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Timed transition systems with independence and timed event structures: an adjunction*

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Introduction

The behaviour of concurrent systems is often specified extensionally by describing their "state-transitions" and the observable behaviours that such transitions produce. The simplest formal model of computation able to express naturally this idea is that of labelled transition systems, where the labels on the transitions represent the observable part of its behaviour.

However, transition systems are an interleaving model of concurrency, which means that they do not allow one to draw a natural distinction between interleaved and concurrent executions of actions. Although for many applications this level of abstraction is appropriate, for several other kinds of analysis a model may be desirable that takes full account of concurrency. Two most popular extensions of transition systems aiming to overcome this limitation and capture the notion of concurrent execution in a precise way are asynchronous transition systems introduced independently by Bednarczyk [2] and Shields [10] and transitions systems with independence proposed by Winskel and Nielsen [11]. These two approaches are based on the simplest idea: endow transition systems with some formal notion of "similarity" of transitions that enables us to distinguish whether or not two transitions represent the same action. Intuitively, this is achieved in both approaches by thinking of transitions as occurrences of events: two transitions represent the same event if they correspond to the same action. However, while in asynchronous transition systems events present explicitly, as part of model specification, in transition systems with independence they are derived from the structure of transitions, upon which the independence relation is directly layered. Hildenbrandt and Sassone have shown that models above are closely related [6] with transition systems with independence being less expressive. In this paper we focus on them due to their simplicity.

Category theory has been used to structure the seemingly confusing world of models for concurrency. Within this framework, objects of categories represent processes and morphisms correspond to behavioural relations between the processes, i.e. to simulations. This approach allows for natural formalization of the fact that one model is more expressive than another in terms

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of an “embedding”, most often taking the form of a coreflection, i.e. an adjunction in which the unit is an isomorphism.

It is generally acknowledged that time plays an important role in many concurrent and distributed systems. This has motivated the lifting of the theory of untimed systems to real-time setting. While timed extensions of interleaving models have been studied thoroughly in last decade, “true concurrent” models received scant attention. For instance, there are several papers dealing with timed variants of transition systems [5, 1], but no timed extensions of “true-concurrent” variants of transition systems are known.

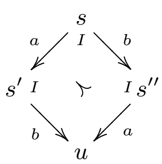
The intention of this paper is to introduce a timed extension of transition systems with independence, and to study their behaviour within the categorical framework. In particular, we show that timed transition systems with independence bear close relationships with timed event structures via a chain of coreflections between categories of the models.

The paper is organized as follows. The first section concerns a timed extension of transition systems with independence and related notions. The next section presents unfolding of timed transition systems and its categorical characterization. In the third section, we show the existence of coreflection between unfolded timed transition systems with independence and timed event structures. In conclusion, we give a short summary of the obtained relations and hints regarding our future work.

1. Timed transition systems with independence

In this section, we describe the basic notions and notations concerning transition systems with independence and define their timed extension. We start with untimed case.

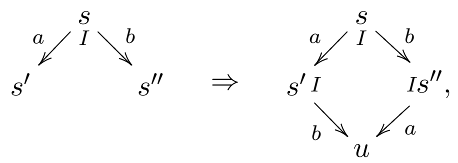
Definition 1. A *transition system with independence* is a tuple $TI = (S, s^I, L, Tran, I)$, where S is a countable set of *states*, $s^I \in S$ is an *initial state*, L is a countable set of *labels*, $Tran \subseteq S \times L \times S$ is a set of *transitions*, and $I \subseteq Tran \times Tran$ is an irreflexive, symmetric *independence relation* on transitions, such that, using $\prec$ to denote the following relation on transitions



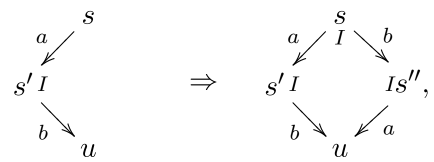
$$\begin{aligned}(s, a, s') \prec (s'', a, u) \iff\ & \exists (s, b, s''), (s', b, u) \in Tran\ . \\ & (s, a, s')\ I\ (s, b, s'') \wedge \\ & (s, a, s')\ I\ (s', b, u) \wedge \\ & (s, b, s'')\ I\ (s'', a, u),\end{aligned}$$

and $\sim$ for the least equivalence relation containing $\prec$, we have

1. $(s, a, s') \sim (s, a, s'') \Rightarrow s = s''$,
2. $(s, a, s')\ I\ (s, b, s'') \Rightarrow \exists (s', b, u), (s'', a, u) \in Tran\ .$
   $(s, a, s')\ I\ (s', b, u) \wedge (s, b, s'')\ I\ (s'', a, u)$,



3. $(s, a, s')\ I\ (s', b, u) \Rightarrow \exists (s, b, s''), (s'', a, u) \in Tran\ .$
   $(s, a, s')\ I\ (s, b, s'') \wedge (s, b, s'')\ I\ (s'', a, u)$,



4. $(s, a, s') \sim (s'', a, u)\ I\ (w, b, w') \Rightarrow (s, a, s')\ I\ (w, b, w')$.

Therefore, transition systems with independence are precisely standard transition systems but with an additional relation expressing when one transition is independent of another. Let

$$\begin{gathered} \mathrm{Diam}_{a,b}(s, s', s'', u) \\ \Longleftrightarrow \\ \exists (s, a, s'), (s, b, s''), (s', b, u), (s'', a, u) \in Tran\ . \\ (s, a, s')\ I\ (s, b, s'') \wedge (s, a, s')\ I\ (s', b, u) \wedge \\ (s, b, s'')\ I\ (s'', a, u). \end{gathered}$$

We say that the transitions above form an *independence diamond*, and denote the $\sim$-equivalence class of a transition $t = (s, a, s')$ as $[t]$.

A transition system functions by performing transitions from one state to another. A *computation* is a sequence of transitions $\pi = t_1, \ldots, t_n$ such that for each $i = 1, \ldots, n$ it holds that $t_i = (s_{i-1}, a_i, s_i)$ and $s_0 = s^I$. We use $|\pi|$ to denote the length of π, $\mathrm{cod}(\pi) = s_n$ to denote its endpoint, and ϵ to denote the *empty* computation. Let $\mathrm{Comp}(TI)$ be the set of computations of TI.

For a computation π and a transition t, define $\mathcal{N}(\pi, [t]) = |\{t' \in \pi \mid t' \in [t]\}|$. A transition $t = (s, a, s')$ is said to be *reachable*, if there exists a computation $\pi \in \mathrm{Comp}(TI)$ such that t appears in π. From now on, we consider only those transition systems with independence in which all transitions are reachable.

Let $\simeq \subseteq \mathrm{Comp}(TI)$ be the least equivalence relation such that

$$\pi_s(s,a,s')(s',b,u)\pi_v \simeq \pi_s(s,b,s'')(s'',a,u)\pi_v$$
$$\Longleftrightarrow$$
$$\mathrm{Diam}_{a,b}(s,s',s'',u).$$

$[\pi]$ stands for the $\simeq$-equivalence class of a computation π.

Before defining morphisms of transition systems with independence, consider the following auxiliary notations. A partial mapping from a set A into a set B will be denoted as $\theta : A \to^* B$. Let $\mathrm{dom}\,\theta = \{a \in A \mid \theta(a) \text{ is defined}\}$. For a subset $A' \subseteq A$, define $\theta A' = \{\theta(a') \mid a' \in A' \cap \mathrm{dom}\,\theta\}$.

Let $TI = (S, s^I, L, Tran, I)$ and $TI' = (S, s^{I'}, L', Tran', I')$ be transition systems with independence. A *morphism* $h : TS \to TS'$ is a pair $h = (\sigma, \lambda)$ of mappings $\sigma : S \to S'$ and $\lambda : L \to^* L'$ such that for any computation $\pi \in \mathrm{Comp}(TI)$ the following holds:

1. $h(\pi) \in \mathrm{Comp}(TI')$ and $\mathrm{cod}(h(\pi)) = \sigma(\mathrm{cod}(\pi))$, where $h(\pi)$ is defined for any $\pi \in \mathrm{Comp}(TI)$ by induction:
   - $h(\epsilon) = \epsilon$;
   - $h(\pi(s,a,s')) = \begin{cases} h(\pi)(\sigma(s),\lambda(a),\sigma(s')), & \text{if } a \in \mathrm{dom}\,\lambda, \\ h(\pi), & \text{otherwise;} \end{cases}$
2. if $\pi' \in \mathrm{Comp}(TI)$ and $\pi \simeq \pi'$, then $h(\pi) \simeq h(\pi')$.

Note that the above definition of morphisms is different from but equivalent to the corresponding definition given in [9].

In the model of timed transition systems with independence under study, it is assumed that time is given by non-negative integers and that there is a global clock that is set to zero at the beginning of system's functioning. Given a transition system with independence, we equip each its transition with a sequence of integer values modelling delays w.r.t. the global clock. Informally speaking, when a transition is enabled for the n-th time, it cannot be executed before the time moment recorded in the n-th item of its sequence of delays. An execution itself is instantaneous, i.e. takes no time. Also, we do not force transitions to execute once their time constraints are met. This is important for modeling the situations where a transition must interact with the external environment, which may result in some time delays, or where the delay of the previous transitions turned out greater than that current delay associated with the transition itself. The advantage of such an approach is that the introduced time characteristics do not violate causality, since the transition is still possible even after its delay time has expired. Thus, the proposed timed extension admits the specification of "true concurrency" and minimal time delays. We do not introduce an explicit notion of passage of time. Instead, similarly to untimed case, we describe the behaviour of a timed transition system with independence in terms of sequences of transitions between its states, but we also record the moment of time at which each transition was executed.

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{N}^+$ be the set of all finite and infinite sequences of elements of $\mathbb{N}$. The elements of $\mathbb{N}^+$ are denoted as $\delta = \langle \delta(1), \ldots, \delta(n), \ldots \rangle$ and the length of a sequence δ is denoted as $|\delta|$.

Definition 2. A *timed transition system with independence* is a tuple $TTI = (S, s^I, L, Tran, I, \delta)$, where $[\![TTI]\!] = (S, s^I, L, Tran, I)$ is an underlying transition system with independence, and $\delta : Tran \to \mathbb{N}^+$ is a *delay function* that agrees with $\sim$, i.e $\delta(t) = \delta(t')$ for any $t, t' \in Tran$ such that $t \sim t'$.

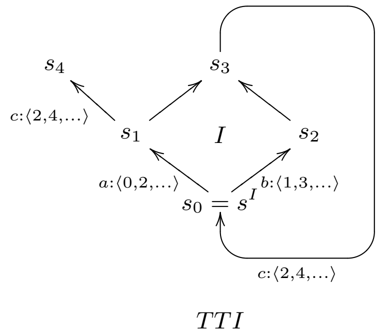


Figure 1

Example 1. Figure 1 shows a timed transition system with independence. We use $s_0, \ldots, s_4$ to depict the states and the arrows with labels to show transitions between the states. In addition to the labels, the arrows denoting the transitions also carry the associated delays. Note that for the sake of brevity the labels and delays are omitted for all but single representative of each class of $\sim$-equivalent transitions. The symbol I marks the diamond of independence.

A *timed computation* is a sequence $\Pi = (t_1, d_1), \ldots, (t_n, d_n)$ $(n \geq 0)$ such that for all $1 \leq i, j \leq n$ we have:

1. $[\![\Pi]\!] = t_1, \ldots, t_n \in \mathrm{Comp}([\![TTI]\!])$,
2. $i \leq j \Rightarrow d_i \leq d_j$,
3. $k_i \leq |\delta(t_i)|$, where $k_i = \mathcal{N}([\![\Pi_i]\!], [t_i])$ and $\Pi_i = (t_1, d_1) \ldots (t_i, d_i)$,
4. $\delta(t_i)(k_i) \leq d_i$.

The rationale behind the definition above is simple. Item 1 guarantees that a timed computation is well-defined from the untimed point of view. Item 2 asserts that the recorded time moments agree with the order of transitions. Items 3 and 4 guarantee that the number of times a transition is executed is limited by the number of the delays associated with it (recall that $\sim$-equivalent transitions have equal sequences of delays), and that the i-th occurrence of the transition must respect the i-th delay.

We will denote the empty timed computation as ϵ and the set of all timed computations of TTI as $\mathrm{TComp}(TTI)$. Let $\Pi \simeq \Pi' \overset{def}{\Longleftrightarrow} [\![\Pi]\!] \simeq [\![\Pi']\!]$. It is easy to see that $\simeq$ is an equivalence relation; the $\simeq$-class of a timed computation Π is denoted as $[\Pi]$.

Example 2. Consider the timed transition system with independence TTI from example 1. The sequences $(a,0),(b,1),(c,2),(a,2),(b,3)$ and $(a,4)$, $(b,4),(c,4),(a,4),(b,4)$[1] are timed computations of the TTI.

Now we are ready to introduce morphisms of timed transition systems with independence. Let $TTI = (S, s^I, L, Tran, I, \delta)$ and $TTI' = (S', s'^I, L'$, $Tran', I', \delta')$ be timed transition systems with independence. A *morphism* $h : TTI \to TTI'$ is a pair of mappings $h = (\sigma : S \to S', \lambda : L \to^* L')$ such that for any $\Pi \in \mathrm{TComp}(TTI)$ the following is true:

1. $h(\Pi) \in \mathrm{TComp}(TTI')$ and $\mathrm{cod}(h(\Pi)) = \sigma(\mathrm{cod}(\Pi))$, where $h(\Pi)$ is inductively defined by

$$
\begin{aligned}
&h(\epsilon) = \epsilon;\\
&h\big(\Pi((s,a,s'),d)\big) =\\
&\qquad\qquad = \begin{cases} h(\Pi)\big((\sigma(s),\lambda(a),\sigma(s')),d\big), & \text{if } a \in \mathrm{dom}\,\lambda \\ h(\Pi), & \text{otherwise.} \end{cases}
\end{aligned}
$$

2. if $\Pi' \in \mathrm{TComp}(TTI)$ and $\Pi \simeq \Pi'$, then $h(\Pi) \simeq h(\Pi')$.

It is easy to see that morphisms of timed transition systems with independence are essentially morphisms of underlying (untimed) transition systems with independence that respect timing constraints.

Timed transition systems with independence with morphisms between them form a category **TTSI** with unit morphisms $\mathbf{1}_{TTI} = (\mathbf{1}_S, \mathbf{1}_L) : TTI \to TTI$ for any $TTI = (S, s^I, L, Tran, I, \delta)$.

We end this section with the following example.

[1] We used only labels instead of full transition specifications for brevity.

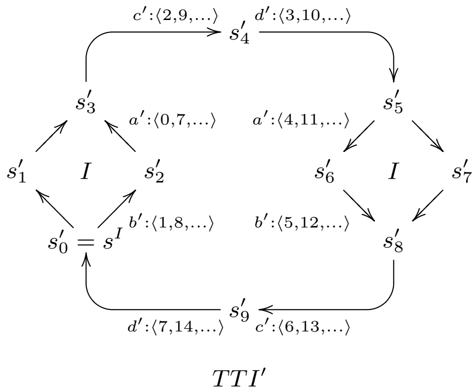


Figure 2

Example 3. Consider timed transition systems with independence TTI and TTI' depicted in figures 1 and 2, respectively. It is easy to see that a pair of mappings (σ, λ) defined by

$$\begin{aligned}
&\sigma(s'_0) = \sigma(s'_5) = s_0,\\
&\sigma(s'_1) = \sigma(s'_6) = s_1,\\
&\sigma(s'_2) = \sigma(s'_7) = s_2,\\
&\sigma(s'_3) = \sigma(s'_8) = s_3,\\
&\sigma(s'_4) = \sigma(s'_9) = s_0,\\
&\lambda(a') = a,\\
&\lambda(b') = b,\\
&\lambda(c') = c,\\
&\lambda(d') \text{ undefined,}
\end{aligned}$$

is a morphism $(\sigma, \lambda) : TTI' \to TTI$.

2. Unfolding of timed transition systems with independence

The aim of this section is to study unfolding of timed transition systems with independence. To that end, we first define a subclass of timed transition systems with independence that serves as a target of unfolding. After that, we construct an unfolding's mapping and show that together with the inclusion functor it defines a coreflection.

A *timed occurrence transition system with independence* is an acyclic timed transition system with independence $OTTI = (S, s_0, L, Tran, I, \delta)$ such that $(s'', a, u) \neq (s', b, u) \in Tran \Rightarrow \exists s \in S \,.\, \mathrm{Diam}_{a,b}(s, s', s'', u)$ and $|\delta(t)| = 1$ for all $t \in Tran$. Let **oTTSI** $\subset$ **TTSI** be the full subcategory[2] of timed occurrence transition systems with independence.

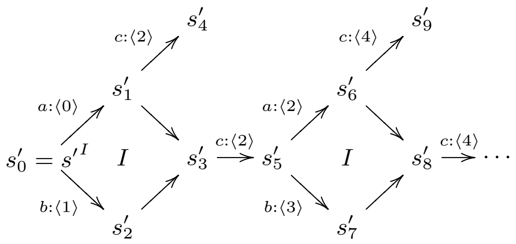
Define a mapping *ttsi.ottsi* : **TTSI** $\to$ **oTTSI** as follows. For each timed transition system with independence $TTI = (S, s^I, L, Tran, I, \delta)$, $ttsi.ottsi(TTI) = (S_{\simeq}, [\epsilon], L, Tran_{\simeq}, I_{\simeq}, \delta_{\simeq})$, where

1. $S_{\simeq} = \{[\Pi] \mid \Pi \in \mathrm{TComp}(TTI)\}$;
2. $([\Pi], a, [\Pi']) \in Tran_{\simeq} \iff \exists t = (s, a, s') \in Tran, \exists d \in \mathbb{N} \,.\, \Pi' \simeq \Pi(t, d)$,

[2]Recall that a subcategory **B** $\subset$ **A** is *full*, if **B** contains exactly same the morphisms between any two objects $X, Y \in$ **B** as the category **A**.

3. $([\Pi], a, [\Pi(t,d)]) I_{\simeq} ([\bar{\Pi}], b, [\bar{\Pi}(\bar{t}, \bar{d})]) \iff t I \bar{t}$;
4. $\delta_{\simeq}([\Pi], a, [\Pi(t,d)]) = \langle \delta(t)(k) \rangle$ with $k = \mathcal{N}(\Pi(t,d), [t])$.

It is easy to see that *ttsi.ottsi* maps timed transition systems with independence to timed occurrence systems with independence.



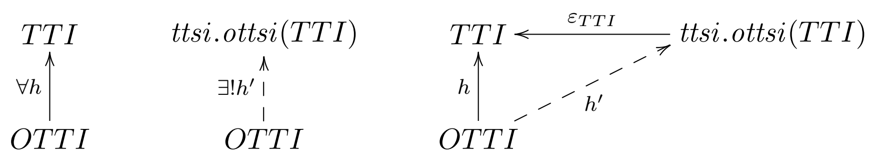
$OTTI'$

Figure 3

Example 4. Figure 3 depicts the unfolding of the timed transition system with independence shown in Figure 1.

In order to demonstrate that the mapping *ttsi.ottsi* is adjoint to the inclusion functor **oTTSI** $\hookrightarrow$ **TTSI**, we define the following morphism and prove that it is the unit of this adjunction. For a transition system with independence TTI, let $\varepsilon_{TTI} = (\sigma_\varepsilon, 1_L) : ttsi.ottsi(TTI) \to TTI$, where $\sigma_\varepsilon([\Pi]) = \mathrm{cod}(\Pi)$ for any $[\Pi] \in Tran_{ttsi.ottsi(TTI)}$. It is easy to see that ε_{TTI} is a morphism in **TTSI** for any TTI.

Lemma 1 (ε_{TTI} is universal). For any timed transition system with independence TTI, any timed occurrence transition system with independence $OTTI$ and any morphism $h : OTTI \to TTI$ there exists a unique $h' : OTTI \to ttsi.ottsi(TTI)$ such that $h = \varepsilon_{TTI} \circ h'$:



Proof. It is easy to see that if $h = (\sigma, \lambda)$, then h' is of the form $h' = (\bar{\sigma}, \lambda)$. Define $\bar{\sigma}(s) = [h(\Pi_s)]$, where $\Pi_s \in \mathrm{TComp}(OTTI)$ is such that $\mathrm{cod}(\Pi_s) = s$. This definition is correct because if $\Pi, \Pi' \in \mathrm{TComp}(OTTI)$ and $\mathrm{cod}(\Pi) = \mathrm{cod}(\Pi')$, then $\Pi \simeq \Pi'$ (see lemma 4.2 [9]), and because h preserves the $\simeq$-equivalence, according to the definition of morphisms in **TTSI**.

Next, we show that h' is a morphism in **TTSI**.

1. Let $\Pi \in \text{TComp}(OTTI)$. We proceed by induction on the length of Π. If $\Pi = \epsilon$, the result is trivial. Consider a timed computation $\Pi = \Pi'((s, a, s'), d)$. By the induction hypothesis, we have $h'(\Pi') \in \text{TComp}(TTI)$ and $\bar{\sigma}(\text{cod}(\Pi')) = \text{cod}(h'(\Pi'))$. Then, either $a \notin \text{dom}\,\lambda$ or $a \in \text{dom}\,\lambda$. If $a \notin \text{dom}\,\lambda$, then it is sufficient to consider the following equations:

$$\begin{aligned}\text{cod}(h'(\Pi)) &= \text{cod}(h'(\Pi')) = \bar{\sigma}(\text{cod}(\Pi')) = \\ &= \quad [h(\Pi')] = [h(\Pi)] \quad = \bar{\sigma}(\text{cod}(\Pi)).\end{aligned}$$

   Suppose now that $a \in \text{dom}\,\lambda$. Since h is a morphism in **TTSI**, then $h(\Pi) = h(\Pi')(t', d) \in \text{TComp}(TTI)$, where $t' = (\sigma(s), \lambda(a), \sigma(s'))$. Note that $\text{cod}(\Pi') = s$ and $\text{cod}(\Pi) = s'$. Hence $h'(\Pi) = h'(\Pi')\big(t_\simeq, d\big)$, where $t_\simeq = ([h(\Pi')], \lambda(a), [h(\Pi)])$ and $\text{cod}(h'(\Pi)) = [h(\Pi)] = \bar{\sigma}(s') = \bar{\sigma}(\text{cod}(\Pi))$. By the definition of *ttsi.ottsi*, we have $\delta_\simeq(t_\simeq) = \delta_{TTI}(t')(k)$ with $k = \mathcal{N}(h(\Pi), [t'])$, and $\delta_{TTI}(t')(k) \leq d$, because $h(\Pi)$ is a timed computation. Therefore, $h'(\Pi) \in \text{TComp}(ttsi.ottsi(TTI))$.

2. To show that h' preserves $\simeq$, it is sufficient to show that it preserves the independence relation. Let $t = (s, a, s')$ and $\bar{t} = (\bar{s}, b, \bar{s}')$ be two independent transitions. Since h is a morphism in **TTSI**, we get $h(t) I_{TTI} h(\bar{t})$. Therefore, by the definition of *ttsi.ottsi*, for any transitions $\tilde{t} = ([\Pi], \lambda(a), [\Pi(h(t), d)])$ and $\tilde{\bar{t}} = ([\bar{\Pi}], \lambda(b), [\bar{\Pi}(h(\bar{t}), \tilde{d})])$ in $Tran_\simeq$, we have $\tilde{t} I_\simeq \tilde{\bar{t}}$. But this is how the images of t and $\bar{t}$ under h' look like. Hence, $h'(t) I_\simeq h'(\bar{t})$.

To prove that the diagram above commutes, we only need to note that if $\bar{\sigma}(s) = [h(\Pi_s)]$, then $\text{cod}(h(\Pi_s)) = \sigma(s)$ and, therefore, $\sigma_\varepsilon \circ \bar{\sigma} = \sigma$. It remains to show that h' is unique. Observe that since the diagram commutes, then the image of each $s \in S_{OTTI}$ is a $\simeq$-class of timed computations leading to s. But such a class is unique (see lemma 4.2 [9] for more details). □

The next theorem concludes this section and presents a categorical characterization of unfolding.

Theorem 1 ($\hookrightarrow \dashv$ *ttsi.ottsi*). The map *ttsi.ottsi* extends to a functor from **TTSI** $\rightarrow$ **oTTSI** which is right adjoint to the functor $\hookrightarrow$: **oTTSI** $\rightarrow$ **TTSI**. Moreover, this adjunction is a coreflection.

Proof. The existence of the adjunction directly follows from Lemma 1. To prove that this adjunction is a coreflection, we need to identify the unit of the adjunction and show that it is an isomorphism.

Construct a mapping $\eta_{OTTI} = (\sigma_\eta, 1_L) : OTTI \to \mathit{ttsi}.\mathit{ottsi}(OTTI)$ as follows. For each $OTTI$ and $s \in S_{OTTI}$ define $\sigma_\eta(s) = [\Pi_s]$, where $\Pi_s \in \mathrm{TComp}(OTTI)$ is such that $\mathrm{cod}(\Pi_s) = s$. It is easy to see that η_{OTTI} is a morphism in **oTTSI**. Next, if $\varepsilon_{OTTI} = (\sigma_\varepsilon, 1_L) : \mathit{ttsi}.\mathit{ottsi}(OTTI) \to OTTI$, then $\sigma_\varepsilon \circ \sigma_\eta = 1_{S_{OTTI}}$ and $\sigma_\eta \circ \sigma_\varepsilon = 1_{S_{\mathit{ttsi}.\mathit{ottsi}(OTTI)}}$. Therefore, η_{OTTI} is an isomorphism.

In order to show that η_{OTTI} is the unit of adjunction, we need to show that the following diagram commutes (see [8, p. 83]).

$$\begin{array}{ccc} OTTI & \xrightarrow{\eta_{OTTI}} & \mathit{ttsi}.\mathit{ottsi}(OTTI) \\ & \searrow\!\!\!\!= & \downarrow \varepsilon_{OTTI} \\ & & OTTI \end{array}$$

But its commutativity directly follows from the fact that $\sigma_\epsilon \circ \sigma_\eta = 1_S$. $\square$

3. Timed transition systems with independence and timed event structures

In this section, we study the relationships between timed transition systems with independence and timed (prime) event structures [3]. We start with the definition of (untimed) event structures.

Definition 3. An *event structure* is a triple $E = (E, \leq, \#)$, where

- E is a countable set of *events*;
- $\leq \subseteq E \times E$ is a partial order on E modelling causality and satisfying the following principle of "finite causes": for each event $e \in E$, $\{e' \in E \mid e' \leq e\}$ is a finite set;
- $\# \subseteq E \times E$ is a symmetric irreflexive *conflict relation* satisfying the "conflict heredity" principle, i.e., $e \# e' \leq e'' \Rightarrow e \# e''$.

A set of events $C \subseteq E$ is said to be a *configuration* of an event structure E iff it is *left-closed*: $\forall e \in C \,.\, {\downarrow}e$[3] $\subseteq C$, and *conflict-free*: $\forall e, e' \in C \,.\, \neg(e \# e')$. We say that distinct events $e, e' \in E$ are *concurrent* and write $e \smile e'$ if $\neg(e \leq e' \wedge e' \leq e' \wedge e \# e')$. Introduce the concept of a *reflexive conflict* as $e \mathrel{\vee\!\!\!\vee} e' \iff e \# e' \vee e = e'$.

Next, we consider the concept of timed event structures from [3]. By analogy with the model of timed transition systems with independence, there is a global non-negative integer-valued clock. Each event in an event structure is associated with a time delay with respect to the initial time moment; i.e., if an event e is associated with a time delay t, then e may occur not earlier

[3]Here ${\downarrow}e$ is a lower cone of event e, i.e. ${\downarrow}e = \{e' \in E \mid e' \leq e\}$

than all the predecessors of the event occur and the clock shows time t. In this case, the event itself occurs instantaneously. The states of the timed event structures are described in terms of timed configurations consisting of sets of past events and the current clock value. In what follows, we assume that the events are not "urgent" (it is not required that the events occur immediately after all their predecessors occurred and the global clock showed time equal to the time delay of the event). This approach is very close to that used in the incorporation of the concept of time in the semantic models of timed event structures [7, 4].

Definition 4. A timed event structure is a pair $TE = (E, \leq, \#, \Delta)$, where $(E, \leq, \#)$ is an event structure and $\Delta : E \to \mathbb{N}$ is a function of time delays such that $e' \leq e \Rightarrow \Delta(e') \leq \Delta(e)$.

Let $TE = (E, \leq, \#, \Delta)$ be a timed event structure. Then, a *timed configuration* in TE is a pair (C, t), where C is a configuration of $(E, \leq, \#)$ and $t \in \widetilde{\mathbb{N}}$[4] such that $\Delta(e) \leq t$ for each $e \in C$. The set of timed configurations of a timed event structure TE is denoted as $TConf(TE)$. A sequence $(e_0, t_0), \ldots, (e_n, t_n)$, where $e_i \in E$ and $t_i \in \mathbb{N}$ $(i = 0, \ldots, n)$, is called a *securing sequence* for the timed configuration (C, t), if $(\{e_0, \ldots, e_i\}, t_i)$ is a timed configuration for any $i = 0, \ldots, n$ and $(\{e_0, \ldots, e_i\}, t_n) = (C, t)$.

Let $TE = (E, \leq, \#, \Delta)$ and $TE' = (E', \leq', \#', \Delta')$ be event structures. A partial mapping $\theta : E \to^* E'$ is a *morphism* if

- $\downarrow\theta(e) \subseteq \theta(\downarrow e)$
- $\theta(e) \mathrel{\mathrm{W}} \theta(e') \Rightarrow e \mathrel{\mathrm{W}} e'$, for all $e, e' \in \operatorname{dom}\theta$,
- $\Delta'(\theta(e)) \leq \Delta(e)$, for all $e \in \operatorname{dom}\theta$.

Timed event structures with their morphisms define a category **TES** with unit morphisms $\mathbf{1}_{TS} = \mathbf{1}_E : TS \to TS$ for all $TS = (E, \leq, \#, \Delta)$.

In order to establish the categorical relationships between timed event structures and timed occurrence transition systems with independence, we define a mapping *tpes.ottsi* : **TPES** $\to$ **oTTSI**. Informally speaking, the states and the transitions of *tpes.ottsi*(TE) correspond to particular timed configurations and the occurrences of events of TE, respectively.

For an event structure $TE = (E, \leq, \#, \Delta)$, let *tpes.ottsi*$(TE) = (S_{TE}, s^I_{TE}, L_{TE}, Tran_{TE}, I_{TE}, \delta_{TE})$, where

- $S_{TE} = \big\{(C, \Delta(C)) \in TConf(TE) \mid |C| < \infty \wedge \Delta(C) = \max\{\Delta(e) \mid e \in C\}\big\}$;
- $s^I_{TE} = (\varnothing, 0)$;

[4] $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

- $L_{TE} = E$;
- $\big((C, \Delta(C)), e, (C', \Delta(C'))\big) \in Tran_{TE} \iff C' \setminus C = \{e\}$;
- $\big((C, \Delta(C)), e, (C', \Delta(C'))\big) I_{TE} \big((\bar{C}, \Delta(\bar{C})), \bar{e}, (\bar{C}', \Delta(\bar{C}'))\big) \iff e \smile \bar{e}$;
- $\delta_{TE}((C, \Delta(C)), e, (C', \Delta(C'))) = \langle \Delta(e) \rangle$.

It is easy to see that the above definition is correct, i.e. *tpes.ottsi* maps timed event structures to timed occurrence transition systems.

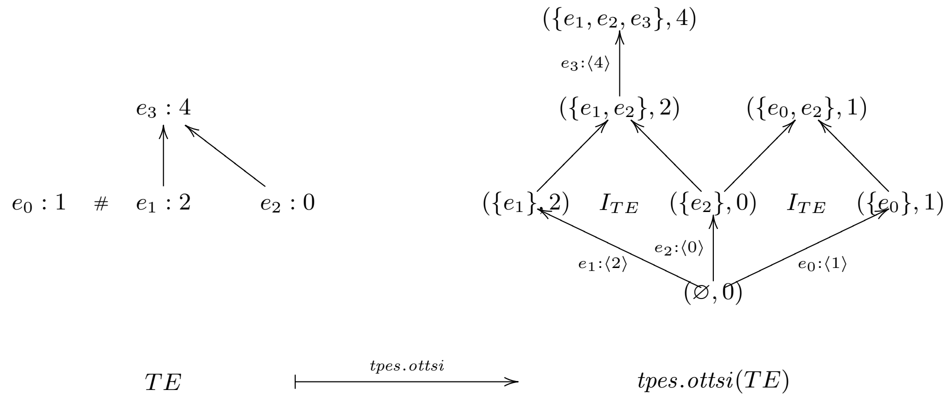


Figure 4

Example 5. Figure 4 shows an event structure TE and its image under the mapping *tpes.ottsi*.

Next, we construct a mapping *ottsi.tpes* : **oTTSI** $\to$ **TPES** acting in the opposite direction and mapping timed occurrence transition systems with independence to timed event structures. Intuitively speaking, the image of a timed occurrence transition system with independence $OTTI$ is a timed event structure *ottsi.tpes*$(OTTI)$ whose events are classes of $\sim$-equivalent transitions of $OTTI$. During the translation, we also make sure that the time delays and the causality order of events agree with each other.

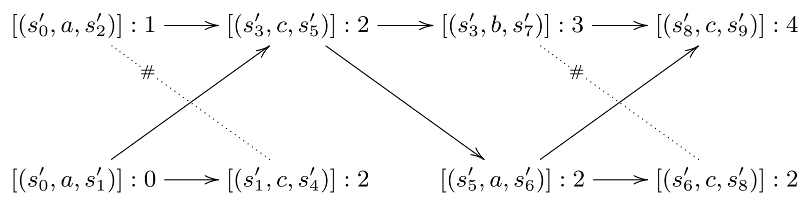
Let $OTTI = (S, s^I, L, Tran, I, \delta)$ be a timed occurrence transition system with independence. Then *ottsi.tpes*$(OTTI) = (Tran_{\sim}, \leq, \#, \Delta)$, where

1. $Tran_{\sim} = \{[t] \mid t \in Tran\}$;
2. $[t] < [t'] \iff$
   $\forall \Pi(\bar{t}', d) \in \mathrm{TComp}(OTTI) \,.\, \big(\bar{t}' \sim t' \Rightarrow (\exists (\bar{t}, \bar{d}) \in \Pi \,.\, \bar{t} \sim t)$; $\leq = (<)^*$;
3. $[t] \# [t'] \iff$
   $\forall \Pi \in \mathrm{TComp}(OTTI)), \forall \bar{t} \in [t], \forall \bar{t}' \in [t'] \,.\, (\bar{t}, \bar{d}) \in \Pi \Rightarrow (\bar{t}', \bar{d}') \notin \Pi$ for all $\bar{d}' \in \mathbb{N}$;

4. $\Delta([t]) = \max\{\delta(t')(1) \mid [t'] \leq [t]\}$.

For a morphism $h = (\sigma, \lambda) : OTTI \to OTTI'$ in **oTTSI**, let $ottsi.tpes(h) = \theta$, where

$$\theta([(s, a, s')]) = \begin{cases} [(\sigma(s), \lambda(a), \sigma(s')], & \text{if } a \in \text{dom}\,\lambda \\ \text{undefined}, & \text{otherwise.} \end{cases}$$



$ottsi.tpes(OTTI')$

Figure 5

Example 6. Figure 5 shows how *ottsi.tpes* acts on the timed occurrence transition system $OTTI'$ from example 4.

The below fact follows from the definition of *ottsi.tpes*.

Proposition 1. *ottsi.tpes* : **oTTSI** $\to$ **TPES** is a functor.

Now we show that *ottsi.tpes* and *tpes.ottsi* define a coreflection. First, we construct a mapping $\mathcal{L}_S : S_{ottsi.tpes(OTTI)} \to S_{OTTI}$ for each timed occurrence transition system with independence $OTTI$. Let $\mathcal{L}_S(C, \Delta(C)) = \text{cod}(\Pi(\zeta))$, where $\zeta = ([t_1], d_1), \ldots, ([t_n], d_n)$ is a securing sequence for $(C, \Delta(C))$ and $\Pi(\zeta) = ((s_0, a_1, s_1), d_1), \ldots, ((s_{n-1}, a_n, s_n), d_n)$ is a timed computation constructed as follows:

- If $i = 1$, then (s_0, a_1, s_1) is the unique representative in $[t_0]$ whose source state is s^I_{OTTI}. Its existence can be shown using Lemma 4.18 [9] and it is unique because of property 4 of the definition of transition systems with independence. Directly from the definition of *ottsi.tpes*, it follows that $\delta(s_0, a_1, s_1)(1) = \Delta([t_1]) \leq d_1$
- If $n > 1$, then (s_{i-1}, a_i, s_i) is the unique representative in $[t_i]$ whose source state is s_{i-1}. Notice, $\delta(s_{i-1}, a_i, s_i)(1) \leq \Delta([t_i]) \leq d_i$.

Using Lemma 4.20 [9], it is easy to see that the above definition is correct as it does not depend on the choice of ζ. Next, define a mapping $\mathcal{L}_L : E_{ottsi.tpes(OTTI)} \to L_{OTTI}$ as $\mathcal{L}_L([(s, a, s')]) = a$ for any event

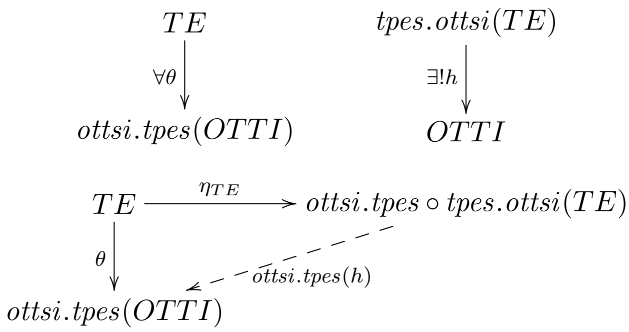
$[(s, a, s')] \in E_{ottsi.tpes(OTTI)}$. It is easy to see that $\mathcal{L} = (\mathcal{L}_S, \mathcal{L}_L) : tpes.ottsi \circ ottsi.tpes(OTTI) \to OTTI$ is a morphism in **oTTSI**.

Finally, we define a unit of the adjunction. For an event structure TE, let $\eta_{TE} : E_{TE} \to E_{ottsi.tpes \circ tpes.ottsi(TE)}$ be a mapping such that

$$\eta_{TE}(e) = [(C, \Delta(C)), e, (C \cup \{e\}, \Delta(C \cup \{e\}))].$$

It is straightforward to show that η_{TE} is an isomorphism in **TPES**. In order to show the existence of the adjunction, we need to check that η_{TE} is a unit indeed, i.e. it is universal.

Lemma 2 (η_{TE} is universal). For any timed event structure TE, any timed occurrence transition system $OTTI$, and any morphism $\theta : TE \to ottsi.tpes(OTTI)$ there exists a unique morphism $h : tpes.ottsi(TE) \to OTTI$ such that $\theta = ottsi.tpes(h) \circ \eta_{TE}$:



Proof. Let $TE' = ottsi.tpes(OTTI)$ and $h = (\sigma, \lambda)$, where

- $\sigma(C, \Delta(C)) = \mathcal{L}_S(\theta C, \Delta_{TE'}(\theta C))$;
- $\lambda(e) = \begin{cases} a, & \text{if } e \in \operatorname{dom}\theta \text{ and } \theta(e) = [(s, a, s')], \\ \text{undefined}, & \text{otherwise.} \end{cases}$

It is easy to see that $h = \mathcal{L} \circ tpes.ottsi(\theta)$ and, hence, h is well-defined.

In order to show that $\theta = ottsi.tpes(h) \circ \eta_{TE}$, i.e. the diagram commutes, consider an event $e \in E_{TE}$. Suppose that $e \notin \operatorname{dom}\theta$. Then we have $e \notin \operatorname{dom}\lambda$. Hence, both sides of the equation are undefined. If $e \in \operatorname{dom}\theta$, then:

$$e \overset{\eta_{TE}}{\longmapsto} \big[\big((C, \Delta(C)), e, (C' = C \cup \{e\}, \Delta(C'))\big)\big]$$

$$\big\downarrow{\scriptstyle ottsi.tles(h)}$$

$$\big[\big(\mathcal{L}_S(\theta C, \Delta_{TE'}(\theta C)), \lambda(e), \mathcal{L}_S(\theta C', \Delta_{TE'}(\theta C'))\big)\big]$$

$$\|$$

$$\big[\big(\sigma(C, \Delta(C)), \lambda(e), \sigma(C', \Delta(C'))\big)\big].$$

Notice, $\big((\theta C, \Delta_{TE'}(\theta C)), \lambda(e), (\theta C', \Delta_{TE'}(\theta C'))\big)$ is a transition of $tpes.ottsi \circ ottsi.tpes(OTTI)$ corresponding to an occurrence of the event $\theta(e)$. Using lemma 4.22 [9], it is easy to show that

$$\big(\mathcal{L}_S(\theta C, \Delta_{TE'}(\theta C)), \lambda(e), \mathcal{L}_S(\theta C', \Delta_{TE'}(\theta C'))\big)$$

is a transition in $OTTI$. Since $C' = C \cup \{e\}$, then

$$\big[\big(\mathcal{L}_S(\theta C, \Delta_{TE'}(\theta C)), \lambda(e), \mathcal{L}_S(\theta C', \Delta_{TE'}(\theta C'))\big)\big] = \theta(e).$$

It remains to show that h is unique. Suppose the contrary, i.e. there exists $h' \neq h$ such that $\theta = ottsi.tpes(h) \circ \eta_{TE}$. It is obvious that h' is of the form (σ', λ). Since the above diagram commutes, for each event $e \in \operatorname{dom}\theta$ the following holds:

$$\begin{aligned} ottsi.tpes(h)\left(\big[(C, \Delta(C)), e, (C', \Delta(C'))\big]\right) = \\ \big[\big(\sigma(C, \Delta(C)), \lambda(e), \sigma(C', \Delta(C'))\big)\big] = \\ \theta(e) = \\ \big[\big(\sigma'(C, \Delta(C)), \lambda(e), \sigma'(C', \Delta(C'))\big)\big]. \end{aligned}$$

By induction on the size of C, it is easy to show that $\sigma = \sigma'$. □

The next theorem establishes the existence of a coreflection.

Theorem 2 ($tpes.ottsi \dashv ottsi.tpes$). Map $tpes.ottsi$ can be extended to a functor $tpes.ottsi$: **TPES** $\rightarrow$ **oTTSI**, which is left adjoint to the functor $ottsi.tpes$. Moreover, this adjunction is a coreflection.

Proof. Follows immediately from Lemma 2 and the fact that η_{TE} is an isomorphism for any TE. □

4. Conclusion

We have defined and studied a timed extension of a well-known "true concurrent" model of transition systems with independence and have shown that there exists a chain of coreflections from their category to the category of timed event structures.

$$\mathbf{TTSI} \underset{\longleftarrow}{\overset{ttsi.ottsi}{\underset{\top}{\longrightarrow}}} \mathbf{oTTSI} \underset{tpes.ottsi}{\overset{ottsi.tpes}{\underset{\top}{\rightleftarrows}}} \mathbf{TPES}$$

We plan to extend obtained results to the case of dense time.

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