Bull. Nov. Comp. Center, Math. Model. in Geoph., 8 (2003), 113–125

Spectral inversion method of wave equations*

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The statements of inverse problems of two types are considered within the framework of the scalar wave equation. The inverse problems include the reconstruction of sources of the wave field (the right-hand side of the equation) and the determination of the unknown coefficient (wave velocity). In spite of some difference in the statements of these problems, they are closely interrelated: the solution of one of the problems follows from the solvability of the other problem, and vice versa.

The spectral approach is used to construct and analyze the solution. Within the framework of this approach, this generality and uniformization of the solution to these problems manifest themselves most clearly. The Fourier transforms, which form the basis of the spectral method, will be used in the following form:

$$\left.\begin{aligned} \hat{f}(\boldsymbol{k},\omega) &= \iiiint\limits_{-\infty}^{\infty} f(\boldsymbol{x},t)e^{-i(\omega t+\boldsymbol{k}\cdot\boldsymbol{x})}\,dt\,d\boldsymbol{x} \equiv F[f], \\ f(\boldsymbol{x},t) &= \frac{1}{(2\pi)^4}\iiiint\limits_{-\infty}^{\infty} \hat{f}(\boldsymbol{k},\omega)e^{+i(\omega t+\boldsymbol{k}\cdot\boldsymbol{x})}\,d\omega\,d\boldsymbol{k} \equiv F^{-1}[\hat{f}] \end{aligned}\right\}. \qquad (1)$$

It is assumed that there exist the Fourier integrals that are understood in terms of the theory of generalized functions. The class of slowly increasing singular functions turns out to be suitable for this purpose [3].

1. First, let us consider the following model problem of radiation and wave propagation:

$$\left.\begin{aligned} &\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)u(\boldsymbol{x},t) = f(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t>0, \\ &u(\boldsymbol{x},t)\equiv 0 \qquad\qquad\qquad\qquad\quad\ \text{for } t<0,\ \boldsymbol{x}\in\mathbb{R}^3 \end{aligned}\right\}. \qquad (2)$$

Here we assume that $c_0 = \text{const}$, and the carrier $\bar{B}_0 = \operatorname{supp} f = \bar{D}_0 \times [0, t_1]$ is compact. Besides, $f(\boldsymbol{x},t) \equiv 0$ for $t < 0$, that is, the source function satisfies

*Supported by the Russian Foundation for the Basic Research under Grant 03-05-64081.

the causality conditions, and $f(\boldsymbol{x},t) \equiv 0$ for $t > t_1$, where t_1 is the time of termination of the sources' action.

It should be noted that problem (2) is formulated for the entire space $\mathbb{R}^3$, but not for a half-space. This is done to simplify further computations, although allowance for the boundary conditions on the plane $z = 0$ does not introduce any significant difficulties: this can always be done for the wave equation and half-space, for instance, by the mirror reflection method [3, 5, 7].

The inverse problem is in determining the source function $f(\boldsymbol{x},t)$ by using the field $u_0(\boldsymbol{x}',t)$ and its normal derivative $\mu(\boldsymbol{x}',t)$, known at the closed surface S containing a source domain $\bar{D}_0$. Hereafter, we use the following notation:

$$\boldsymbol{x}' \equiv \boldsymbol{x} \in S, \quad u_0(\boldsymbol{x}',t) \equiv u(\boldsymbol{x} \in S,t), \quad \mu(\boldsymbol{x}',t) \equiv \partial_n u(\boldsymbol{x} \in S,t).$$

The inverse problem formulated in this way does not have a unique solution. The set of sources $\{f\}$, that give, at the surface S, the same values of the field $u_0(\boldsymbol{x}',t)$ and its normal derivative $\mu(\boldsymbol{x}',t)$, are sources of so-called "non-radiating type" [1, 13].

However, there are sources of special structure

$$f(\boldsymbol{x},t) = f(\boldsymbol{x})\delta'(t). \tag{3}$$

They have the form of instantly "actuated" sources (the Cauchy data). For such sources, the solution to the inverse problem of reconstruction of the function $f(\boldsymbol{x})$ is unique, and this solution can be obtained by the method of field continuation (reverse in time) [1].

For this purpose, we introduce into our consideration a fundamental solution G_* for the wave operator, which represents the "difference" between the solutions of advanced and retarded types:

$$G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = G_-(\boldsymbol{x},\boldsymbol{\xi},t-\tau) - G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau), \tag{4}$$

where

$$\begin{aligned} &G_-(\boldsymbol{x},\boldsymbol{\xi},t-\tau) \equiv 0 \quad \text{for} \quad t < \tau \ \text{(retarded solution)}, \\ &G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau) \equiv 0 \quad \text{for} \quad t > \tau \ \text{(advanced solution)}. \end{aligned}$$

$G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau)$ is the solution of the following Cauchy problem:

$$\left.\begin{aligned} &\Box\, G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = 0 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1;\ \boldsymbol{\xi} \in D_0,\ \tau > 0; \\ &G_*(\boldsymbol{x},\boldsymbol{\xi},0) = 0 \quad \text{for } t = \tau; \\ &\frac{\partial}{\partial t}G_*(\boldsymbol{x},\boldsymbol{\xi},0) = -c_0^2\delta(\boldsymbol{x}-\boldsymbol{\xi}) \quad \text{for } t = \tau. \end{aligned}\right\} \tag{5}$$

In (5), $\Box$ denotes the wave operator for the variables $(\boldsymbol{x}, t)$, and $(\boldsymbol{\xi}, \tau)$ are the parameters.

In a homogeneous medium, $c_0 = \text{const}$, the solution G_* has the following explicit form:

$$G_*(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau) = -\frac{1}{4\pi}\Big\{|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\Big(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0}\Big) - |\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\Big(t-\tau+\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0}\Big)\Big\}. \tag{6}$$

The spectral representation in the space $(\boldsymbol{k}, \omega)$ of solution (6) [5] is as follows:

$$\begin{aligned}\hat{G}_*(\boldsymbol{k}, \omega) \equiv \hat{G}_*(k, \omega) &= -i2\pi(\operatorname{sign}\omega)\delta\Big(\frac{\omega^2}{c_0^2} - |\boldsymbol{k}|^2\Big)\\ &= -i\frac{\pi}{k}\Big\{\delta\Big(\frac{\omega}{c_0} - |\boldsymbol{k}|\Big) - \delta\Big(\frac{\omega}{c_0} + |\boldsymbol{k}|\Big)\Big\},\end{aligned} \tag{7}$$

where $\boldsymbol{k} \equiv (k_x, k_y, k_z)$ and $|\boldsymbol{k}| = k$, $k = |\omega|/c_0$ is wave number, $k > 0$.

Let us apply Green's formula [5, 14] to the pair of functions u and G_*. Using zeroes for all $\mathbb{R}^{3+1}$ in the source function carrier $\overline{B_0}$, we obtain the following basic integral equation of the convolution type in $\mathbb{R}^{3+1}$, which relates the continued field $w_*(\boldsymbol{x}, t)$ to the source function $f(\boldsymbol{x}, t)$:

$$\begin{gathered} f(\boldsymbol{x}, t) * G_*(\boldsymbol{x}, t) = w_*(\boldsymbol{x}, t) \\ \int_{-\infty}^{\infty} d\tau \iiint_{-\infty}^{\infty} f(\boldsymbol{\xi}, \tau) G_*(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\, d\boldsymbol{\xi} = w_*(\boldsymbol{x}, t), \end{gathered} \tag{8}$$

where the continued field w_* is determined by the surface integral in Green's formula:

$$w_*(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_S \Big\{u_0(\boldsymbol{\xi}', \tau)\frac{\partial}{\partial n_\xi} G_*(\boldsymbol{x}-\boldsymbol{\xi}', t-\tau) - \mu(\boldsymbol{\xi}', \tau) G_*(\boldsymbol{x}-\boldsymbol{\xi}', t-\tau)\Big\} dS_\xi, \tag{9}$$

where $\boldsymbol{\xi}' \equiv \boldsymbol{\xi} \in S$, $u_0(\boldsymbol{\xi}', \tau) \equiv u(\boldsymbol{\xi} \in S, \tau)$, $\mu(\boldsymbol{\xi}', \tau) \equiv \partial_n u(\boldsymbol{\xi} \in S, \tau)$.

The solution to equation (8) in the spectral space $(\boldsymbol{k}, \omega > 0)$ is as follows:

$$\frac{1}{i\omega}\hat{f}(\boldsymbol{k}, \omega)\delta\Big(\frac{\omega}{c_0} - |\boldsymbol{k}|\Big) = \frac{1}{\pi c_0}\hat{W}_*(\boldsymbol{k}, \omega). \tag{10}$$

Equation (10) shows that the spectrum of the source $\hat{f}(\boldsymbol{k}, \omega)$ can be determined only at points of a conoid:

$$|\boldsymbol{k}| = \frac{\omega}{c_0}, \qquad \omega > 0. \tag{11}$$

At other points of the 4D space, the spectrum $\hat{f}(\boldsymbol{k},\omega)$ remains undetermined. This is the "secret of non-uniqueness" of the solution to the inverse problem on the reconstruction of sources of the field $f(\boldsymbol{x},t)$: one can continue the spectrum from the conical surface (11) to the entire $(\boldsymbol{k},\omega)$-space in different ways, not changing values of the field $u_0(\boldsymbol{x}',t)$ and its normal derivative $\mu(\boldsymbol{x}',t)$ at the observation surface S. The set of spectra obtained in this way is for sources of "non-radiating type". Their radiation is "invisible" to an observer on S.

In the case of instantly acting sources of the type (3), the situation is different:

$$\hat{f}(\boldsymbol{k})\delta\left(\frac{\omega}{c_0} - |\boldsymbol{k}|\right) = \frac{1}{\pi c_0}\hat{W}(\boldsymbol{k},\omega). \tag{12}$$

Now, at continuous frequency variation from 0 to ∞, the sphere $|\boldsymbol{k}| = \omega/c_0$ passes through all points of the 3D space (k_x, k_y, k_z), allowing us to determine the 3D spectrum of the source $\hat{f}(\boldsymbol{k})$ by using the known 4D spectrum of the continued field $\hat{W}(\boldsymbol{k},\omega)$.

2. The inverse problem was considered above in a somewhat refined statement: the trace of the field u_0 and its normal derivative μ, "observed" at the closed surface S were considered as "initial data". A model more realistic from the point of view of geophysical applications is that in which the plane $z = 0$ dividing $\mathbb{R}^3$ into two half-spaces serves as an observation surface, at whose points only the trace of the field $u_0(x_1, x_2, 0, t)$ is recorded. It turns out [12] that in this case of homogeneous half-space all reasoning of Section 1 is repeated. The basic integral equation (8) remains the same, except that the source function $f(\boldsymbol{x},t)$ must be oddly extended to the second half-space:

$$\tilde{f}(\boldsymbol{x},t) = f(\boldsymbol{x},t) - f(x_1, x_2, -x_3, t). \tag{13}$$

The second remark concerns the field continuation algorithm (9). It is simplified considerably in the model under consideration, and takes the form of the 3D convolution of the variables (x, y, t). This, in turn, admits the use, at their numerical realization, of fast Fourier transform algorithms [4]:

$$\begin{aligned} w_*(\boldsymbol{x},t) &= 2\int_{-\infty}^{\infty} d\tau \iint_S u_0(\boldsymbol{\xi}',\tau)\frac{\partial}{\partial n_\xi}G_*(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\, dS_\xi \\ &= -\frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\partial}{\partial x_3}\left\{\frac{u_0(\boldsymbol{\xi}', t + \frac{|\boldsymbol{x}-\boldsymbol{\xi}'|}{c_0}) - u_0(\boldsymbol{\xi}', t - \frac{|\boldsymbol{x}-\boldsymbol{\xi}'|}{c_0})}{|\boldsymbol{x}-\boldsymbol{\xi}'|}\right\} d\xi_1\, d\xi_2 \\ &= u_0(\boldsymbol{x}',t) * H(x,y,t;z), \end{aligned} \tag{14}$$

where the symbol $*$ is the convolution of the variables $(x_1 \equiv x, x_2 \equiv y, t)$, and $x_3 \equiv z$ plays the role of a transformation parameter;

$$\boldsymbol{x}' \equiv (x_1, x_2, 0); \qquad \boldsymbol{\xi}' \equiv (\xi_1, \xi_2, 0);$$

$$|\boldsymbol{x} - \boldsymbol{\xi}'| = \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2\right]^{1/2};$$

$$H(x, y, t; z) = -\frac{1}{2\pi}\frac{\partial}{\partial x_3}\left\{|\boldsymbol{x}|^{-1}\delta(t + |\boldsymbol{x}|/c_0) - |\boldsymbol{x}|^{-1}\delta(t - |\boldsymbol{x}|/c_0)\right\}. \tag{15}$$

It is shown in [10, 11] that the Fourier transform of the variables (x, y, t) for the kernel (15) is

$$F[H] = +2i\sin(zk_z^0), \tag{16}$$

where

$$k_z^0 = +\sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2}. \tag{17}$$

It should be noted that here only homogeneous plane waves are of interest to us. Therefore, radical (17) is real (Figures 1 and 2).

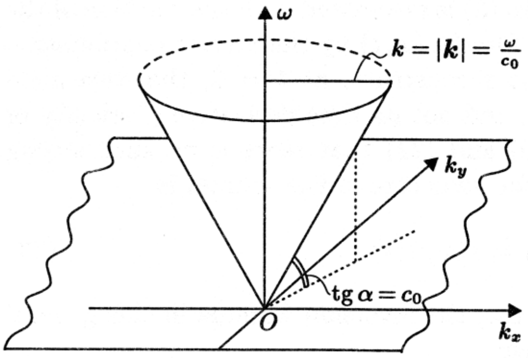

Figure 1. Space (ω, k_x, k_y)

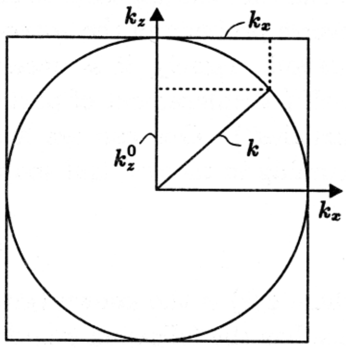

Figure 2. Section $k_y = 0$

Now, applying the Fourier transform of the coordinate z to both parts of equality (14) and taking into account the known formula [3], we have

$$F_z[2i\sin(zk_z^0)] = \delta(k_z - k_z^0) - \delta(k_z + k_z^0), \tag{18}$$

and obtain the following expression for the continued field spectrum:

$$\hat{W}_*(\boldsymbol{k}; \omega) = \hat{U}_0(k_x, k_y; \omega)\{\delta(k_z - k_z^0) - \delta(k_z + k_z^0)\}. \tag{19}$$

On the other hand, the spectrum of fundamental solution (7) also has a special structure: it is concentrated at points of conical surface (11). In expression (7), we transform the surface delta-function

$$\delta\left(\frac{\omega^2}{c_0} - k_x^2 - k_y^2 - k_z^2\right) = \frac{1}{2k_z^0}\{\delta(k_z - k_z^0) + \delta(k_z + k_z^0)\},$$

where k_z^0 is given by (17), and obtain the following solution of integral equation (8) in the spectral space for sources of the type (3): for $\omega > 0$, $k_z > 0$

$$\hat{\tilde{f}}(k_x, k_y, k_z) = \frac{1}{(i\omega)\pi} i k_z^0 \hat{U}_0(k_x, k_y, \omega). \tag{20}$$

Here, we restrict our consideration to the domain $k_z > 0$: as for real functions, this information is sufficient for the unique reconstruction of $f(\boldsymbol{x})$. It should also be noted that in expression (3) the time derivative of the delta-function can be removed, and we can consider sources of the following form:

$$f(\boldsymbol{x}, t) = f(\boldsymbol{x})\delta(t), \tag{21}$$

for which the form of the inversion formula is very simple:

$$\hat{\tilde{f}}(k_x, k_y, k_z) = \frac{1}{\pi}(i k_z^0)\hat{U}_0(k_x, k_y, \omega). \tag{22}$$

Recall that the appearance of $\delta'(t)$ in (3) is associated with the solution of the inverse problem in the space-time domain $(\boldsymbol{x}, t)$ by the inverse continuation method, namely, it is necessary to reconstruct, at $t = 0$, the wave field itself ("displacement of particles"), and not its time derivative ("velocity of particles"). One can see from (20) and (22) that there is no such strong coupling in the spectral space. If the structure of the sources is

$$f(\boldsymbol{x}, t) = f(\boldsymbol{x})\varphi(t), \tag{23}$$

where $\varphi(t)$ is the known (given) pulse, the inversion formula in this general case has the following standard form:

$$\hat{\tilde{f}}(k_x, k_y, k_z) = \frac{1}{\pi}\hat{\varphi}^{-1}(\omega)(i k_z^0)\hat{U}_0(k_x, k_y, \omega), \tag{24}$$

here $\hat{\varphi}^{-1}(\omega)$ is the deconvolution of the signal $\varphi(t)$ in the frequency domain.

The standardized form of the solution to the inverse problem of sources' reconstruction in the $(\boldsymbol{k}, \omega)$-space is an attractive feature of the entire spectral approach. For instance, to pass over to the 2D problem, it is sufficient to assume that $k_y \equiv 0$, and all above formulas remain valid in the plane case: it is known [3] that in the space-time domain $(\boldsymbol{x}, t)$ the structure of the 2D and the 3D solutions differs considerably, and the "transition" to the 2D space will require special adjustment of the algorithm.

One should comment briefly on the algorithm for the solution to the inverse problem. It follows from comparison (20) with similar formula (12) of the preceding section that, in contrast to (12), in the case under consideration there is no field continuation stage, which is most labor-consuming from the point of view of computations. Instead of it, a 3D Fourier transform is performed. First, the input data observed in the space (x, y, t), are transformed to the spectral domain (k_x, k_y, ω). Then they are transferred through half-spheres (17) to the space (k_x, k_y, k_z). Since we assume that these transforms can be performed with the use of the FFT algorithms [4], we obtain an obvious increase in the speed and accuracy of calculations in comparison to the direct calculation of integrals of the type (9) or (14).

3. In the final part of the paper, we consider the inverse problem of wave scattering from medium's inhomogeneities. Also, we show that the spectral inversion formulas presented in the preceding sections are suitable in this case. Now we consider, instead of (2), the following problem statement:

$$\left.\begin{aligned} &\left(\Delta - \frac{1}{(c(\boldsymbol{x}))^2}\frac{\partial^2}{\partial t^2}\right)u(\boldsymbol{x}, t) = \delta(t)\delta(\boldsymbol{x} - \boldsymbol{x}_0) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1, \\ &u(\boldsymbol{x}, t) \equiv 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0. \end{aligned}\right\} \tag{25}$$

In this statement, the wave field is excited by a concentrated source acting at the point $\boldsymbol{x}_0$ of the plane $x_3 \equiv z = 0$. The source comes into action at $t = 0$, and the wave velocity is represented as the sum

$$c^{-2}(\boldsymbol{x}) = c_0^{-2} + m(\boldsymbol{x}). \tag{26}$$

Here, $c_0 = \text{const}$ is considered to be known (the reference model), and the sought-for addition is considered to be small. It is assumed that the anomaly $m(\boldsymbol{x})$ occupies the local domain $\bar{D}_0$ (in the preceding sections, it served as a carrier of the source function of the field), which is entirely located in the half-space $x_3 \equiv z > 0$.

As in Section 2, the plane $x_3 = 0$ is a fictitious surface, at whose points only the trace of the wave field $u_0(\boldsymbol{x}', t; \boldsymbol{x}_0)$ is fixed, and the boundary conditions are absent. It is assumed that the variable velocity $c(\boldsymbol{x})$ has appropriate smoothness, so that there are no boundaries with jumpwise velocity variation. Therefore, it is not necessary to formulate additional conditions for the contact of media.

The inverse problem of scattering is in determining the velocity anomaly $m(\boldsymbol{x})$ (and, hence, the velocity $c(\boldsymbol{x})$) by using the field $u_0(\boldsymbol{x}', t; \boldsymbol{x}_0)$, observed in the plane $x_3 = 0$, to which points of the receiver $\boldsymbol{x}' \equiv (x_1, x_2, 0)$ and the source $\mathbf{x}_0 \equiv (x_{01}, x_{02}, 0)$ belong.

In accordance with the idea of linearization, the full field is represented as the sum of two terms:

$$u(\boldsymbol{x}, t; \boldsymbol{x}_0) = u_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{x}_0) + u_{\text{s}}(\boldsymbol{x}, t; \boldsymbol{x}_0). \tag{27}$$

Here, the "incident" field (sounding signal) u_{in} is caused by the action of the concentrated source in the reference medium c_0 (without anomaly), and the scattered field u_{s} is determined by the following conditions:

$$\left.\begin{aligned}&\left(\Delta - \frac{1}{c_0^2}\frac{\partial}{\partial t^2}\right)u_{\mathrm{s}}(\boldsymbol{x},t;\boldsymbol{x}_0) = m(\boldsymbol{x})\frac{\partial^2}{\partial t^2}u_{\mathrm{in}}(\boldsymbol{x},t;\boldsymbol{x}_0) \quad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ &u_{\mathrm{s}}(\boldsymbol{x},t;\boldsymbol{x}_0)\equiv 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t<0.\end{aligned}\right\} \tag{28}$$

The problem of linearization consists in the fact that there is the "incident" field u_{in} instead of the full field u, in the right-hand side of (28), which would be produced at the exact transition from (25) to (28). Physically, this means that we consider only single-scattered waves.

Below we consider an observation system with a fixed source. A point of this source can be taken as the origin of coordinates:

$$\left.\begin{aligned}&\boldsymbol{x}_0 = 0; \quad \boldsymbol{x} = r\boldsymbol{e}_x;\\ &r = |\boldsymbol{x}|; \quad \boldsymbol{e}_x = \frac{\boldsymbol{x}}{r}.\end{aligned}\right\} \tag{29}$$

A more detailed problem statement is given in [2]. In this paper, it is shown that the inverse problem of scattering in the space-time representation is reduced to the Radon transform:

$$\iiint\limits_{-\infty}^{\infty} f(\boldsymbol{\xi})\delta\left(\frac{r}{2} - \boldsymbol{\xi}\cdot\boldsymbol{e}_x\right)d\boldsymbol{\xi} = w(r\boldsymbol{e}_x), \tag{30}$$

Here, the sought-for function $f(\boldsymbol{x})$ is related to the anomaly $m(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) = -\frac{c_0}{(4\pi)^2}\frac{m(\boldsymbol{x})}{|\boldsymbol{x}|^2}, \tag{31}$$

and the continued field is taken at $t = 0$:

$$w(\boldsymbol{x}) \equiv w_*(\boldsymbol{x}, 0). \tag{32}$$

In this case, not the trace of the full field u from (27), but that of the scattered field u_{s} (and only its low-frequency part) is continued into the medium:

$$\left.\begin{aligned}&u_0(\boldsymbol{x}',t) = u_{\mathrm{s}}(\boldsymbol{x}',t);\\ &v_0(\boldsymbol{x}',t) = t_+ * u_0(\boldsymbol{x}',t),\end{aligned}\right\} \tag{33}$$

where $t_+ = tH(t) = H(t) * H(t)$, $H(t)$ is the Heaviside function, and the symbol $*$ denotes time convolution. Taking into account these remarks, the field continuation algorithm is given by the following equalities:

$$\begin{aligned} w(\boldsymbol{x}) &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\frac{\partial}{\partial x_3}\left\{\frac{v_0(\boldsymbol{\xi}', \frac{|\boldsymbol{x}-\boldsymbol{\xi}'|}{c_0})}{|\boldsymbol{x}-\boldsymbol{\xi}'|}\right\} d\xi_1 d\xi_2 \\ &= [v_0(\boldsymbol{x}', t) * H(\boldsymbol{x}', t; z)]_{t=0} \\ &= \frac{1}{(2\pi)^3}\iiint\limits_{\Omega_0} 2i\sin(zk_z^0)\hat{V}_0(k_x, k_y; \omega)e^{+i(k_x x + k_y y)}\, dk_x\, dk_y\, d\omega. \end{aligned} \quad (34)$$

Here, the symbol $*$ denotes the convolution of the variables (x, y, t), the function H is given by (15), k_z^0 is given by (17), and Ω_0 is the domain of existence of homogeneous plane waves:

$$k_x^2 + k_y^2 \leq \frac{\omega^2}{c_0^2}. \quad (35)$$

The solution to the inverse problem in the statement under consideration is, in principle, given by the inversion formula of the Radon transform (30). The purpose of this paper, however, is a more detailed development of an alternative spectral approach to the solution of this problem outlined in [2]. First, let us introduce additional notation. Let $\hat{D}(k_x, k_y, k_z)$ denote the 3D spectrum of the field $w(x, y, z)$, continued at $t = 0$, that is,

$$\hat{D}(\boldsymbol{k}) = \iiint\limits_{-\infty}^{\infty} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} w(\boldsymbol{x})\, d\boldsymbol{x} \quad (36)$$

with the corresponding inversion formula

$$w(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{D}(\boldsymbol{k})\, d\boldsymbol{k} \quad (36b)$$

We need the same spectral transforms in spherical coordinates:

$$\left.\begin{aligned} \hat{D}(k, \theta, \varphi) &= \int_0^{\infty} dr\, r^2 \iint\limits_{S_1} w(r, \alpha, \beta) e^{-ikr(\boldsymbol{e}_k\cdot\boldsymbol{e}_x)}\, d\Omega_x, \\ w(r, \alpha, \beta) &= \frac{1}{(2\pi)^3}\int_0^{\infty} dk\, k^2 \iint\limits_{S_1} \hat{D}(k, \theta, \varphi) e^{+ikr(\boldsymbol{e}_k\cdot\boldsymbol{e}_x)}\, d\Omega_k. \end{aligned}\right\} \quad (37)$$

Here

S_1 is the sphere of directions of a unit radius;

$d\Omega_x = \sin\alpha\, d\alpha\, d\beta$ is an element of the solid angle in the coordinate space (x, y, z);

$d\Omega_k = \sin\theta\, d\theta\, d\varphi$ is an element of the solid angle in the spectral space (k_x, k_y, k_z);

$\boldsymbol{e}_k \cdot \boldsymbol{e}_x = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\varphi - \beta)$ is the "cosine of angle" given by the directions of the basis vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_k$;

$\boldsymbol{e}_x = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ is the basis vector of direction in the coordinate space (x, y, z);

$\boldsymbol{e}_k = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the basis vector of direction in the spectral space (k_x, k_y, k_z);

$\boldsymbol{x} \equiv (x, y, z) = r\boldsymbol{e}_x$;

$\boldsymbol{k} \equiv (k_x, k_y, k_z) = k\boldsymbol{e}_k$.

On the basis of the remark [8, 9] that the N-dimensional Fourier transform is a composition of the 1D Fourier transform and the Radon transform, let us introduce into consideration the 1D spectra, $\hat{W}(k; \alpha, \beta)$, of the same continued field $w(r; \alpha, \beta)$, calculated by using the following rule:

$$\hat{W}(k; \alpha, \beta) = \int\limits_{-\infty}^{\infty} e^{-ikr} w(r; \alpha, \beta)\, dr, \tag{38}$$

that is, for each direction given by the angles (α, β) in the coordinate space (x, y, z), we take the 1D Fourier transform of the parameter r. In this way, we formally extend the function evenly to negative values of this parameter:

$$w(r; \alpha, \beta) = w(-r; \alpha, \beta).$$

Substituting the second formula from (37) into (38), we obtain:

$$\hat{W}(k; \alpha, \beta) = \frac{1}{(2\pi)^3} \int\limits_{-\infty}^{\infty} dk'\, k'^2 \iint\limits_{S_1} d\Omega_k\, \hat{D}(k', \theta, \varphi) \left\{ \int\limits_{-\infty}^{\infty} e^{+ir[k'(\boldsymbol{e}_x \cdot \boldsymbol{e}_k) - k]}\, dr \right\}.$$

Owing to the properties of the delta-function [3]:

$$\int\limits_{-\infty}^{\infty} e^{+ir[k'(\boldsymbol{e}_x \cdot \boldsymbol{e}_k) - k]} dr = 2\pi\delta(k'\boldsymbol{e}_x \cdot \boldsymbol{e}_k - k), \tag{39}$$

we obtain the following relation between the 1D- $\hat{W}$ and the 3D- $\hat{D}$ spectrum of the same continued field w:

$$\hat{W}(k; \alpha, \beta) = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} dk'\, k'^2 \iint\limits_{S_1} \hat{D}(k', \theta, \varphi)\delta(k'\boldsymbol{e}_k \cdot \boldsymbol{e}_x - k)\, d\Omega_k, \tag{40}$$

or, in a compact way, in the Cartesian coordinates:

$$\hat{W}(\boldsymbol{k}) = \frac{1}{(2\pi)^2} \iiint_{-\infty}^{\infty} \hat{D}(\boldsymbol{k}')\delta(\boldsymbol{k}' \cdot \boldsymbol{e}_x - k)\, d\boldsymbol{k}', \tag{41}$$

where

$$\boldsymbol{k} \equiv (k_x, k_y, k_z) = k\boldsymbol{e}_x, \quad \boldsymbol{k}' \equiv (k'_x, k'_y, k'_z) = k'\boldsymbol{e}_k.$$

Thus, the 1D spectrum $\hat{W}$ is related to the vector of space frequencies $\boldsymbol{k}$, and the 3D spectrum $\hat{D}$ corresponds to the vector $\boldsymbol{k}'$. There is an integral relationship, (41), of the type of the Radon transform, between them, but in the spectral space, because the argument of the delta-function (41) includes the expression for the plane

$$\boldsymbol{k}' \cdot \boldsymbol{e}_x = k \tag{42}$$

of the variables (k'_x, k'_y, k'_z), through the point $\boldsymbol{k}$ orthogonally to the radius-vector $\boldsymbol{k} = k\boldsymbol{e}_x$ (Figure 3).

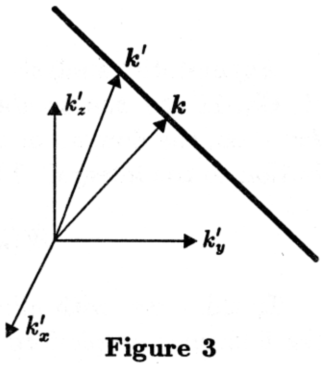

Figure 3

Now, we apply the Fourier operator $\int\limits_{-\infty}^{\infty} e^{-ikp}dp$, where $p = r/2$, to both sides of (30) and obtain the following equalities.

In the right-hand side of (30), we have

$$\int_{-\infty}^{\infty} e^{-ikp} w(\boldsymbol{e}_x, 2p)\, dp = \frac{1}{2}\hat{W}(\boldsymbol{e}_x, k/2). \tag{43}$$

In the left-hand side of (30), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} dp\, e^{-ikp} \left\{ \iiint_{-\infty}^{\infty} d\boldsymbol{\xi} f(\boldsymbol{\xi}) \delta(p - \boldsymbol{\xi} \cdot \boldsymbol{e}_x) \right\} \\
&\quad = \iiint_{-\infty}^{\infty} d\boldsymbol{\xi}\, f(\boldsymbol{\xi}) \left\{ \int_{-\infty}^{\infty} e^{-ikp} \delta(p - \boldsymbol{\xi} \cdot \boldsymbol{e}_x)\, dp \right\} \\
&\quad = \iiint_{-\infty}^{\infty} f(\boldsymbol{\xi}) e^{-ik(\boldsymbol{e}_x \cdot \boldsymbol{\xi})}\, d\boldsymbol{\xi} = \hat{f}(k\boldsymbol{e}_x).
\end{aligned} \tag{44}$$

It follows from a comparison of (43) and (44) that the solution to the inverse problem of scattering in the spectral domain has the following form:

$$\hat{f}(\boldsymbol{k}) = \frac{1}{2}\hat{W}\left(\frac{1}{2}\boldsymbol{k}\right), \tag{45}$$

and in the coordinate space the sought function is found by the Fourier inversion (1):

$$f(\boldsymbol{x}) = F^{-1}[\hat{f}(\boldsymbol{k})], \tag{46}$$

where $\boldsymbol{k} = k\boldsymbol{e}_x$, $k = \omega/(2c_0)$.

In the solution (45), the function $\hat{W}(k/2, \alpha, \beta)$ is a 1D spectrum of the continued field $w(r; \alpha, \beta)$, calculated by using rule (38) and related to the 3D spectrum $\hat{D}(k, \theta, \varphi)$ by integral transform (40). In these formulas, the length of the vector of space frequencies $\boldsymbol{k}$ is related to the frequency ω and velocity c_0 as follows:

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{2c_0}\right)^2. \tag{47}$$

Asymptotic analysis of an integral of the type (40) performed in [6, Ch. 11] by the saddle-point method has shown that at large values, $kr \to \infty$, the domain of angles (θ, φ), close to (α, β) make the main contribution to the integral. Thus, in the far observation zone we have

$$\hat{W}(k, \alpha, \beta) \approx \hat{D}(k, \theta = \alpha, \varphi = \beta). \tag{48}$$

In this case, with allowance for (14)–(19) and (33), solution (45) takes the following simple form:

$$\text{for } \omega > 0,\ k_z > 0 \qquad \hat{f}(\boldsymbol{k}) = \frac{1}{2}\hat{V}_0\left(k_x, k_y; \frac{\omega}{2}\right)\delta(k_z - k_z^0), \tag{49}$$

where $k_z^0 = \sqrt{\dfrac{\omega^2}{4c_0^2} - k_x^2 - k_y^2}$ and is in agreement with (47).

As in the case of algorithm (20) from the preceding section, the most labor-consuming stage from the point of view of computations associated with wave continuation is also avoided in (49). Instead of it, there is a sequence of direct and inverse 3D Fourier transforms (observation data), which make it possible to reconstruct the "source" spectrum $\hat{f}(\boldsymbol{k})$, the function $f(\boldsymbol{x})$ itself from the spectrum and, hence, the anomaly $m(\boldsymbol{x})$ from (31).

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