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Compressibility effects in lake hydrodynamics

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A non-hydrostatic version of mathematical model of deep temperate lake is formulated. Water is treated as compressible. Nonlinear relation corresponds density with temperature and pressure. Numerical procedure is based on variational principle in the combination with splitting methods. Two-dimensional variant of the model is used in the numerical experiment which presents the influence of differential compressibility of water on the lateral flow through gravitational adjustment. This experiment simulates the possible way of the formation of the winter mid-depth temperature maximum in deep lakes. Such phenomenon is regularly observed in lake Baikal.

1. Introduction

The pressure in deep lakes is high enough to compress water significantly to lower the temperature of maximum density (TMD). TMD diminishes at the rate of $0.021°\mathrm{C} \cdot \mathrm{bar}^{-1}$. The bottom pressure in lake Baikal, which has the maximum depth more than 1600 m, is about 160 bars. Such a pressure decrease TMD to 0.625° C while it is 4° C at the normal atmospheric pressure. Decreasing of TMD versus depth has crucial consequences in lake hydrodynamics because it controls the mixing processes.

There are some phenomena in lakes which cannot be explained another way than the influence of compressibility. As an example, partial spring and autumn mixing of deep lakes can be considered. One more example is the existence of winter mid-depth temperature maximum. These phenomena are regularly observed in lake Baikal [1–3].

The subject under consideration, that is, the pressure influence on density, has not received much attention in limnology, (this remark, in particular, concerns numerical models), perhaps because most lakes are not deep enough, although the question is still open, which lake should be treated as a deep one. It appears that a lake having the depth of 200 m or more will feel compressibility effects. Therefore the understanding of the processes taking place in deep lakes has both theoretical and practical significance. In this case, our goal is to construct numerical models which could reproduce the physical events realistically. Various versions of numerical models have been constructed and are being used for the investigation of hydrodynamics and transport of contaminants in lake Baikal [4–6]. Two most important approximations were used there. These are hydrostatic pressure distribution and Boussinesq assumption. Hereby the non-hydrostatic model taking into account the compressibility of water is presented.

2. Mathematical model

The governing system of equations expresses the balance laws of momentum, mass and energy

$$\frac{du}{dt} - lv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\nu\frac{\partial u}{\partial z} + \frac{\partial}{\partial x}A\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}A\frac{\partial u}{\partial y}, \tag{1}$$

$$\frac{dv}{dt} + lv = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\nu\frac{\partial v}{\partial z} + \frac{\partial}{\partial x}A\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}A\frac{\partial v}{\partial y}, \tag{2}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g + \frac{\partial}{\partial z}\nu\frac{\partial w}{\partial z} + \frac{\partial}{\partial x}A\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}A\frac{\partial w}{\partial y}, \tag{3}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z} = 0, \tag{4}$$

$$\rho c_p\frac{dT}{dt} - \alpha\bar{T}\frac{dp}{dt} = \frac{\partial}{\partial z}\rho c_p\nu_T\frac{\partial T}{\partial z} + \frac{\partial}{\partial x}\rho c_p\mu\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}\rho c_p\mu\frac{\partial T}{\partial y} + q, \tag{5}$$

$$\rho = \rho(p, T, S_o). \tag{6}$$

Here u, v, w are the components of velocity vector $\vec{u}$ in the x, y, z directions respectively, l is the Coriolis parameter, p is the pressure, ρ is the density, A, ν, μ, ν_T are the turbulent momentum and thermal diffusivities in the horizontal and vertical directions, respectively, T and $\bar{T}$ are the ordinary and absolute temperatures, c_p is the specific heat at constant pressure, q is the energy flux produced by any distributed source, S_o is the salinity, α is the coefficient of thermal expansion. In the given model the state equation adapted to the limnological problems is used [7]. In it density is the function of temperature, pressure and salinity. As the parameters of the model have been chosen according to lake Baikal conditions, the salinity S_o is supposed to be constant and its value is equal to 96 mg/l.

The governing system of equations (1)–(6) are considered in the domain $D_t = D\times[0,\bar{t}]$ where D is the domain of spatial variables (x, y, z) and $[0,\bar{t}]$ is the time interval. In the real problems D describes the volume of the lake. Its lateral boundary and the bottom are defined by the configuration of the lake. They are the given functions of x, y, z. The boundary and

initial conditions are as follows:
at the lake surface $z = 0$

$$\nu\frac{\partial u}{\partial z} = -\frac{\tau_x}{\rho}, \quad \nu\frac{\partial u}{\partial z} = -\frac{\tau_y}{\rho}, \quad \nu_T\frac{\partial T}{\partial z} = -\frac{Q}{\rho c_p}, \tag{7}$$

$$w = 0, \quad p = p_a(x, y, t);$$

at the bottom $z = H(x, y)$

$$u = v = w = 0, \quad \frac{\partial T}{\partial N} = 0; \tag{8}$$

at the rigid lateral boundaries

$$u = v = w = 0, \quad \frac{\partial T}{\partial N} = 0; \tag{9}$$

initial conditions in D at $t = 0$

$$\varphi = \varphi^o(x, y, z), \tag{10}$$

where

$$\varphi = (u, v, w, T, p, \rho).$$

Here $\partial/\partial N$ means the co-normal derivative

$$\frac{\partial}{\partial N} = \mu \ \cos(n, x)\frac{\partial}{\partial x} + \mu \ \cos(n, y)\frac{\partial}{\partial y} + \nu_T \ \cos(n, z)\frac{\partial}{\partial z}, \tag{11}$$

n is external normal vector to the boundary of the area, p_a is the atmospheric pressure, τ_x, τ_y are wind stresses in the x and y directions, Q is the heat flux on the lake surface.

As for the coefficient of thermal expansion α, it should be noted that in some definite temperature intervals it may change its sign. Such situation occurs in the vicinity of the temperature of maximum density. This is due to the specific fresh water properties that are described in the nonlinear state equation.

The non-hydrostatic model, based on the the equations (1)–(10), is the development of the hydrostatic model [4–6]. Here the water compressibility, variations of density and its dependence on pressure and temperature are explicitly taken into account in the model. The main stages of the construction of the numerical model and an example of its use for the investigation of hydrodynamic processes in a deep basin are described here.

3. Method of the solution of 3D problem

Let us make some transformations in the initial system of equations for the convenience of the algorithmic realization of the model. At first, let us represent density and pressure as the sum

$$p = \bar{p}(z) + p', \quad \rho = \bar{\rho}(z) + \rho', \tag{12}$$

where $\bar{p}$, $\bar{\rho}$ are the background values and p', ρ' are the deviations from non-perturbed values for the corresponding fields. It is supposed that $\bar{p}$ and $\bar{\rho}$ depend only on z and are related by means of the hydrostatic equation

$$\frac{\partial \bar{p}}{\partial z} = g\bar{\rho}. \tag{13}$$

Under these assumptions the pressure derivatives in the equations (1) and (2) does not change their form. In the equation (3) the first two terms on the right-hand side are transformed by means of (11) and (12) in the following way

$$-\frac{1}{\rho}\,\frac{\partial p}{\partial z} + g = \; -\frac{1}{\rho}\left(\frac{\partial p'}{\partial z} - g\rho'\right). \tag{14}$$

Function p' is the dynamic part of pressure. Further p' is a desired function and for convenience the prime in its notation is omitted. In terms $(1/\rho)$ grad p of the equations (1)–(3) and in the term div $\rho\vec{u}$ of the equation (4) all transformations and approximations for the function ρ should be made identical everywhere. This requirement follows from the specific properties of the computational algorithms.

Discrete approximations of the model and the algorithms of its realization are constructed on the basis of variational principle in the combination with the splitting method [8]. Following the idea of splitting with respect to physical essence of processes, we should introduce two stages on each time step: transport with turbulent exchange and dynamic adjustment of the fields [9].

Let us define the structure of approximations on fractional time steps in accordance with the symmetric two-cyclic splitting scheme [10, 11]. We approximate advective-diffusive transport operators apearing in the equations of the model by monotonic numerical schemes using analytical solution of local adjoint problems [12]. Realization of the stage of transport and turbulent exchange is carried out according to the scheme that is typical for the splitting method. It means that local one-dimensional problems in the direction of spatial coordinates x, y, z are solved successively at each fractional time step.

Specific character of the present problem manifests itself reflected first of all at the stage of dynamic adjustment of the fields. Taking this fact into account, the algorithm of realization of this stage will be described in more details. Let us define the problem of dynamic adjustment of the velocity and pressure fields as follows:

$$\frac{\partial u}{\partial t} - lv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{15}$$

$$\frac{\partial v}{\partial t} + lu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{16}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z} - g\rho^{'}\right), \tag{17}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\rho\ \vec{u} = 0. \tag{18}$$

This problem is solved on the time interval $t_j \leq t \leq t_{j+1}$ with the length Δt provided that at $t = t_j$ values of all functions u, v, w, p, ρ, $\rho^{'}$ are given. It is supposed also that the temperature does not change locally within the time interval of the dynamic adjustment. Depending on the assumptions on the approximation ρ in the interval $[t_j, t_{j+1}]$ one may obtain several modifications of the adjustment problem. In this case the function ρ is calculated by means of the state equation, and equations (14)–(17) are used for the calculation of pressure and velocity components.

The system (15)–(18) is linearized with respect to $\rho = \rho^j$ and $\partial\rho/\partial t$ in (18) is excluded by means of the relation $\partial p/\partial t = c^2 \partial\rho/\partial t$, where $c^2 = \partial p/\partial \rho$.

Then, the system is approximated with respect to time by the implicit scheme and the following system of equations on $[t_j, t_{j+1}]$ is obtained

$$\frac{u^{j+1} - u^j}{\Delta t} - lv^{j+1} = -\frac{1}{\rho^j}\frac{\partial p^{j+1}}{\partial x}, \tag{19}$$

$$\frac{v^{j+1} - v^j}{\Delta t} + lu^{j+1} = -\frac{1}{\rho^j}\frac{\partial p^{j+1}}{\partial y}, \tag{20}$$

$$\frac{w^{j+1} - w^j}{\Delta t} = -\frac{1}{\rho^j}\left(\frac{\partial p^{j+1}}{\partial z} - g\rho^{'j}\right), \tag{21}$$

$$\frac{1}{c^2}\frac{p^{j+1} - p^j}{\Delta t} + \mathrm{div}\ \rho^j \vec{u}^{j+1} = 0. \tag{22}$$

Here and further the notations for the derivatives with respect to spatial variables are formally used for the convenience. In actual computations their discrete analogues are present. Let us solve the equations (18)–(20) with respect to u^{j+1}, v^{j+1}, w^{j+1}

$$u^{j+1} = -C\left(\frac{\partial p}{\partial x} + l\Delta t\frac{\partial p}{\partial y}\right)^{j+1} + f_1, \tag{23}$$

$$v^{j+1} = -C\left(\frac{\partial p}{\partial y} - l\Delta t\frac{\partial p}{\partial x}\right)^{j+1} + f_2, \tag{24}$$

$$w^{j+1} = -\frac{\Delta t}{\rho^j}\frac{\partial p^{j+1}}{\partial z} + f_3, \tag{25}$$

where

$$f_1 = a(u^j + l\Delta t v^j), \quad f_2 = a(v^j - l\Delta t u^j),$$
$$f_3 = (\Delta t g \rho'^j)/\rho^j + w^j, \quad a = 1/(1 + (l\Delta t)^2),$$
$$C = \Delta t a/\rho^j.$$

The equation for pressure p^{j+1} is obtained by the substitution of u^{j+1}, v^{j+1}, w^{j+1} into equation (22)

$$\frac{1}{c^2}\frac{p^{j+1} - p^j}{\Delta t} - \left[\frac{\partial}{\partial x}\left(R\frac{\partial p}{\partial x} + B\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial y}\left(R\frac{\partial p}{\partial y} - B\frac{\partial p}{\partial x}\right) - \Delta\frac{\partial^2 p}{\partial z^2}\right]^{j+1} = F, \tag{26}$$

where

$$R = a\Delta t, \qquad B = al\Delta t^2,$$
$$F = -\mathrm{div}\rho^j\vec{f}, \quad \vec{f} = (f_1, f_2, f_3).$$

Boundary conditions for the closure of the system of discrete equations approximating (19)–(25) are the consequence of conditions (7)–(9) for the initial problem.

For relatively small water basins it is possible to take l = const. Then the equation (26) is simplified because the sum of terms with the coefficient B vanishes. The algorithm structure for the solution of equation (26) depends on the form of domain D and its discrete approximations. The iterative procedure is used for the water basins with real shorelines and non-homogeneous bottom relief. After the determination of function p^{j+1} the calculations of the functions $\vec{u}^{j+1} = (u, v, w)^{j+1}$ are provided by (23)–(25). Then the discrete analog of equation (5) with the given functions $\vec{u}^{j+1}$ and p^{j+1} is solved. For this purpose the same splitting scheme as for the stage of transport with turbulent exchange for the functions u, v, w is used. Finally, the density ρ^{j+1} is calculated by the equation of state (6). At the next time steps the whole cycle of computations is repeated.

4. Two-dimensional version of the model

For the description of the non-hydrostatic processes in lakes the numerical model should have fine spatial and temporal resolution of discrete approximations, because it is necessary to simulate the processes, horizontal and vertical scales of which are comparable. The corresponding sizes of domain D itself for real lakes may differ greatly. It is obvious that to carry out a set of numerical experiments on the basis of the three-dimensional model with high resolution is problematic. An admissible compromise between the wish to use a sufficiently complex model and the real possibilities of computing numerical experiments in the desired volume can be reached, if to construct both the 3D model and its 2D modification. It is important that the both versions should have the similar properties. Experience shows that 2D models are a convenient scientific instrument for the solution of the methodological questions with relatively low expenses [6, 13, 14].

It is proposed to construct the 2D non-hydrostatic version of the model (1)–(10) in the same way as the corresponding hydrostatic 2D model described in [6]. As a domain D let us define a 2D transverse section of the 3D domain on the vertical plane (y, z). It is supposed that all the functions are homogeneous along the coordinate x, and their derivatives with respect to x in the 3D model are equal to zero.

The realization of such a model requires not so many calculations as for the basic 3D model, though the general structure of algorithms and splitting schemes is almost the same for the both models. In particular, at the stage of transport and turbulent exchange the solution of local 1D problems in the direction x is excluded.

In the problem of dynamic adjustment in all the formulae (15)–(26) the substitutions $\partial p/\partial x = 0$, $\partial(\rho u)/\partial x = 0$, $\partial f_1/\partial x = 0$ are made.

The 2D variant of the equation (25) at $l =$ const, has the form

$$\frac{1}{c^2}\frac{p^{j+1} - p^j}{\Delta t} - R\frac{\partial^2 p}{\partial y^2}^{j+1} - \Delta t\frac{\partial^2 p}{\partial z^2}^{j+1} = F. \qquad (27)$$

From (24)–(25) it follows that, using the boundary conditions

$$v^{j+1} = 0 \quad \text{at} \quad y = a, b, \qquad w^{j+1} = 0 \quad \text{at} \quad z = 0, H \qquad (28)$$

one obtains the Neumann conditions for the function p. The simplest algorithm for the solution of the problem (27) is obtained when $H =$ const, and the grid domain in D is regular and homogeneons in the direction z. Under these conditions the variables are separated and the algorithm can be direct. The main elements of this algorithm are spectral expansion on eigenfunctions of the vertical operator from (27) and the solution of the discrete equations in the direction y in order to find the corresponding Fourier coefficients of the function p.

5. The results of numerical experiment and conclusions

Now let us consider the numerical experiment realized on the basis of the 2D non-hydrostatic model. Domain D is supposed to be rectangular in the plane (y, z). The grid domain with 49 levels in the vertical direction and 121 points along y is used in the calculations. The grid lengths are: horizontal $\Delta y = 30$ m, vertical $\Delta z = 20$ m, time step $\Delta t = 30$ sec. The chosen parameters of the domain D and discrete approximations of the model make it possible to describe the local fluid motions with the same scales in both directions. The depth H of D is chosen according to the conditions of lake Baikal.

The example suggested by E.C. Carmack and R.F. Weiss [3] was taken as the basis for the design of the simulation scenario. The same experiment was fulfilled in hydrostatic version of 2D model [6]. From the physical point of view the essence of the problem is in the following. It is supposed that at the initial time moment the water in the domain D is conditionally vertically divided into two parts. The temperature in each part is constant from the surface to the bottom and is equal to 3.0^o C in the left part and to 4.0^o C in the right part.

Such distribution of the temperature is characterized by the fact that its values are in the interval which is critical in the definite sense. It is connected with the changing of the temperatures of maximum density with the increase of the depth or pressure. Really, near the surface warm water is heavier than cold water and in the deep layers the situation is *vice versa*. And the second fact is the relatively opposite position of actual temperature values in the cells of D with respect to the temperature of maximum density curve.

The values of the density itself increase with the increase of the depth due to the influence of pressure and its contribution is rather great. At the time moment $t = t_o$ imaginary wall between the cells is taken away and the process of mutual adjustment of currents, pressure and temperature in the continuously stratified fluid in the gravity field begins. One more condition is given additionally, that is on the lake surface the wind stress $\tau_y = 1$ dyne/cm^2 is given. Fragments of the computational scenario presented here illustrate the behaviour of the processes of the field adjustment simulated by the model.

In the figures one can see distributions of temperature and velocity components u, v, w at $t = 125$ min (Figures 1–4) and at $t = 335$ min (Figures 5–8). The figures are provided with the identification tables which show the correspondence between numbers designating isolines (1–15) and values of the displayed functions – temperature (^oC) and velocity components u, v, w (cm/sec), respectively.

The analysis of the results of numerical simulation shows that the formation of the mid-depth maximum takes place at the initial stage of the processes in the temperature field. Figures 1, 5 illustrate the temporal development of this phenomenon. The temperature maximum appears at the depths of 200–300 m in that part of the domain where due to the nonlinearity of state equation the densities of the cold water and the warm water are equal. This resembles the situation when the warm water wedges into the cold water in the shape of a "tongue". Simultaneously cold water moves near to the bottom and the surface of the basin. In the vicinity of the bottom the effects of the boundary layer appear, and the propagation of cold water intensifies under the influence of wind stress on the surface of the basin.

Convective cells are formed in the current field. This is clearly seen in the figures of the distribution of the vertical velocity component w. In Figure 4 one can see two cells near the boundary of the separation of the water mass, the left cell being an upward flux and the right cell being a downward flux. In Figure 8 there are three such cells, the central one being downward motions, and two lateral ones being upward motions. In the fields of horizontal velocity components (Figures 2, 3 and 6, 7) the cells with opposite by directed fluxes are formed, too.

Change of the signs takes place in the transition region of the temperature contrasts. With the increase of the time interval the cells are deformed and their number increases. At the same time the influence of the boundary conditions is felt. And if the sizes of the experimental domain D are smaller than the horizontal sizes of the simulated basin it is also necessary to extend the domain for the model definition when the time interval increases.

The temperature field tends to the steady state. This means that the temperature comes to such a state that the lighter water is distributed over the heavier water, and the fluid is in the hydrostatic stable state. Mutual adjustment of the fields takes place under the influence of wind stress on the surface, and this results in asymmetry of the temperature and velocities.

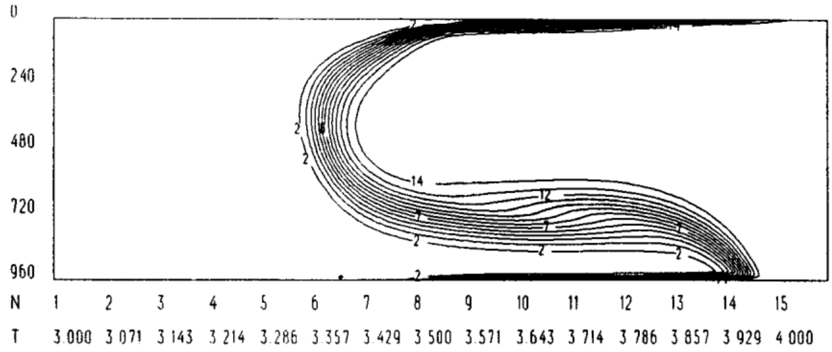

Figure 1. Temperature at $t = 125$ min

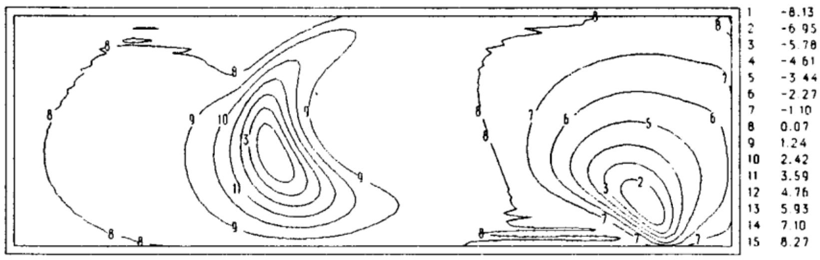

Figure 2. Vertical velocity component w at $t = 125$ min

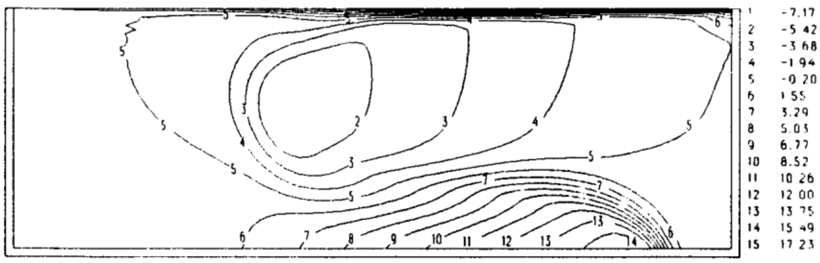

Figure 3. Horizontal velocity component u at $t = 125$ min

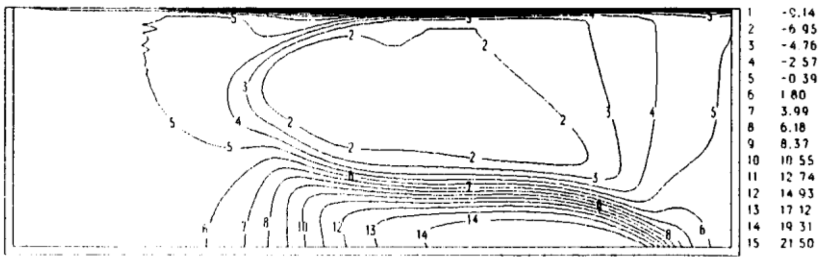

Figure 4. Horizontal velocity component v at $t = 125$ min

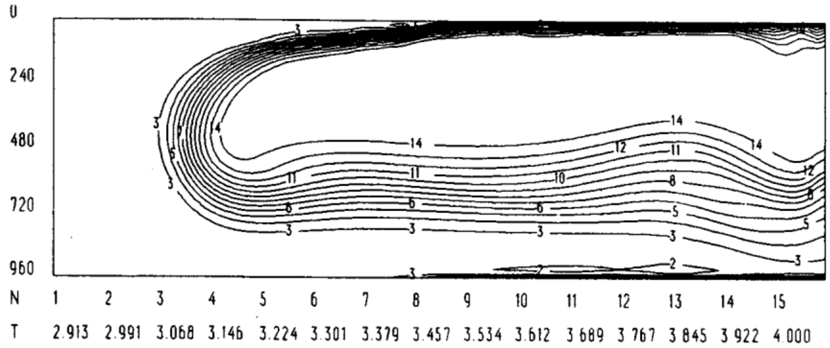

Figure 5. Temperature at $t = 335$ min

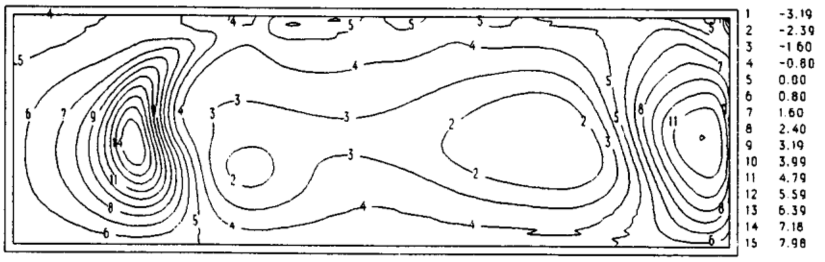

Figure 6. Vertical velocity component w at $t = 335$ min

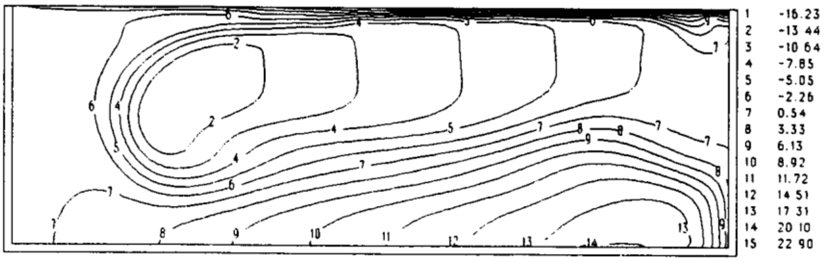

Figure 7. Horizontal velocity component u at $t = 335$ min

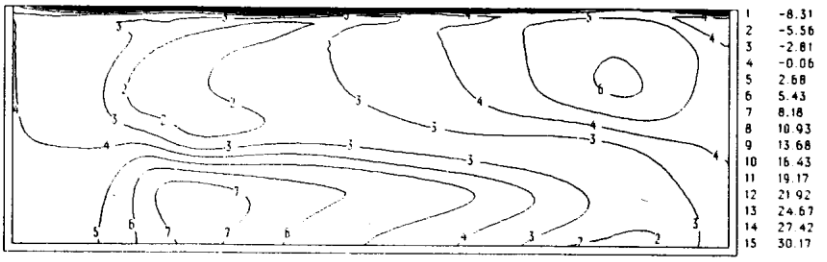

Figure 8. Horizontal velocity component v at $t = 335$ min

In general, the results of this numerical experiment are in agreement with the theoretical assumptions in [1–3] about the role of water compressibility in the appearance of the mesothermal maximum in deep lakes and the role of the gravitational factor of its formation. There is also a good qualitative agreement with the results of simulation by the hydrostatic model [6]. However, the spatial configuration and horizonal sizes of the domains in these two models are different. That is why the results of simulation may be comparéd only in the neighbourhood of the interface between the water masses and only in a relatively short period of time from the initial point. The combined account of non-hydrostatic effects and water compressibility in the models makes it possible to describe in more details the transient behaviour of the adaptation processes when unstable situations appear.

In the computations using the non-hydrostatic model the mechanism of the formation of mesothermal maximum and temperature front propagation manifests itself better. For the adequate model presentation of such phenomena it is very important to use monotonic numerical approximations with the transportivity property for advective-diffusive operators. The development of this model for the investigation of natural phenomena in deep lakes is in progress now. The model of a variable structure combining hydrostatic and non-hydrostatic approximations in the governing equations seems to be usefull in future. It will make us possible to have an agreed description of the processes with different scales. In the framework of such a hybrid system, control over the simulation processes is provided taking into account the relations between the sizes of the lake domain and the scales of the investigated phenomena.

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