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r-solution of an operator equation in Hilbert space and its application to solve ill-posed problem

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The notion of the r-solution for a system of linear equation in the finite dimensional Euclidean space generalizes for an equation in Hilbert space with a compact operator. The Hadamard correctness of r-solution is proved (i.e., its stability with respect to perturbation of right-hand side and operator).

1. Introduction

Let us remind what is the r-solution for a system of N linear algebraic equations

$$Ax = f, \tag{1}$$

for M unknowns $x = (x_1, x_2, \dots, x_M)^T$ (see [1]). Let $s_1 \geq s_2 \geq \dots \geq s_{N_0} \geq 0$ be the singular values of the matrix A and

$$A = VDU^*,$$

its singular value decomposition, where D is rectangular $N \times M$ matrix such that

$$D = \begin{cases} [\Sigma : 0] & \text{for } N < M, \\ \Sigma & \text{for } N = M, \\ \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} & \text{for } N > M, \end{cases} \tag{2}$$

while Σ is $N_0 \times N_0$ diagonal matrix with singular values of A decreasing along the diagonal. If now for some integer r $(1 \leq r \leq N_0)$ such that $s_r > s_{r+1}$, one will introduce the diagonal matrix Σ_r with elements coinciding with ones for Σ for $j \leq r$ and vanishing for $j > r$ and will determine $A_r = VD_rU^*$, (D_r is the same as in (2) but with $\Sigma = \Sigma_r$), then,

Definition 1. r-solution of (1) is the generalized normal solution of the system of linear equations

$$A_r x = f.$$

Remark 1. For $s_r = s_{r+1}$ r-solution of (1) is not determined.

If x is any least-squared solution of (1), then r-solution is given by

$$x_r = P_r x, \quad P_r = U I_{M,r} U^*,$$

where $I_{M,r}$ is the diagonal matrix with its first r diagonal elements equal to 1 and all others equal to 0.

The mapping $f \to x_r$ is continuous and its numerical calculation is stable if *the condnumber of the matrix A*

$$\mu_r(A) = \frac{\sigma_1(A)}{\sigma_r(A)},$$

the parameter of the rupture in the singular spectrum

$$d_r(A) = \frac{\sigma_1(A)}{s_r(A) - s_{r+1}(A)},$$

and *the parameter of the non-coincidence*

$$\theta_r(A, f) = \frac{\|Ax_r - f\|}{s_r\|x_r\|}$$

are rather small [1]. This mapping defines the continuous r-pseudoinverse operator $A_r = UD_r^+V^* : R^N \to R^M$, where

$$D = \begin{cases} \begin{bmatrix} \Sigma_r \\ 0 \end{bmatrix} & \text{for } N < M, \\ \Sigma_r & \text{for } N = M, \\ [\Sigma_r : 0] & \text{for } N > M, \end{cases}$$

with

$$\Sigma_r = \begin{pmatrix} \sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ & & & \sigma_r^{-1} \end{pmatrix}$$

and the vector x_r (r-solution of (1)) may be presented as

$$x_r = A_r f.$$

The main goal of this paper is to generalize the notion of r-solution for an operator equation

$$Ax = f, \tag{3}$$

with a compact linear operator from the Hilbert space H_1 to the Hilbert space H_2 and to justify the advantage to use it to regularize (3).

2. The geometry of finite-dimensional subspaces in a Hilbert space

For any two N-dimensional subspaces X_1, X_2 of a separable Hilbert space H one can introduce the matrix

$$G = \left[(x_i^{(1)}, x_j^{(2)})\right]_{i,j=1}^{N},$$

where $\{x_i^{(1)}\}_{i=1}^N$, $\{x_i^{(2)}\}_{i=1}^N$ are the orthonormal bases in these subspaces. Any orthogonal transformation of these bases leads to the transformation of the matrix $G \to VGU$ by means of the orthogonal matrices V, U and one can choose them to come to the diagonal matrix

$$G = \Sigma = \mathrm{diag}(\sigma_j),$$

with $\sigma_j = (x_j^{(1)}, x_j^{(2)}) = \cos\phi_j$, where $\phi_j \geq 0$ – angles between $x_j^{(1)}$ and $x_j^{(2)}$ $((x_i^{(1)}, x_j^{(2)}) = 0$ for $i \neq j)$. Then, for any two vectors

$$x = \sum_{j=1}^{N} c_j x_j^{(1)} \in X_1, \qquad y = \sum_{j=1}^{N} c_j x_j^{(2)} \in X_2,$$

$$\frac{\|x-y\|}{\|x\|} \leq \sqrt{2(1-\sigma_1)} = \sqrt{2(1-\cos\phi_1)} = 2\sin\frac{\phi_1}{2}.$$

Hence, for orthogonal projectors Π_{X_1} and Π_{X_2} on X_1 and X_2

$$\Pi_{X_1} = \sum_{j=1}^{N} (\,\cdot\,, x_j^{(1)}) x_j^{(1)}, \qquad \Pi_{X_2} = \sum_{j=1}^{N} (\,\cdot\,, x_j^{(2)}) x_j^{(2)}$$

the equality $\|\Pi_{X_1} - \Pi_{X_2}\| = \sin\phi_1$ is valid.

Let $S : H \to H$ be a non-negative compact self-adjoint operator, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots$ – its eigenvalues and $\{u_1, u_2, \ldots, u_n, \ldots\}$ – its eigenvectors. This operator generates invariant finite dimensional subspaces

$$U_m = \left\{ \sum_{j=1}^{m} c_j u_j,\ \ c_j \in C \right\}$$

and orthogonal projectors $\Pi_m = \Pi_{U_m}$ onto these subspaces

$$\Pi_m = \sum_{j=1}^{m} P_j, \quad P_j = (\cdot, u_j) u_j.$$

Remark 2. Π_m is determined correctly only if $\lambda_m > \lambda_{m+1}$.

Let us suppose now that $\tilde{S}$ is some other nonnegative compact self-adjoint operator on the same Hilbert space (below we will write $\tilde{}$ for all objects connected with this operator) and

$$\|S - \tilde{S}\| \leq \delta \|S\|, \tag{4}$$

then [2]

$$|\lambda_j - \tilde{\lambda}_j| \leq \delta \|S\|,$$

and to provide correctness of the invariant subspace $\tilde{U}_m$ and its orthogonal projector $\Pi_{\tilde{U}_m} = \tilde{\Pi}_m$ it is necessary to claim

$$\lambda_{m+1} < \lambda_m - 2\delta \|S\|. \tag{5}$$

By means of dimensionless parameter (*the m-th relative parameter of the gap in the spectrum of the operator S*)

$$d_m = \begin{cases} \dfrac{\|S\|}{\lambda_m - \lambda_{m+1}} & \text{for } \lambda_m > \lambda_{m+1}, \\ \infty & \text{for } \lambda_m = \lambda_{m+1}, \end{cases}$$

inequality (5) is rewritten as $2\delta d_m < 1$.

The question is how close to each other are invariant subspaces of these operators?

To answer it one has to estimate the value $\|\Pi_m - \tilde{\Pi}_m\|$. But as

$$\Pi_m - \tilde{\Pi}_m = \Pi_m \tilde{\Pi}_m^{\perp} - \Pi_m^{\perp} \tilde{\Pi}_m,$$

where

$$\Pi_m^{\perp} = I - \Pi_m \quad \tilde{\Pi}_m^{\perp} = I - \tilde{\Pi}_m,$$

and for any $x \in H$

$$\begin{aligned} \|(\Pi_m - \tilde{\Pi}_m)x\|^2 &= \|\Pi_m \tilde{\Pi}_m^{\perp} x\|^2 + \|\Pi_m^{\perp} \tilde{\Pi}_m x\|^2, \\ \|\Pi_m \tilde{\Pi}_m^{\perp} x\| &\leq \|\Pi_m \tilde{\Pi}_m^{\perp}\| \cdot \|\tilde{\Pi}_m^{\perp} x\|, \\ \|\Pi_m^{\perp} \tilde{\Pi}_m x\| &\leq \|\Pi_m^{\perp} \tilde{\Pi}_m\| \cdot \|\tilde{\Pi}_m x\|, \end{aligned}$$

it is enough to estimate $\|\Pi_m \tilde{\Pi}_m^{\perp}\|$ and $\|\Pi_m^{\perp} \tilde{\Pi}_m\|$. To perform this let us denote that S and $\tilde{S}$ commutate with Π_m and $\tilde{\Pi}_m^{\perp}$ respectively, and $Y = \Pi_m \tilde{\Pi}_m^{\perp}$ satisfies the Sylvester equation

$$S_1Y - YS_2 = \Phi, \tag{6}$$

where $S_1 = \Pi_m S \Pi_m$, $S_2 = \tilde{\Pi}_m^{\perp} \tilde{S} \tilde{\Pi}_m^{\perp}$, $\Phi = \Pi_m(S - \tilde{S})\tilde{\Pi}_m^{\perp}$ ($\|\Phi\| \leq \delta\|S\|$).

Lemma 1. *For* $2d_m\delta < 1$ *equation* (6) *has the unique solution under relation* $\Pi_m Y = Y$. *For this solution the estimation*

$$\|Y\| \leq \frac{d_m\delta}{1 - d_m\delta}$$

is hold.

Proof. Let us choose ρ to provide $\rho\tilde{\lambda}_{m+1} < 1 < \rho\lambda_m$, then $\|\rho S_2\| < 1$ and suppose that $SY_1 = YS_2$ (i.e., Y is a non-trivial solution of (6)). Then for any n

$$(\rho S_1)^n Y = Y(\rho S_2)^n. \tag{7}$$

The right-hand side in (7) turns to zero for $n \to \infty$ as Y is bounded, which is possible only if $S_1Y = 0$, i.e., if $Y(H) \in \operatorname{Ker} S_1$. But, as $\operatorname{Ker} S_1 = \operatorname{Ker} \Pi_m$, that means that $\Pi_m Y = Y = 0$.

If $(\rho S_1)^+$ is m-pseudoinverse operator

$$(\rho S_1)^+ = \sum_{j=1}^{m} \frac{1}{\rho\lambda_j} P_j,$$

with the one-dimensional eigen-orthogonal projectors P_j and $\|(\rho S_1)^+\| = 1/\rho\lambda_m < 1$, then

$$((\rho S_1)^+)^k \to 0, \quad (\rho S_2)^k \to 0 \quad \text{for} \quad k \to \infty,$$

and the operator

$$Y = \sum_{k=0}^{\infty} \left((\rho S_1)^+\right)^{k+1} \rho\Phi(\rho S_2)^k$$

exists, satisfies equation (6), condition $\Pi_m Y = Y$, and is estimated by

$$\begin{aligned}\|Y\| &\leq \sum_{k=0}^{\infty} \frac{(\rho\tilde{\lambda}_{m+1})^k}{(\rho\lambda_m)^{k+1}} \rho\|S\|\delta = \frac{\delta}{\lambda_m} \sum_{k=0}^{\infty} \left(\frac{\tilde{\lambda}_{m+1}}{\lambda_m}\right)^k \|S\| = \frac{\delta\|S\|}{\lambda_m - \tilde{\lambda}_{m+1}} \\ &\leq \frac{\delta\|S\|}{\lambda_m - \lambda_{m+1} - \delta\|S\|} = \frac{\delta(\|S\|/(\lambda_m - \lambda_{m+1}))}{1 - \delta(\|S\|/(\lambda_m - \lambda_{m+1}))} = \frac{\delta d_m}{1 - \delta d_m}.\end{aligned}$$

To complete the proof of the lemma, let us note that $\|\Pi_m \tilde{\Pi}_m^{\perp}\|$ may be estimated by the same technique. □

Summarizing, we can formulate the theorem

Theorem 1. *Let $S : H \to H$ be a nonnegative self-adjoint compact operator, d_m - its m-th relative parameter of the gap in the spectrum, Π_m - its orthogonal projector onto invariant subspace generating by its first m eigenvectors. Next, let an operator $\tilde{S} : H \to H$ be also nonnegative self-adjoint compact and*

$$||S - \tilde{S}|| \le \delta ||S||,$$

with $d_m \delta < \frac{1}{2}$. Then the orthogonal projector $\tilde{\Pi}_m$ is determined correctly and the inequality

$$||\Pi_m - \tilde{\Pi}_m|| \le \frac{\delta d}{1 - \delta d}$$

is hold.

Let us consider now a compact operator $A : X \to Y$, where X and Y are the separable Hilbert spaces with singular values $s_1 \ge s_2 \ge \ldots \ge s_n \ldots \ge 0$ and right and left singular vectors $\{x_i\}$ and $\{y_i\}$:

$$Ax_i = s_i y_i, \quad A^* y_i = s_i x_i,$$

($A^* : Y \to X$ is adjoint operator for A) and introduce the Hilbert space $H = X \times Y$ (the Cartesian product of X and Y). Then the operator

$$\mathcal{A} = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} : H \to H$$

is compact and self-adjoint. One can generalize all previous consideration for this operator [3] and formulate the theorem

Theorem 2. *Let $A : X \to Y$ be a compact operator from the separable Hilbert space X in the separable Hilbert space Y, d_m - its m-th relative parameter of the gap in the spectrum, Π_m^X, Π_m^Y - its orthogonal projectors onto invariant subspaces for generating by its first m right and left singular vectors. If $\tilde{A} : X \to Y$ is any compact operator such that*

$$||A - \tilde{A}|| \le \delta ||A||,$$

then for $2\delta d_m < 1$ the orthogonal projectors $\tilde{\Pi}_m^X$, $\tilde{\Pi}_m^Y$ are determined correctly and

$$||\Pi_m^X - \tilde{\Pi}_m^X|| \le \frac{\delta d}{1 - \delta d}, \qquad ||\Pi_m^Y - \tilde{\Pi}_m^Y|| \le \frac{\delta d}{1 - \delta d}.$$

3. Generalized normal r-solution of the operator equation in a Hilbert space

Any compact operator $A : X \to Y$ from the Hilbert space X to the Hilbert space Y generates decomposition of the spaces X and Y ([4])

$$X = \operatorname{Ker} A \oplus \bar{R}(A^*), \quad Y = \operatorname{Ker} A^* \oplus \bar{R}(A), \tag{8}$$

(as A and its adjoint A^* are compact, their ranges $R(A)$ and $(R(A^*)$ may be non-closed).

On the basis of this decomposition it is possible to describe the structure of $R(A)$.

Lemma 2.

$$f \in R(A) \iff \sum_{s_i>0} \frac{(f, y_i)^2}{s_i^2} < \infty$$

and

$$(f, z) = 0 \quad \forall z \in \operatorname{Ker} A^* \tag{9}$$

(here y_i are left singular vectors of A).

Proof. $\forall f \in R(A)\, \exists x \in X$: $f = Ax$, but $\forall x \in X$

$$x = x^0 + \sum_{s_i>0} (x, x_i)x_i, \quad x^0 \in \operatorname{Ker} A,$$

(x_i are right singular vectors of A) and

$$f = Ax = \sum_{s_i>0} s_i(x, x_i)y_i = \sum_{s_i>0} (f, y_i)y_i,$$

but that means

$$\sum_{s_i>0} \frac{(f, y_i)^2}{s_i^2} = \|x - x_0\|^2 \leq \|x\|^2.$$

The equality (9) follows from (8).

If f is orthogonal to $\operatorname{Ker} A^*$, it is decomposed as $f = \sum (f, y_i)y_i$, where the sum performs with respect to left singular vectors with nonzero singular values only. But by the assumption the vector

$$x = \sum_{s_i>0} \frac{(f, y_i)}{s_i} x_i \in X,$$

and $Ax = f$. □

This lemma allows to determine the solution $x \in X$ of the equation $Ax = f$ for $f \in R(A)$ as

$$x = \sum_{s_i>0} \frac{(fi)}{s_i} x_i,$$

but it is not valid $\forall f \in Y$ as the relation

$$\sum_{s_i>0} \frac{(f, y_i)^2}{s_i^2} < \infty$$

may be not hold. Nevertheless, one can generalize it for f with nonzero projection onto $\operatorname{Ker} A^*$.

Lemma 3. *$\forall f \in R(A) \oplus \operatorname{Ker} A^*$ exists the unique vector $\bar{x} \in (\operatorname{Ker} A)^\perp$ such that*

$$\bar{x} = \arg\min_{x \in X} \|Ax - f\|^2$$

and

$$\bar{x} = \sum_{s_i>0} \frac{(f, y_i)}{s_i} x_i. \tag{10}$$

Proof. $\forall f \in Y$ $f = f^{(1)} + f^{(0)}$, $f^{(0)} \in R(A)^\perp$, $f^{(1)} \in \bar{R}(A)$, therefore,

$$\|Ax - f\|^2 = \|Ax - f^{(1)}\|^2 + \|f^{(0)}\|^2 \geq \|f^{(0)}\|^2 \quad \forall x \in X.$$

But, as we supposed $f^{(1)} \in R(A)$, there exists the unique $\bar{x} \in X$ such that $A\bar{x} = f^{(1)}$. □

Definition 2. The vector x determined by (10) is called generalized normal solution of the equation

$$Ax = f, \tag{11}$$

for $f \in R(A) \oplus R(A)^\perp$ [5].

The mapping $f \to \bar{x}$ produces the linear operator $A^+ : Y \to X$ with the domain $R(A) \oplus R(A)^\perp$ – pseudo-inverse operator with respect to A or generalized inverse ([6–8]). If $R(A)$ is not closed, this operator is not bounded. As one can easily check

$$A^+ y_j = \frac{1}{s_j} x_j, \quad A^+ y_i^0 = 0.$$

The domain of A^+ is dense in Y, so the operator $(A^+)^* : Y \to X$ exists and

$$(A^+)^* x_j = \frac{1}{s_j} y_j, \quad (A^+)^* x_k^0 = 0.$$

If

$$\Pi_X : X \to X, \qquad \Pi_Y : Y \to Y,$$
$$\Pi_X = \sum_{s_j>0} (\cdot, x_j)x_j, \qquad \Pi_Y = \sum_{s_j>0} (\cdot, y_j)y_j,$$

are the orthogonal projectors onto $\bar{R}(A^*)$ and $\bar{R}(A)$ respectievly, then

$$A^+A = \Pi_X, \quad AA^+ = \Pi_Y.$$

The second relation means that the operator Π_Y is the unique continuous extension of the operator AA^+ with dense domain in Y. If there is a gap within the singular values of the operator A, i.e., $\exists r : s_r > s_{r+1}$, then one can introduce the finite-dimensional operator

$$A_r = \sum_{j=1}^{r} s_j(\cdot, x_j)y_j.$$

Definition 3. The generalized normal r-solution (r-solution) of the equation (11) is the general normal solution of the equation

$$A_r x = f.$$

The operator

$$A_r^+ = \sum_{j=1}^{r} \frac{1}{s_j}(\cdot, y_j)x_j$$

is the r-pseudo inverse with respect to A.

Remark 3. A_r, A_r^+, r-solution etc. are not defined if $s_r = s_{r+1}$.

The r-solution and the r-pseudoinverse operator are continuous with respect to the right-hand side f of the equation and the operator itself. Really, let us consider two operator equation

$$Ax_r^{(1)} = f^{(1)}, \quad Ax_r^{(2)} = f^{(2)},$$

with the same operator. Then the r-solutions of these equations are:

$$x_r^{(1)} = \sum_{j=1}^{r} \frac{(f^{(1)}, y_j)}{s_j} x_j, \quad x_r^{(2)} = \sum_{j=1}^{r} \frac{(f^{(2)}, y_j)}{s_j} x_j,$$

and

$$\|x_r^{(1)} - x_r^{(2)}\| \le \frac{1}{s_r}\Big(\sum_{j=1}^{r}(f^{(1)} - f^{(2)}, y_j)^2\Big)^{1/2} \le \|A_r^+\|\,\|f^{(1)} - f^{(2)}\|.$$

To get the low estimation of $\|x_r\|$ let us introduce *the parameter of unconsistency* of an operator equation (for a system of linear algebraic equation such parameter was defined in [1])

$$\theta_r(A,f) = \frac{1}{s_r}\cdot\frac{\big(\sum_{j=r+1}^{\infty}(f,y_j)^2 + \sum(f,y_i^0)^2\big)^{1/2}}{\big(\sum_{j=1}^{r}\frac{(f,y_j)^2}{s_j^2}\big)^{1/2}}.$$

Then, we have

$$\begin{aligned}\|x_r^{(1)}\|^2 &= \sum_{j=1}^{r}\frac{(f^{(1)},y_j)^2}{s_j^2} \ge \frac{1}{s_r^2}\sum_{j=1}^{r}(f^{(1)},y_j)^2\\ &= \frac{1}{s_r^2}\Big(\|f\|^2 - \sum_{j=r+1}^{\infty}(f^{(1)},y_j)^2 - \sum(f,y_k^0)^2\Big)\\ &= \frac{1}{s_r^2}(\|f\|^2 - \theta^2 s_r^2\|x_r^{(1)}\|^2),\end{aligned}$$

from it follows that

$$\|x_r^{(1)}\| \ge \frac{\|f\|}{\|A\|\big(1+\frac{\theta_r^2(A,f)}{\mu_r^2}\big)^{1/2}},$$

and the relative error of the r-solution with respect to a perturbation of the right-hand side is given by

$$\begin{aligned}\frac{\|x_r^{(1)} - x_r^{(2)}\|}{\|x_r^{(1)}\|} &\le \|A\|\,\|A_r^+\|\Big(1+\frac{\theta_r^2(A,f)}{\mu_r^2}\Big)^{1/2}\frac{\|f^{(1)} - f^{(2)}\|}{\|f^{(1)}\|}\\ &= (\theta_r^2(A,f)+\mu_r^2)^{1/2}\frac{\|f^{(1)} - f^{(2)}\|}{\|f^{(1)}\|}.\end{aligned}$$

Now let us consider two equations:

$$\begin{aligned}&Ax = f, \quad \tilde{A}\tilde{x} = \tilde{f},\\ &\|A-\tilde{A}\| \le \delta\|A\|, \quad \|f-\tilde{f}\| \le \varepsilon\|\tilde{f}\|,\end{aligned} \tag{12}$$

i.e., there is a perturbation not of the right-hand side only, but of the operator itself too. Then the difference of two r-solutions is

$$x_r - \tilde{x}_r = A_r^+ f - \tilde{A}_r^+ \tilde{f} = (A_r^+ - \tilde{A}_r^+) f + \tilde{A}_r^+ (f - \tilde{f}). \qquad (13)$$

It is necessary to suppose again that $2d_r\delta < 1$, because otherwise one cannot provide the existence of the r-pseudoinverse operator $\tilde{A}_r^+$ for any $\tilde{A}$ satisfying (12).

Let us introduce the biorthonormal bases

$$\begin{aligned}
&\xi_1, \xi_2, \ldots, \xi_r \ - \text{ the basis in } (\operatorname{Ker} A_r)^\perp,\\
&\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_r \ - \text{ the basis in } (\operatorname{Ker} \tilde{A}_r)^\perp,\\
&\eta_1, \eta_2, \ldots, \eta_r \ - \text{ the basis in } (\operatorname{Ker} A_r^*)^\perp,\\
&\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r \ - \text{ the basis in } (\operatorname{Ker} \tilde{A}_r^*)^\perp,
\end{aligned}$$

where

$$\begin{aligned}
&(\eta_i, \tilde{\eta}_j) = 0, \quad (\xi_i, \tilde{\xi}_j) = 0 \quad \text{for } i \neq j,\\
&(\eta_i, \tilde{\eta}_i) = \delta_i, \quad (\xi_i, \tilde{\xi}_i) = \sigma_i,
\end{aligned}$$

and

$$\sigma_i \geq \sigma_1 = \cos\phi_1 = (1 - \sin^2\phi_1)^{1/2} \geq \Big(1 - \frac{d_r^2\delta^2}{(1 - d_r\delta)^2}\Big)^{1/2} = \frac{\sqrt{1 - 2d_r\delta}}{1 - d_r\delta},$$

$$\delta_i \geq \frac{\sqrt{1 - 2d_r\delta}}{1 - d_r\delta}.$$

Then r-solutions x_r, $\tilde{x}_r$ may be presented as

$$x_r = \sum_{j=1}^{r} c_j \xi_j, \qquad \tilde{x}_r = \sum_{j=1}^{r} \tilde{c}_j \tilde{\xi}_j$$

and, taking into account that $A : (\operatorname{Ker} A_r)^\perp \to (\operatorname{Ker} A_r^*)^\perp$, and

$$A\xi_j = \sum_{j=1}^{r} a_{ij} \eta_i, \qquad \tilde{A}\tilde{\xi}_j = \sum_{j=1}^{r} \tilde{a}_{ij} \tilde{\eta}_i$$

let us rewrite (12) as two systems of linear algebraic equations:

$$\sum_{j=1}^{r} a_{ij} c_j = d_i, \quad \sum_{j=1}^{r} \tilde{a}_{ij} \tilde{c}_j = \tilde{d}_i, \quad i = 1, \ldots, r,$$

with $d_i = (f, \eta_i)$, $\tilde{d}_i = (\tilde{f}, \tilde{\eta}_i)$, or, in the matrix form

$$\mathcal{A}c = d, \quad \tilde{\mathcal{A}}\tilde{c} = \tilde{d}. \tag{14}$$

Now let us estimate the right-hand sides in (14)

$$\|d - \tilde{d}\|^2 = \sum_{j=1}^{r}(d_j - \tilde{d}_j)^2 = \Big\|\sum_{j=1}^{r} d_j\eta_j - \sum_{j=1}^{r}\tilde{d}_j\tilde{\eta}_j + \sum_{j=1}^{r}\tilde{d}_j(\eta_j - \tilde{\eta}_j)\Big\|^2.$$

But, as the bases $\{\eta_i\}$ and $\{\tilde{\eta}_i\}$ are biorthonormal:

$$\begin{aligned}\|d - \tilde{d}\| &\le \|\Pi_r f - \tilde{\Pi}_r\tilde{f}\| + \Big(\sum_{i=1}^{r} d_i^2\Big)^{1/2}\max_j\|\eta_j - \tilde{\eta}_j\| \\ &\le \|(\Pi_r - \tilde{\Pi}_r)f\| + \|\tilde{\Pi}_r(f - \tilde{f})\| + \max_j\|\eta_j - \tilde{\eta}_j\|\,\|\tilde{\Pi}_r\tilde{f}\|,\end{aligned}$$

where Π_r ($\tilde{\Pi}_r$) is the orthogonal projector onto $(\operatorname{Ker} A_r^*)^\perp$ $((\operatorname{Ker}\tilde{A}^*)^\perp)$,

$$\|\eta_j - \tilde{\eta}_j\|^2 = 2(1 - \delta_j) \le 2(1 - \cos\phi_1) = 2\sin^2\phi_1/2,$$

therefore,

$$\|d - \tilde{d}\| \le \Big(\frac{d_r\delta}{1 - d_r\delta} + \epsilon + \beta\delta\Big)\|f\|, \tag{15}$$

with

$$\beta = \frac{\sqrt{2}d_r}{\sqrt{1 - d_r\delta}\cdot\sqrt{1 - d_r\delta\sqrt{1 - 2d_r\delta}}}.$$

To estimate $\|\mathcal{A} - \tilde{\mathcal{A}}\|$ let us consider how $\tilde{A}$ acts on the vector $p = \sum_{j=1}^{r} p_j\tilde{\xi}_j$

$$\tilde{A}\sum_{j=1}^{r} p_j\tilde{\xi}_j = A\sum_{j=1}^{r} p_j\xi_j + (\tilde{A} - A)\sum_{j=1}^{r} p_j\xi_j + \tilde{A}\sum_{j=1}^{r} p_j(\tilde{\xi}_j - \xi_j).$$

Thus,

$$\Big\{\sum_{k=1}^{r}\Big(\sum_{j=1}^{r}(\tilde{a}_{jk} - a_{jk})p_j\Big)^2\Big\}^{1/2} \le \|\tilde{A} - A\|\,\|p\| + \|A\|\,\|p\| \times \Big(\max_j\|\tilde{\xi}_j - \xi_j\| + \max_j\|\tilde{\eta}_j - \eta_j\|\Big).$$

And, taking into account the special choice of the bases $\{\xi_j\}$, $\{\tilde{\xi}_j\}$ and $\{\eta_j\}$, $\{\tilde{\eta}_j\}$, we get the estimation

$$\|\mathcal{A} - \tilde{\mathcal{A}}\| \le (\delta + 2\beta\delta)\,\|A\|.$$

The singular values of the matrices $\mathcal{A}$ and $\tilde{\mathcal{A}}$ coincide with the first r singular values of the operators A and $\tilde{A}$ respectievly, i.e., $\|\mathcal{A}\| = \|A\|$, $\|\tilde{\mathcal{A}}\| = \|\tilde{A}\|$ and the last estimation may be rewritten as

$$\|\mathcal{A} - \tilde{\mathcal{A}}\| \le \rho\|\mathcal{A}\|, \quad \text{where} \quad \rho = \delta(1 + 2\beta).$$

To rewrite the estimation (15) in the relative form, one can use the parameter of the inconsistency $\theta_r(A, f)$ again:

$$\begin{aligned}\|f\|^2 &= \|d\|^2 + \|Ax_r - f\|^2 = \|d\|^2 + \theta^2 s_r^2\|x\|^2 \\ &= \|d\|^2 + \theta^2 s_r^2\|\mathcal{A}^{-1}d\|^2 \le (1 + \theta^2)\|d\|^2.\end{aligned}$$

Therefore,

$$\|d - \tilde{d}\| \le \tau\|d\|$$

with

$$\tau = \sqrt{1+\theta^2}\left[\epsilon + \left(\frac{d_r\delta}{1 - d_r\delta} + \frac{\sqrt{2}d_r\delta}{\sqrt{1 - d_r\delta}\cdot\sqrt{1 - d_r\delta\sqrt{1 - 2d_r\delta}}}\right)\right].$$

If the condnumber of the matrix $\mathcal{A}$ is not rather big, i.e.,

$$\rho\mu(\mathcal{A}) = \rho\mu_r(A) = \frac{s_1}{s_r}\rho \le 1,$$

then

$$\frac{\|c - \tilde{c}\|}{\|c\|} \le \mu(\mathcal{A})\frac{\rho + \tau}{1 - \mu(\mathcal{A})\rho},$$

and, taking into account that

$$\Big\|\sum_j \tilde{c}_j(\xi_j - \tilde{\xi}_j)\Big\| \le \max_j \|\xi_j - \tilde{\xi}_j\|\,\|\tilde{x}_r\| \le \beta\delta\|\tilde{x}_r\|,$$

$$\|c\| = \Big(\sum c_j^2\Big)^{1/2} = \|x_r\|,$$

we come to the final estimation

$$\begin{aligned}\|x_r - \tilde{x}_r\| &\le \mu(\mathcal{A})\frac{\rho + \tau}{1 - \mu(\mathcal{A})\rho}\|x_r\| + \beta\delta\|\tilde{x}_r\| \\ &\le \mu_r(A)\frac{(\rho + \tau)(1 + \beta\delta)}{1 - \mu_r(A)\rho}\|x_r\|.\end{aligned}$$

Summarizing, we can formulate the theorem

Theorem 3. *If $Ax = f$, $\tilde{A}\tilde{x} = \tilde{f}$ are equations with compact operators from the Hilbert space X to the Hilbert space Y and are closed to each other*

$$\|A - \tilde{A}\| \leq \delta\|A\|, \quad \|f - \tilde{f}\| \leq \epsilon\|f\|,$$

and there is a rupture in the spectrum of the operator A

$$2d_r\delta < 1, \tag{16}$$

then the generalized normal r-solutions x_r and $\tilde{x}_r$ of these equations exist and

$$\frac{\|x_r - \tilde{x}_r\|}{\|x_r\|} \leq \mu_r(A)\frac{(\rho+\tau)(1+\beta\delta)}{1-\mu_r(A)\rho}, \tag{17}$$

under the constraint

$$\rho\mu_r(A) < 1. \tag{18}$$

Here

$$\begin{aligned}
\theta &= \theta_r(A,f) = \frac{\|Ax_r - f\|}{s_r\|x_r\|}, \quad \mu_r(A) = \frac{s_1}{s_r}, \\
\beta &= \frac{\sqrt{2}d_r}{\sqrt{1-d_r\delta}\sqrt{1-d_r\delta\sqrt{1-2d_r\delta}}}, \quad \rho = \delta(1+2\beta), \\
\tau &= \sqrt{1+\theta^2}\Big[\epsilon + \Big(\frac{d_r\delta}{1-d_r\delta} + \delta\beta\Big)\Big].
\end{aligned}$$

Remark 4. The inequalities (16) and (18) depend from each other. Usually the first one is more stiff.

Remark 5. We do not pretend these estimations to be optimal, but would like to note that they are similar with the same for linear algebraic systems ([1]) and, so, are not very rough at least.

Remark 6. The technique we applied here to get the estimation (17) is different with respect to the same from [1], as to realize the last it would be necessary to attract the singular value decomposition not for the compact operator only, but for the bounded (with the bounded inverse) ones also. We suppose, one can prefer one technique to another only by means of comparison of the final estimations.

Remark 7. The estimations (17) are rather far from the estimations with "guaranted accuracy", as

- the algorithm how to calculate r-solution is not pointed out;
- the possibility to simulated all possible errors by means of equivalent perturbations of input data for the systems of linear algebraic equations is not checked.

But, nevertheless, if the way how to approximate an operator equation by means of a system of linear algebraic equations is chosen, one has to estimate the accuracy of this approximation and to attract next (17) to estimate the accuracy of r-solution.

4. Regularization on the basis of r-solution

Theorem 4. *$\forall f \in R(A) \oplus \mathrm{Ker}(A^*)$ there exists a sequence $\{r_j\}$ ($r_j \to \infty$ for $j \to \infty$), such that $\|A^+_{r_j} f - A^+ f\|_X \to 0$ for $j \to \infty$.*

Proof. As

$$A^+_r = A^+_r \Pi_Y, \quad A^+ = A^+ \Pi_Y,$$

it is enough to prove this theorem for $f \in R(A)$. But then one can apply Lemma 3 and the property of the compact operator (there exists a finite quantity of the linear independent singular vectors with the same singular value) which provides the existence of the sequence of integers $\{r_j\}$ for which operator $A^+_{r_j}$ is determined. □

Corollary. *The linear operator A^+ is regularized on $R(A) \oplus \mathrm{Ker}\, A^*$ by means of the sequence of the operators $\{A_r\}$.*

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