eta).$$

Next, (7) leads to

$$\mathrm{Re}\sum_{j=1}^{N}\int_{\omega_1}^{\omega_2}\bar{F}^S(\omega)\omega^2\left[u_j(\omega)-F^S(\omega)G(z_j;0;\omega)\right]\bar{G}(z;0;\omega)\bar{G}(z;z_j;\omega)d\omega\equiv 0,$$

where $F^S(\omega)$ is from (8).

Let us introduce the function

$$\Psi(\xi,\zeta)=2\,\mathrm{Re}\sum_{j=1}^{N}\int_{\omega_1}^{\omega_2}A_j(\omega)\bar{G}(\xi;0;\omega)\bar{G}(\zeta;z_j;\omega)d\omega, \tag{9}$$

where $A_j(\omega)=\bar{F}^S(\omega)\omega^2(u_j(\omega)-F^S(\omega)G(z_j;0;\omega))$. As one can check, for $\{\xi>0,\zeta>0\}$, this function satisfies the equation

$$n_S^{-2}(\zeta)\frac{\partial^2\Psi}{\partial\zeta^2} - n_S^{-2}(\xi)\frac{\partial^2\Psi}{\partial\xi^2} = 2\sum_{j=1}^{N}\frac{\delta(\zeta - z_j)}{n_S^2(z_j)}\operatorname{Re}\int_{\omega_1}^{\omega_2} A_j(\omega)\bar{G}(\xi;0;\omega)d\omega. \tag{10}$$

which, for new variables

$$y = \int_0^\zeta n_S(t)dt; \quad x = \int_0^\xi n_S(t)dt,$$

transforms to

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} - \frac{d\ln n_S(x)}{dx}\frac{\partial\Psi}{\partial x} + \frac{d\ln n_S(y)}{dy}\frac{\partial\Psi}{\partial y}$$
$$= 2\sum_{j=1}^{N}\frac{\delta(y - y_j)}{n_S(y_j)}\operatorname{Re}\int_{\omega_1}^{\omega_2} A_j(\omega)\bar{G}(\zeta(y);z_j;\omega)d\omega.$$

Taking into account (10), the identity $n_S(z) \equiv n_H$ for $z > H$ and the fact that $z_j \in [0, H]$, one comes to the equality $\Psi(x, y) \equiv 0$ for $\{x \geq H_1, y \geq H_1\}$. This follows $\Psi(x, y) \equiv 0$ in Domain II (Figure 1) as a solution of the

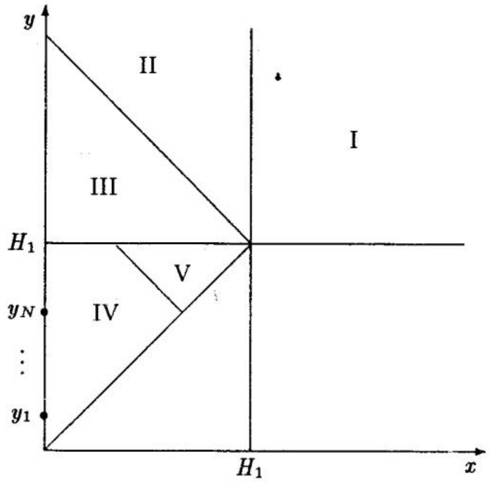

Figure 1

Cauchy problem with zero data on the ray $x = H_1$, $y \geq H_1$. So, $\Psi(0, y) \equiv 0$ for $y \geq 2H_1$. But for $y \geq H_1$ one has $G(\zeta(y); z_j; \omega) = S_j(\omega)\exp(i\omega y)$ (wave, going to the infinity), from what follows that

$$\Psi(0, y) = 2\operatorname{Re}\sum_{j=1}^{N}\int_{\omega_1}^{\omega^2} \exp(i\omega y)G(0;0;\omega)A_j(\omega)\bar{S}_j(\omega)d\omega,$$

but it means that, for $y \geq H_1$, $\Psi(0,y)$ is the analytic function of the real variable y, and if it vanishes for $y \geq 2H_1$, then it vanishes for $y \geq H_1$, too. So, in Domain III $\Psi(x,y)$ satisfies the equation

$$\frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} + \frac{d \ln n_S(x)}{dx} \frac{\partial \Psi}{\partial x} = 0,$$

and conditions $\Psi(0,y) = \Psi(2H_1 - y, y) = 0$, $0 \leq y \leq H_1$. But, as was proved in [5], it follows $\Psi(x,y) \equiv 0$ in Domain III.

Let us consider now $\Psi(x,y)$ within the square $\Pi = \{0 \leq x \leq H_1; 0 \leq y \leq H_1\}$. It satisfies there the following equation:

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} - \frac{d \ln n_S(x)}{dx} \frac{\partial \Psi}{\partial x} + \frac{d \ln n_S(y)}{dy} \frac{\partial \Psi}{\partial y} \\ = 2 \sum_{j=1}^{N} \frac{\delta(y - y_j)}{n_S(y_j)} \operatorname{Re} \int_{\omega_1}^{\omega_2} A_j(\omega) \bar{G}(\xi(x); 0; \omega) d\omega,\end{aligned} \tag{11}$$

Cauchy's data

$$\Psi(x, H_1) = \left. \frac{\partial \Psi}{\partial y} \right|_{y=H_1} = 0, \quad 0 \leq x \leq H_1, \tag{12}$$

and additional condition $\Psi(x,x) = 0$ for $0 \leq x \leq H_1$.

To solve the Cauchy problem (11), (12) within Triangle IV is the same as to solve the system of integral equations:

$$\begin{aligned}\Psi(x,y) = \frac{1}{2} \int_y^{H_1} d\eta \int_{x-(\eta-y)}^{x+(\eta-y)} \left[a(\xi) p(\xi,\eta) - a(\eta) q(\xi,\eta) \right] d\eta + \\ \sum_{j=1}^{N} \frac{\theta(y_j - y)}{n_S(y_j)} \int_{x-(y_j-y)}^{x+(y_j-y)} F_j(\xi) d\xi;\end{aligned} \tag{13}$$

$$\begin{aligned}p(x,y) = \frac{1}{2} \int_y^{H_1} \Big[a(x + (\eta - y)) p(x + (\eta - y), \eta) - a(\eta) q(x + (\eta - y), \eta) - \\ a(x - (\eta - y)) p(x - (\eta - y), \eta) + a(\eta) q(x - (\eta - y), \eta) \Big] d\eta + \\ \sum_{j=1}^{N} \frac{\theta(y_j - y)}{n_S(y_j)} \left[F_j(x + (y_j - y)) - F(x - (y_j - y)) \right];\end{aligned} \tag{14}$$

$$q(x,y) = \frac{1}{2}\int_y^{H_1}\Big[a(x-(\eta-y))p(x-(\eta-y),\eta) - a(\eta)q(x-(\eta-y),\eta) - a(x+(\eta-y))p(x+(\eta-y),\eta) + a(\eta)q(x+(\eta-y),\eta)\Big]d\eta + \sum_{j=1}^{N}\frac{\theta(y_j-y)}{n_S(y_j)}\big[F_j(x-(y_j-y)) - F_j(x+(y_j-y))\big], \tag{15}$$

where

$$p(x,y) = \frac{\partial\Psi(x,y)}{\partial x}; \quad q(x,y) = \frac{\partial\Psi(x,y)}{\partial y}; \quad a(x) = \frac{d\ln n(x)}{dx}; \qquad F_j(\xi) = \mathrm{Re}\int_{\omega_1}^{\omega_2} A_j(\omega)\bar{G}\big(\xi_1(\xi);0;\omega\big)d\omega. \tag{16}$$

Additional relation $\Psi(x,x)=0$ for $\frac{H_1}{2} \le x \le H_1$ may be rewritten as

$$\int_x^{H_1} d\eta \int_{2x-\eta}^{\eta}\big[a(\xi)p(\xi,\eta) - a(\eta)q(\xi,\eta)\big]d\xi + \sum_{j=1}^{N}\frac{\theta(y_j-x)}{n_*(y_j)}\int_{2x-y_j}^{y_j}F_j(\xi)d\xi = 0. \tag{17}$$

Within Triangle V (below the last geophone), (13)–(17) is a system of homogeneous Volterra's integral equations of the second kind which has the only trivial solution $\Psi(x,y)\equiv 0$. After differentiating this identity $2n$ times with respect to x within this triangle and taking into account the equation for $G(\xi(x);0;\omega)$ one comes to the relation

$$\int_{\omega_1}^{\omega_2}\omega^{2n}\,\mathrm{Re}\Big[G(\xi(x);0;\omega)\sum_{j=1}^{N}A_j(\omega)G\big(\zeta(y);z_j;\omega\big)\Big]d\omega = 0,$$

from what follows that

$$\phi(x,y) \equiv \mathrm{Re}\Big[G(\xi(x);0;\omega)\sum_{j=1}^{N}A_j(\omega)G(\zeta(y);z_j;\omega)\Big] = 0 \quad \forall\omega\in(\omega_1,\omega_2),$$

within Triangle V. But, as $\phi(x,y)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{d\ln n_*(x)}{dx}\frac{\partial\phi}{\partial x} + \omega^2\phi = 0, \tag{18}$$

it is identically equal to zero for $\{y_N < y \le H_1; x \le y\}$ as a solution of the ordinary differential equation (18) with zero Cauchy data. So, we come to the same Cauchy problem with zero data for $\Pi_1 = \{0 \le x \le y_N; 0 \le y \le y_N\}$ as we had before for Π.

Let us analyze (13)–(17) within the square Π_1 for $y_{N-1} < y \le y_N$. Here the sums in the right-hand sides of (13)–(15) have only the last term and (17) leads to the relation

$$\int_{2x-y_N}^{y_N} F_N(\xi)d\xi = -\int_x^{H_1} d\eta \int_{2x-\eta}^{\eta} \big[a(\xi)p(\xi,\eta) - a(\eta)q(\xi,\eta)\big]d\eta,$$

which allows to find $F_N(x)$ over (y_{N-1}, y_N) as a function of p and q. So, (13), (14) is again a homogeneous system of Volterra's integral equations with trivial solution only, i.e., $\Psi(x,y) \equiv 0$ within the triangle with the side coinciding with the line $y = x$ over the interval (y_{N-1}, y_N). And in the same way as before this fact leads to the identity $\mathrm{Re}[A_N(\omega)\bar{G}(\xi(x);0;\omega)] \equiv 0$, which may be hold only if $A_N(\omega) \equiv 0$, because Green's function $G(\xi(x);0;\omega)$ must not be identical zero over any contineous interval of x.

So, the right-hand side of (11) has only $N-1$ first terms and one comes to the Cauchy problem with zero data for square Π_2. Next, repeating this procedure one will finally come to the equalities $A_j(\omega) \equiv 0$; $j = 1, \dots, N$; $\omega \in (\omega_1, \omega_2)$.

Summarizing, we can formulate the theorem:

Theorem. *Let us suppose a solution $u(z;\omega)$ of the boundary value problem (1), (2) be known in the points $z_1, \dots, z_N$, ($z_j \in (0,H)$; $j = 1, \dots, N$), the wave propagation velocity $c(z) \in C^2(0,\infty)$ and for $z \ge H c_*(z) \equiv c_\infty$. Then gradient of the cost function $\Phi[F;n^2]$ is equal to zero only in the point $\{F^S(\omega), n_S^2(z)\}$, such that*

$$u_j(\omega) \equiv F^S(\omega)G(z_j;0,\omega); \quad j = 1, \dots, N; \quad \omega \in (\omega_1, \omega_2), \qquad (19)$$

where $G(z_j;0;\omega)$ is a solution to the boundary value problem (1), (2) with $F(\omega) \equiv 1$, $n^2(z) \equiv n_S^2(z)$, calculated in the point z_j.

Corollary. *If all assumptions of the theorem are valid, and the wave propagation velocity is known within the interval $(0,h)$, $h < H$, where at least two geophonesare situated, then the cost function (4) has only one stationary point $\{F^S(\omega), n_S^2(z)\}$ which coincides with the real source function and the real slowness.*

Proof. As the real slowness $n(z)$ is known over the interval (z_1, z_2) one can resolve the boundary value problem

$$\frac{d^2u}{dz^2} + \omega^2 n^2(z)u = 0, \tag{20}$$

$$u(z_1, \omega) = u_1(\omega), \quad u(z_2, \omega) = u_2(\omega), \tag{21}$$

and find $u(z, \omega)$ within (z_1, z_2). This allows to find $u(z, \omega)$ for the interval $(0, h)$ as a solution of Cauchy's problem for ordinary differential equation (20), calculate its derivative with respect to z for $z = 0$ and to find a source function $F(\omega)$ (as the spectrum of (20), (21) is discrete, $F(\omega)$ may be reconstructed over the interval (ω_1, ω_2), except of finite quantity of points).

Therefore, we come to a well-known statement of an inverse problem for reconstruction of a wave propagation velocity with known function $F(\omega)$. It is known ([6], [7]), that relations (19) provide coincidence of the slowness $n_S^2(z)$ with the real one. □

Remark. It is necessary to underline that we did not prove that inverse problem has a solution as we did not prove that the cost function has a stationary point. The theorem concludes that for any stationary point (19) is hold.

4. Numerical experiments

To perform minimization of the cost function (4) numerically the following version of method of conjugate gradients was used:

$$n_{k+1}^2(z) = n_k^2(z) - \alpha_k P_k(z), \tag{22}$$

$$\alpha_k = \arg\min \Phi[F_k; n_k^2 - \alpha P_k], \tag{23}$$

$$P_0(z) = \nabla_{n^2} \Phi[F_0; n_0^2], \tag{24}$$

$$P_k(z) = \nabla_{n^2} \Phi[F_k; n_k^2](z) - \beta_k P_{k-1}(z); \quad k \geq 1, \tag{25}$$

$$\beta_k = \big(\nabla_{n^2} \Phi[F_k; n_k^2], \nabla_{n^2} \Phi[F_{k-1}; n_{k-1}^2] - \nabla_{n^2} \Phi[F_k; n_k^2]\big). \tag{26}$$

As the source function $F_k(\omega)$ for $(k+1)$-th step of the process (22)–(26) the function which vanishes the gradient of the cost function $\Phi[F; n^2(z)]$ with respect to for the slowness $n_k^2(z)$ was taken (see (8)). It seems to be more reasonable, than to organize the same process as (22)–(26) for the source function as in [2].

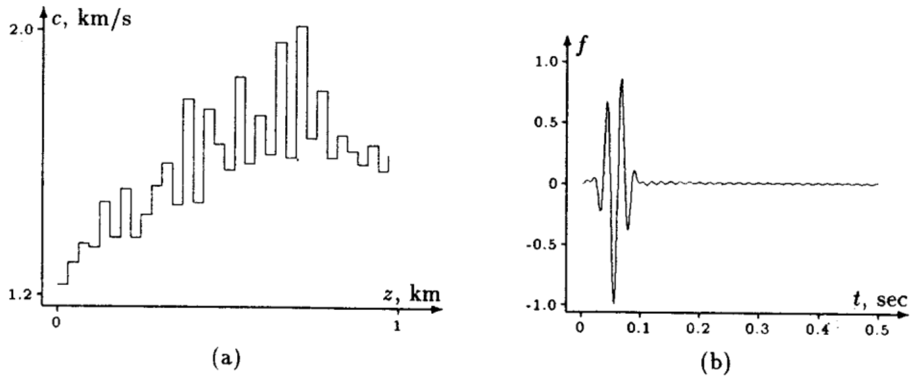

Figure 2

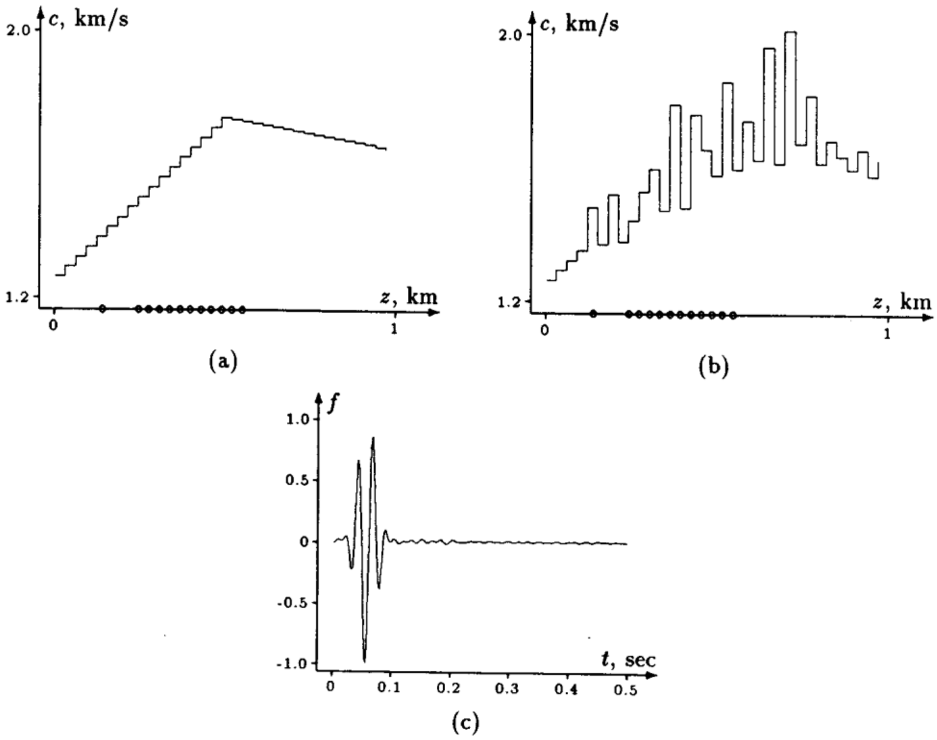

Figure 3

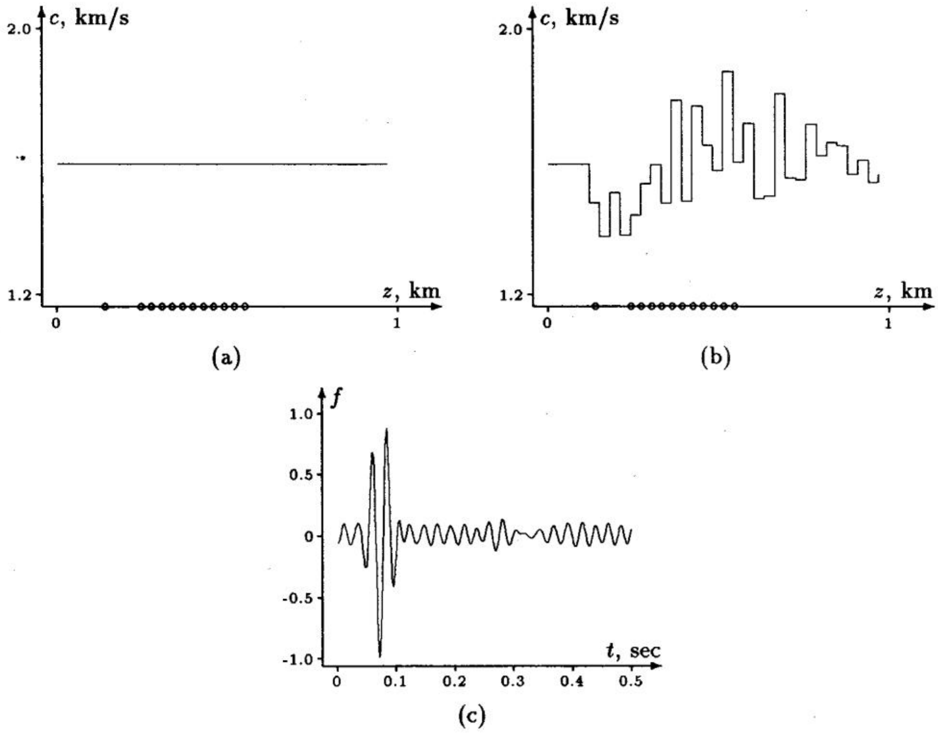

Figure 4

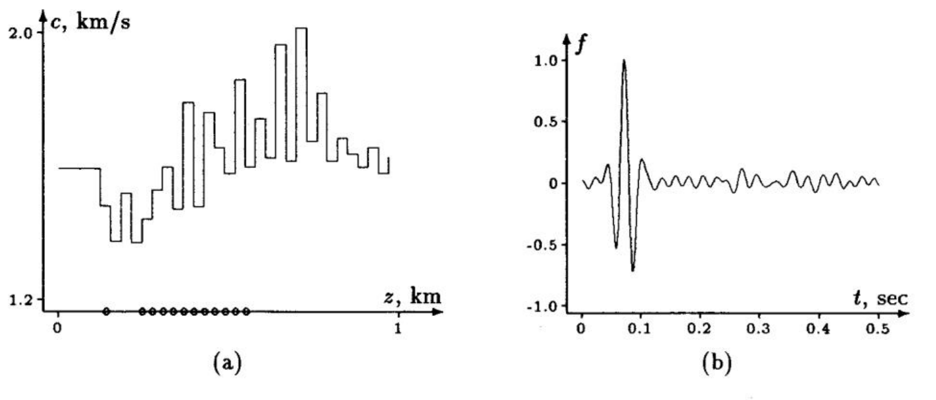

Figure 5

To simulate data (3) the model of the medium presented in Figure 2a was used and as the source function the wavelet with the spectrum

$$F(\omega) = \left[\exp(-(\frac{\omega - 2\pi f}{\pi f})^2) + \exp(-(\frac{\omega + 2\pi f}{\pi f})^2)\right] \times \exp(-1.75 i\omega f^{-1}),$$

where $f = 35$ Hz is the dominant frequency (Figure 2b).

We performed two numerical experiments for the data (3) known within the range of the time frequencies 10 Hz÷60 Hz for two different initial approximations. The first one (Figure 3) was done with the trend component as the initial approximation. One can see an excellent fitting of the reconstructed velocity and impulse with the real ones.

Next we took the constant as the initial approximation for the velocity (Figure 4a). Then there is a rather good accuracy of the reconstruction of the medium only within the location of the antenna of geophones and a very strong deviation out of it (Figure 4b). As far as to the wavelet is concerned, it was reconstructed with high perturbations (as one can see from Figure 4c there is not only deviations of the amplitude, but the convertion of the phase), nevertheless this fact had no influence on the quality of the reconstruction of the medium within the antenna. We suppose that this fact is very important as it follows a possibility to avoid the necessity to reconstruct a source function with high accuracy to provide a rather well resolution of a medium.

To support this hypothesis we would like to refer to Figure 5, which presents the results of wave field inversion within the range of the time frequencies 1Hz÷60Hz for the initial approximation as in Figure 4a (i.e., without the trend component). One can assure there is perfect fitting of the reconstructed medium with the real one, although there are again strong deviation in the source function.

5. Conclusion

On the basis of previous considerations we can conclude that proposed approach to process VSP data allows to detalize a structure of a medium as along a well as below its bottom, if one knows a trend component of a wave propagation velocity, while no information about source function is needed to be available. This approach could be adopted to process the data (3) with a number of source functions $F_j(\omega)$. The matter is to get a VSP data one moves a group of geophones (usually 2÷5) along a well and for every position perform an excitation (an explosion or something else). That means we have to take into account that there will be a lot of unknown functions $F_j(\omega)$. But this trouble may be overcome if one modifies the cost function (4) in the way

$$\Phi(F_1,\dots,F_N;n^2)=\sum_{k=1}^{N}\sum_{j=1}^{J}\int_{\omega_1}^{\omega_2}\left|u_j^k(\omega)-B_j(F_k;n^2)\right|^2 d\omega,$$

where N – a number of explosions, J – a number of geophones in a group.

References

[1] A. Bamberger, G. Chavent, P. Lailly, About stability of the inverse problem in 1-D wave equations - Application to the interpretation of seismic profiles, Appl. Math. Optim., 5, 1979, 1–47.

[2] D. Mace, P. Lailly, Solution of the VSP one-dimensional inverse problem, Geophysical Prospecting, 34, 1986, 1002–1021.

[3] F. Santosa, W.W. Symes, An analysis of a least-squares velocity inversion, Geophysical Monograph Series, 154, 1989.

[4] A.S. Alekseev, A.V. Avdeev, A.G. Fatianov, V.A. Cheverda, Wave processes in vertically-inhomogeneous media: a new strategy for a velocity inversion, Inverse Problems, Vol. 9, No. 3, 1993 (in Russian).

[5] Sh. Njambaa, V.A. Cheverda, An optimizational approach to solve inverse problem of electroprospecting for direct current in layered media, Computing Center, Novosibirsk, 794, 1988 (in Russian).

[6] R. Newton, Inversion of reflection data for layered media; a review of exact methods, Geophys. J.R Astron. Soc. 65, 1981, 191–215.

[7] V.G. Romanov, Inverse problem of mathematical physics, Nauka, Moskva, 1989 (in Russian).