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One-dimensional direct and inverse problems for one system arising in a two-phase medium

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Abstract. A one-dimensional system of the Hopf-type equations is considered. Axial solutions to problems in the field of modeling two-fluid interactions are sought. A nonlinear system of ordinary differential equations is obtained. Direct and inverse problems for the obtained ODE are considered. A theorem on local solvability is proven.

Keywords: two-velocity hydrodynamics, viscous fluid, relative velocity, direct problem, inverse problem, Darcy coefficient.

Introduction

A significant number of mathematical formulations of various problems in natural science have the property of spatial locality, i.e., the most important changes in the characteristics being studied are concentrated in a very limited area of space. In this case, as a rule, the governing dynamic equations with partial derivatives are specified in the entire space, and the corresponding boundary conditions actually determine the asymptotic behavior of the sought solutions at infinity. Such problems arise, for example, in hydrodynamics [1], plasma physics [2], laser physics [3] and other areas of natural science. For example, when spatial locality is generated by the axial symmetry of the statement, i.e., when using cylindrical coordinates, a decrease in solutions in the radial direction is clearly expressed. A similar situation arises in geophysics problems when studying near borehole space. In numerical modeling of this type of phenomena, as a rule, the behavior of functions on the axis is of primary importance (for example, effects such as self-focusing or self-channeling; see [3] and the literature cited there). However, to calculate the axial characteristics it is necessary to carry out calculations of "complete" problems, i.e., in a sufficiently large range of changes in the radial variable compared to the characteristic scales of the process [4].

1. Equations of two-velocity hydrodynamics with one pressure

In papers [5, 6] based on conservation laws, invariance equations for the Galilean transformations and conditions thermodynamic consistency, a nonlinear two-speed model of fluid movement through a deformable porous medium. Two-fluid hydrodynamic theory with the condition for the equilibrium of subsystems under pressure, was constructed in the work [7]. Equations of motion of a two-velocity medium in the dissipative case conditioned by the coefficient of interfacial friction χ with one pressure in the system in the isothermal case has the form [7]

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{u})=0,\qquad \frac{\partial\tilde\rho}{\partial t}+\operatorname{div}(\tilde\rho\tilde{\boldsymbol{u}})=0, \tag{1}$$

$$\begin{aligned}\bar\rho\Big(\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u},\nabla)\boldsymbol{u}\Big) = &-\nabla p+\nu\Delta\boldsymbol{u}+(\nu/3+\mu)\nabla\operatorname{div}\boldsymbol{u}+\\ &\frac{\tilde\rho}{2}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2-b\bar\rho\frac{\tilde\rho}{\rho}(\boldsymbol{u}-\tilde{\boldsymbol{u}})+\bar\rho\boldsymbol{f},\end{aligned} \tag{2}$$

$$\begin{aligned}\bar\rho\Big(\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}+(\tilde{\boldsymbol{u}},\nabla)\tilde{\boldsymbol{u}}\Big) = &-\nabla p+\tilde\nu\,\Delta\tilde{\boldsymbol{u}}+(\tilde\nu/3+\tilde\mu)\nabla\operatorname{div}\tilde{\boldsymbol{u}}-\\ &\frac{\rho}{2}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2+b\bar\rho(\boldsymbol{u}-\tilde{\boldsymbol{u}})+\bar\rho\boldsymbol{f},\end{aligned} \tag{3}$$

where $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}$ are the velocity vectors of the subsystems, components of a two-velocity continuum with corresponding partial densities $\tilde\rho$ and ρ, ν (μ) and $\tilde\nu$ $(\tilde\mu)$ — corresponding shear (volume) viscosity, $b=\chi\,\tilde\rho$, $\bar\rho=\tilde\rho+\rho$ — total density of two-velocity continuum; $p=p(\bar\rho,(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2)$ — equation of two-velocity continuum states; $\boldsymbol{f}$ — mass vector force per unit mass.

Let us rewrite equations (2) and (3) in equivalent form

$$\begin{aligned}\bar\rho\Big(\frac{\partial\boldsymbol{u}}{\partial t}+\frac{1}{2}\nabla(u^2)-\boldsymbol{u}\times\operatorname{rot}\boldsymbol{u}\Big) = &-\nabla p+\nu\,\Delta\boldsymbol{u}+(\nu/3+\mu)\nabla\operatorname{div}\boldsymbol{u}+\\ &\frac{\tilde\rho}{2}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2-b\bar\rho\frac{\tilde\rho}{\rho}(\boldsymbol{u}-\tilde{\boldsymbol{u}})+\bar\rho\boldsymbol{f},\end{aligned} \tag{4}$$

$$\begin{aligned}\bar\rho\Big(\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}+\frac{1}{2}\nabla(\tilde u^2)-\tilde{\boldsymbol{u}}\times\operatorname{rot}\tilde{\boldsymbol{u}}\Big) = &-\nabla p+\tilde\nu\,\Delta\tilde{\boldsymbol{u}}+(\tilde\nu/3+\tilde\mu)\nabla\operatorname{div}\tilde{\boldsymbol{u}}-\\ &\frac{\rho}{2}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2+b\bar\rho\,(\boldsymbol{u}-\tilde{\boldsymbol{u}})+\bar\rho\boldsymbol{f}\end{aligned} \tag{5}$$

From these equations, another equations can be derived that determine the change in vortices over time. To do this, apply to both sides of equations (4), (5) the operator rot. As a result we obtain

$$\begin{aligned}\frac{\partial\boldsymbol{\Omega}}{\partial t}-\operatorname{rot}(\boldsymbol{u}\times\boldsymbol{\Omega}) = &-\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big)+\nu\,\Delta\boldsymbol{\Omega}+\operatorname{rot}\Big(\frac{\nu/3+\mu}{\bar\rho}\nabla\operatorname{div}\boldsymbol{u}\Big)+\\ &\operatorname{rot}\Big(\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2\Big)-b\frac{\tilde\rho}{\rho}(\boldsymbol{\Omega}-\tilde{\boldsymbol{\Omega}})+\operatorname{rot}\boldsymbol{f},\\ \frac{\partial\boldsymbol{\Omega}}{\partial t}-\operatorname{rot}(\tilde{\boldsymbol{u}}\times\tilde{\boldsymbol{\Omega}}) = &-\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big)+\tilde\nu\Delta\tilde{\boldsymbol{\Omega}}+\operatorname{rot}\Big(\frac{\tilde\nu/3+\tilde\mu}{\bar\rho}\nabla\operatorname{div}\tilde{\boldsymbol{u}}\Big)-\\ &\operatorname{rot}\Big(\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{u}}-\boldsymbol{u})^2\Big)+b\,(\boldsymbol{\Omega}-\tilde{\boldsymbol{\Omega}})+\operatorname{rot}\boldsymbol{f}.\end{aligned}$$

2. One-dimensional Hopf type system

Let us consider the process of propagation of nonlinear waves in a two-fluid medium, described by a one-dimensional homogeneous system of equations [8–13]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u - \tilde{u}), \tag{6}$$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} = \varepsilon b(u - \tilde{u}), \tag{7}$$

where $\varepsilon = \frac{\rho}{\tilde{\rho}}$ is a dimensionless positive constant.

The paper considers finding axial solutions to problems in the field of modeling two-fluid interactions. In this case, new direct and inverse problems arise for a nonlinear system of ordinary differential equations. The basis for modeling is a system of hydrodynamic equations for a two-fluid medium [4, 14]. In the plane case, axial symmetry is, of course, not assumed, but for the sake of convenience we will use this name for real solutions of a system of the Hopf-type equations [14] of the form

$$u(t,x) = U(t)x, \quad \tilde{u}(t,x) = \tilde{U}(t)x.$$

Amplitudes $U(t)$, $\tilde{U}(t)$ satisfy a nonlinear system of ordinary differential equations:

$$U' + U^2 = -b(t)(U - \tilde{U}), \tag{8}$$

$$\tilde{U}' + \tilde{U}^2 = \varepsilon b(t)(U - \tilde{U}). \tag{9}$$

3. Direct problem

It is required to determine the functions $U(t)$, $\tilde{U}(t)$ from (8), (9) with known b, ε and the Cauchy data

$$U(0) = U_0, \quad \tilde{U}(0) = \tilde{U}_0. \tag{10}$$

Let us multiply both sides of equation (8) by ε and add them to (9)

$$\frac{d(\varepsilon U + \tilde{U})}{dt} + \varepsilon U^2 + \tilde{U}^2 = 0. \tag{11}$$

It follows that for all $t \in (0,T)$ we have the estimate

$$\varepsilon U(t) + \tilde{U}(t) \leq \varepsilon U_0 + \tilde{U}_0. \tag{12}$$

From equation (8) subtract (9), then obtain

$$\frac{d(U-\tilde{U})}{dt}+(U+\tilde{U})(U-\tilde{U})=-(1+\varepsilon)b(U-\tilde{U}). \qquad (13)$$

The solution to the Cauchy problem for system (12), (13), (10) has the form

$$\varepsilon U(t)+\tilde{U}(t)=\varepsilon U_0+\tilde{U}_0-\int_0^t(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau, \qquad (14)$$

$$U(t)-\tilde{U}(t)=(U_0-\tilde{U}_0)\exp\Big(-(1+\varepsilon)bt-\int_0^t(U(\tau)+\tilde{U}(\tau))d\tau\Big). \qquad (15)$$

Solving system (14) and (15) we obtain a system of nonlinear Volterra equations of the second type

$$U(t)=\frac{\varepsilon U_0+\tilde{U}_0}{1+\varepsilon}-\frac{1}{1+\varepsilon}\int_0^t(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau+ \\ \frac{U_0-\tilde{U}_0}{1+\varepsilon}\exp\Big(-(1+\varepsilon)bt-\int_0^t(U(\tau)+\tilde{U}(\tau))d\tau\Big), \qquad (16)$$

$$\tilde{U}(t)=\frac{\varepsilon U_0+\tilde{U}_0}{1+\varepsilon}-\frac{1}{1+\varepsilon}\int_0^t(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau- \\ \frac{\varepsilon}{1+\varepsilon}(U_0-\tilde{U}_0)\exp\Big(-(1+\varepsilon)bt-\int_0^t(U(\tau)+\tilde{U}(\tau))d\tau\Big). \qquad (17)$$

The solution to system (16), (17) (where it exists) is infinitely smooth. It exists locally according to the Picard theorem [15]. This paper does not discuss the issue of the existence interval.

4. Inverse problem

It is required to determine the functions $U(t)$, $\tilde{U}(t)$ and the coefficient b (the constant ε is assumed to be known and $U_0\neq\tilde{U}_0$) from (3)–(5) for additional information

$$U|_{t=t^*}=U_*, \quad t^*<T. \qquad (18)$$

Assuming in (16) $t=t^*$, we obtain

$$(1+\varepsilon)U_*=\varepsilon U_0+\tilde{U}_0-\int_0^{t^*}(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau+ \\ (U_0-\tilde{U}_0)\exp\Big(-(1+\varepsilon)bt^*-\int_0^{t^*}(U(\tau)+\tilde{U}(\tau))d\tau\Big)$$

or

$$\exp\Big(-(1+\varepsilon)bt^* - \int_0^{t^*}(U(\tau)+\tilde{U}(\tau))d\tau\Big)$$
$$= \frac{\varepsilon(U_* - U_0) + U_* - \tilde{U}_0}{U_0 - \tilde{U}_0} + \frac{1}{U_0 - \tilde{U}_0}\int_0^{t^*}(\varepsilon U^2(\tau) + \tilde{U}^2(\tau))d\tau.$$

From here, taking the logarithm we obtain

$$b = -\frac{1}{(1+\varepsilon)t^*}\int_0^{t^*}(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau -$$
$$\ln\bigg(\frac{\varepsilon(U_* - U_0) + U_* - \tilde{U}_0 + \int_0^{t^*}(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau}{U_0 - \tilde{U}_0}\bigg)^{1/(1+\varepsilon)t^*}. \quad (19)$$

Using this formula from (16), (17) with respect to $U(t)$, $\tilde{U}(t)$ we obtain a system of nonlinear Volterra equations of the second type

$$U(t) = \frac{\varepsilon U_0 + \tilde{U}_0}{1+\varepsilon} - \frac{1}{1+\varepsilon}\int_0^t(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau +$$
$$\frac{(U_0 - \tilde{U}_0)^{(t^*-t)/t^*}}{1+\varepsilon}\bigg(\varepsilon(U_* - U_0) + U_* - \tilde{U}_0 + \int_0^{t^*}(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau\bigg)^{t/t^*} \times$$
$$\exp\Big(-\int_{t^*}^t(U(\tau)+\tilde{U}(\tau))d\tau + \frac{t-t^*}{t^*}\int_0^{t^*}(U(\tau)+\tilde{U}(\tau))d\tau\Big), \quad (20)$$

$$\tilde{U}(t) = \frac{\varepsilon U_0 + \tilde{U}_0}{1+\varepsilon} - \frac{1}{1+\varepsilon}\int_0^t(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau -$$
$$\frac{(U_0 - \tilde{U}_0)^{(t^*-t)/t^*}}{1+\varepsilon}\bigg(\varepsilon(U_* - U_0) + U_* - \tilde{U}_0 + \int_0^{t^*}(\varepsilon U^2(\tau)+\tilde{U}^2(\tau))d\tau\bigg)^{t/t^*} \times$$
$$\varepsilon\exp\Big(-\int_{t^*}^t(U(\tau)+\tilde{U}(\tau))d\tau + \frac{t-t^*}{t^*}\int_0^{t^*}(U(\tau)+\tilde{U}(\tau))d\tau\Big). \quad (21)$$

The local existence of a solution to the system of equations (20), (21) is proved in the same way as in the Picard theorem [15]. After finding the function $U(t)$, $\tilde{U}(t)$, the Darcy coefficient b is determined by formula (19).

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