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# Open maps and weak trace equivalence for timed event structures

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**Abstract.** A timed extension of weak trace equivalence is developed for a model of timed event structures. Moreover, a category-theoretic characterization of this equivalence based on a span of open maps is specified. Finally, the problem of decidability of weak trace equivalence is solved in the setting of finite timed event structures.

## 1. Introduction

The notion of equivalence plays the central role in the theory of concurrency. It allows one to compare systems taking into account particular aspects of their behavior. Now there exists a wide variety of equivalences, represented in literature [9]. Trace equivalences are the most popular among them [14]. This approach is the most simple and natural since such equivalences are defined in terms of the coincidence of languages.

Recently, category-theoretical approaches are actively used in order to describe and investigate different concurrent systems and processes. As a response to some of the numerous models for concurrency proposed in the literature, Winskel and Nielsen have used the category theory as an attempt to understand the relationships between models like event structures, Petri nets, trace languages and asynchronous transition systems [26]. From the algebraic point of view, many operators of CCS like process algebras have been recasted using category-theoretic concepts, such as products and co-products. However, a similar convincing category-theoretic way of adjoining abstract equivalences to the category of models had been missing until Joyal, Nielsen and Winskel proposed the notion of span of open maps [17]. They show how spans of open maps can capture Park and Milner's strong bisimulation and identify a new bisimulation, strong history-preserving bisimulation, on models with independence like event structures and Petri nets. Later in the work [6], Nielsen and Cheng show that spans of open maps can capture not only Park and Milner's strong bisimulation, but a representative selection of well-known bisimulations, such as, e.g., Milner and Sangiorgi's barbed bisimulation and Larsen and Skou's probabilistic bisimulation.

In recent years, great efforts have been made to develop formal methods for real-time and other timing-based systems, i.e. systems whose correctness depends crucially upon real-time considerations. As a result, timed extensions of different equivalences have been defined. The category-theoretic approach came in use for investigations of such equivalences. So in the work [15], Hune and Hielsen got the category-theoretic characterization for a timed interleaving bisimulation in the setting of timed transition systems and proved that this bisimulation is decidable. Moreover, other types of timed equivalences are also characterized on the category for a timed variant of interleaving models, for example, in work [10] a timed variant of Milner and Sangiorgi's barbed bisimulation are investigated for timed transition systems. Later the category-theoretic approach has been applied to analysis of different timed equivalences based on a partial order in the frame of true concurrency models [24].

This work is dedicated to the timed variant of weak trace equivalence defined in the setting of timed event structures — models with a true concurrency semantics. Weak equivalences differ from normal equivalences in at least two aspects. First, a special "invisible" action, usually denoted by $\tau$, is required to be a member of the set of actions. Second, a "visible" action in one model is not required to be simulated exactly by the same action in the other model. It may be preceded and succeeded by several $\tau$ actions. Furthermore, a $\tau$ action need not be simulated by any actions at all.

The contribution of the paper is to show the applicability of the general categorical framework of open maps to the timed variant of weak trace equivalence in the setting of timed extensions of partial order models and to proof the decidability of this equivalences for a subclass of finite models.

The rest of the paper is organized as follows. The basic notions and notations concerning timed event structures are introduced in Section 2. In the next section, a category of timed event structures and a subcategory are defined, and some properties of the categories are established. Moreover, this section contains a definition of open maps. In Section 4, abstract bisimulation is studied and it is shown that it coincides with the timed variant of weak trace equivalence. Further, in Section 5 we provide a proof of decidability of this equivalence based on the Alur's technique of regions [1]. Section 6 contains conclusions and remarks on future works.

## 2. Timed event structures

In this section, we introduce some basic notions and notations concerning timed event structures. First, we recall a notion of event structures [25] which constitute a major branch of partial order models. The main idea behind event structures is to view distributed computations as action occurrences, called events, together with a notion of causality dependency between events (which is reasonably characterized via a partial order). Moreover, in order to model nondeterminism, there is a notion of conflicting (mutually incompatible) events. A labelling function records actions which correspond to events. Let $L_\tau$ be a finite set of actions with a special "invisible" action $\tau$. Further we shall use $L = L_\tau \setminus \{\tau\}$ to denote the set of all "visible" actions.

*A (labelled) event structure* over $L_\tau$ is a tuple $S = (E, \leq, \#, l)$, where $E$ is a set of events; $\leq \subseteq E \times E$ is a partial order (*the causality relation*), satisfying *the principle of finite causes*: $\forall e \in E \diamond \{e' \in E \mid e' \leq e\}$ is finite; $\# \subseteq E \times E$ is a symmetric and irreflexive relation (*the conflict relation*), satisfying *the principle of conflict heredity*: $\forall e, e', e'' \in E \diamond e \# e' \leq e'' \Rightarrow e \# e''$; $l : E \longrightarrow L_\tau$ is a labelling function.

We shall use $\mathcal{O}$ to denote the empty event structure $(\emptyset, \emptyset, \emptyset, \emptyset)$.

For $C \subseteq E$ the *restriction of* $S$ to $C$ (denoted $S\lceil C$) is defined as $(C, \leq \cap (C \times C), \# \cap (C \times C), l \mid_C)$. Moreover, for $C \subseteq E$ we define a subset of visible events $Vis(C)$ as follows: $\{e \in C \mid l(e) \neq \tau\}$.

For an event structure $S = (E, \leq, \#, l)$ we define $\smile = (E \times E) \setminus (\leq \cup \leq^{-1} \cup \#)$ (*the concurrency relation*). Let $C \subseteq E$. Then $C$ is *left-closed* iff $\forall e, e' \in E \diamond e \in C \wedge e' \leq e \Rightarrow e' \in C$; $C$ is *conflict-free* iff $\forall e, e' \in C \diamond \neg(e \# e')$; $C$ is a *configuration of* $S$ iff $C$ is left-closed and conflict-free. Let $\mathcal{C}(S)$ denote the set of all finite configurations of $S$.

We next present a dense time extension of event structures, called timed event structures, because it is well recognized that the dense time approach seems to be more suitable to model realistic systems (see [2] for more explanation). In our model, we add time constraints to event structures by associating their events with the earliest and latest times, w.r.t. a global clock, at which the events can occur. Following [18, 19], the occurrence of an enabled event itself takes no time but it can be suspended for a certain time (between its earliest and latest times) from the start of the system. The reason for not using what is often referred to as local clocks (i.e., each event has its delay timer attached and the timer is set when the event becomes enabled and reset when the event is disabled or started to be executed) is that the operational semantics of timed models is more simple in case of avoiding local clocks (see [18] among others).

Before introducing the concept of a timed event structure, we need to define some auxiliary notations. Let $\mathbf{N}$ be the set of natural numbers, and $\mathbf{R}$ be the set of nonnegative real numbers.

**Definition 1.** *A (labelled) timed event structure over $L_\tau$ is a triple $TS = (S, Eot, Lot)$, where $S = (E, \leq, \#, l)$ is a (labelled) event structure over $L_\tau$; $Eot$, $Lot : E \rightarrow \mathbf{R}$ are functions of the earliest and latest occurrence times of events satisfying $Eot(e) \leq Lot(e)$ for all $e \in E$.*

A timed event structure is said to *have a valid timing*, if $e' \leq_S e \Rightarrow Eot(e') \leq Eot(e)$ and $Lot(e') \leq Lot(e)$ for all $e, e' \in E$. In the following, we will consider only timed event structures having a valid timing and call them simply timed event structures. Let $\mathcal{TO}$ denote the empty timed event structure $(\mathcal{O}, \emptyset, \emptyset)$.

For depicting timed event structures, we use the following conventions. The action labels and timing constraints associated with events are drawn near the events. If no confusion arises, we will often use action labels rather than event identities to denote events. The $<$-relation is depicted by arcs (omitting those derivable by transitivity), and conflicts are also drawn (omitting those derivable by conflict heredity).

**Example 1.** *A trivial example of a timed event structure, labelled over $L_\tau$, is shown in Fig. 1.*

To make our examples easier to understand, we will sometimes give for them algebraic expressions (see [4]) over actions with the time intervals of the corresponding events. The algebraic syntax includes the primitive constructs: sequential composition (;), parallel composition ($\|$), and sum (+). The operation ; ($\|$, +, respectively) may be easily "interpreted" by indicating that all events in one component are in the $<$-relation ($\smile$-relation, #-relation, respectively) with all events in the other.

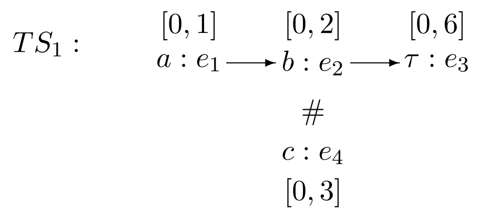


**Figure 1.** The timed event structure $TS_1$ over $L_\tau$

Timed event structures $TS$ and $TS'$ are *isomorphic* (denoted $TS \simeq TS'$), if there exists a bijection $\varphi : Vis(E_{TS}) \longrightarrow Vis(E_{TS'})$ such that $e \leq_{TS} e' \leftrightarrow \varphi(e) \leq_{TS'} \varphi(e')$, $e \#_{TS} e' \leftrightarrow \varphi(e) \#_{TS'} \varphi(e')$, $l_{TS}(e) = l_{TS'}(\varphi(e))$, and $Eot_{TS}(e) = Eot_{TS'}(\varphi(e))$, $Lot_{TS}(e) = Lot_{TS'}(\varphi(e))$, for all $e, e' \in Vis(E_{TS})$.

An execution of a timed event structure is a *timed configuration* which consists of a configuration and a timing function, recording global time moments at which events occur, and satisfies some additional requirements.

**Definition 2.** *Let $TS = (S, Eot, Lot)$ be a timed event structure, $C \in \mathcal{C}(S)$, and $T : C \longrightarrow \mathbf{R}$. Then $TC = (C, T)$ is a timed configuration of $TS$ iff the following conditions hold:*

*(i)* $\forall\ e \in C \ \diamond\ Eot(e) \leq T(e) \leq Lot(e)$,

*(ii)* $\forall\ e, e' \in C \ \diamond\ e \leq_{TS} e' \ \Rightarrow\ T(e) \leq T(e')$.

Informally speaking, the condition (i) expresses that an event can occur at a time when its timing constraints are met; the condition (ii) says that for any two occurred events $e$ and $e'$ if $e$ causally precedes $e'$ then $e$ should temporally precede $e'$.

The *initial timed configuration* of $TS$ is $TC_{TS} = (\emptyset, \emptyset)$. We use $\mathcal{TC}(TS)$ to denote the set of timed configurations of $TS$.

**Example 2.** *To illustrate the concept, consider the set of possible timed configurations of the timed event structure $TS_1$ shown in Figure 1:* $\mathcal{TC}(TS_1) = \{(\emptyset, \emptyset),\ (\{e_1\}, T_1),\ (\{e_4\}, T_2),\ (\{e_1, e_2\}, T_3),\ (\{e_1, e_4\}, T_4),\ (\{e_1, e_2, e_3\}, T_5) \mid T_1(e_1) \in [0,1];\ T_2(e_4) \in [0,3];\ T_3(e_1) \in [0,1],\ T_3(e_2) \in [0,2],\ T_3(e_1) \leq T_3(e_2);\ T_4(e_1) \in [0,1],\ T_4(e_4) \in [0,3];\ T_5(e_1) \in [0,1],\ T_5(e_2) \in [0,2],\ T_5(e_3) \in [0,6],\ T_5(e_1) \leq T_5(e_2) \leq T_5(e_3)\}$.

The semantics of timed event structures is defined by means of timed pomsets. First, we define a *timed partial order set* as a timed event structure $TP = (S_{TP} = (E_{TP}, \leq_{TP}, \#_{TP}, l_{TP}), Eot_{TP}, Lot_{TP})$ over $L_\tau$ with $\#_{TP} = \emptyset$ and $Eot_{TP}(e) = Lot_{TP}(e)$ for all $e \in E_{TP}$. Isomorphic classes of timed partial order sets are called *timed pomsets*.

The *empty pomset* (denoted as $TP_\mathcal{O}$) is an isomorphic class of $(\mathcal{O}, \emptyset, \emptyset)$. We use $\mathcal{TP}om_{L_\tau}$ (or $\mathcal{TP}om_L$) to indicate the set of finite timed pomsets labelled over $L_\tau$ (or $L$).

Let $TS$ be a timed event structure and

$$TC = (C, T), TC' = (C', T') \in \mathcal{TC}(TS).$$

Then the *restriction* of $TS$ to $TC$, denoted as $TS\lceil TC$, is defined as an isomorphic class of $(S\lceil C, T)$.

The set $L_{wtp}(TS) = \{TP \in \mathcal{TP}om_L \mid TP \simeq TS\lceil TC$ for some $TC \in \mathcal{TC}(TS)\}$ is the *weak timed pomset language* of $TS$ (*wtp-language*).

**Example 3.** *To illustrate the concept, consider the wtp-language for $TS_1$ shown in Figure 1: $L_{wtp}(TS_1) = \{TP_\mathcal{O},\ (a : T_1),\ (c : T_2),\ (a : T_3;\ b : T_4),\ (a : T_5 \parallel c : T_6) \mid T_1 \in [0,1],\ T_2 \in [0,3],\ T_3 \in [0,1],\ T_4 \in [0,2],\ T_5 \in [0,1], T_6 \in [0,3]\}$.*

For $TC \in \mathcal{TC}(TS)\}$ we define a visible part of a timed configuration (denoted by $Vis(TC)$) as a pair $(Vis(C), T \mid_{Vis(C)})$. The set of all visible parts of timed configurations for $TS$ we denote as $\mathcal{VISTC}(TS)$. Then the *restriction* of $TS$ to $Vis(TC)$, denoted as $TS\lceil Vis(TC)$, is defined as an isomorphic class of $(S\lceil Vis(C), T \mid_{Vis(C)})$.

## 3. The category of timed event structures $\mathcal{CTES}_{weak}$

In this section, we define and study a category of timed event structures $\mathcal{CTES}_{weak}$. The morphisms of our model categories will be the simulation morphisms, following the approach of [16].

We start with introducing the notion of a morphism.

**Definition 3.** *Let $TS = (E, \leq, \#, l, Eot, Lot)$ and $TS' = (E', \leq', \#', l', Eot', Lot')$ be timed event structures over $L_\tau$. The map $\mu : TS \to TS'$ is called a morphism, if $\mu : Vis(E) \to Vis(E')$ is a function such that $l' \circ \mu = l$ and for all $Vis(TC) \in \mathcal{VISTC}(TS)$ it holds:*

- *$\mu\ Vis(TC) \in \mathcal{VISTC}(TS')$, where* *$\mu\ Vis(TC) = (\mu\ Vis(C), T')$ with $T' \circ \mu = T\ |_{Vis(C)}$;*
- *$\forall e, e' \in Vis(C) \diamond \mu(e) = \mu(e') \Rightarrow e = e'$;*
- *$\forall e, e' \in Vis(C) \diamond \mu(e) < \mu(e') \Leftrightarrow e < e'$.*

**Example 4.** *As an illustration, consider the morphism $\mu$ from the timed event structure $TS_2$, shown in Figure 2, to the timed event structure $TS_1$, shown in Figure 1, mapping events in the following way: $\mu(e'_1) = e_1$, $\mu(e'_2) = e_2$ and $\mu(e'_3) = e_4$. It is easy to check that the constraints of Definition 3 are satisfied.*

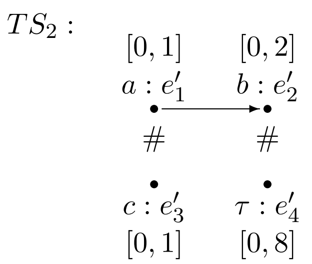


**Figure 2.** The timed event structure $TS_2$

Let us consider a simulation property of a morphism defined above.

**Proposition 1.** *Let $\mu : TS \longrightarrow TS'$ be a morphism and $TC \in \mathcal{TC}(TS)$. Then $TS\lceil TC \simeq TS'\lceil TC'$ where $Vis(TC') = \mu\ Vis(TC)$.*

Now we define the category $\mathcal{CTES}_{weak}$ of timed event structures.

**Definition 4.** *Timed event structures (labelled over $L_\tau$) with morphisms between them form a category of timed event structures, $\mathcal{CTES}_{weak}$, in which the composition of two morphisms $\mu_1 : TS_0 \longrightarrow TS_1$ and $\mu_2 : TS_1 \longrightarrow TS_2$ is $(\mu_2 \circ \mu_1) : TS_0 \longrightarrow TS_2$ and the identity morphism is the identity function.*

Following the standards of timed event structures and the paper [16], we would like to choose "observation objects" with morphisms between them so as to form subcategories of the categories of timed event structures over $L_\tau$.

**Definition 5.** *With respect to a set of actions $L_\tau$, let $\mathcal{TP}_{L_\tau}$ denote the full subcategory of the category $\mathcal{CTES}_{weak}$ with objects from $\mathcal{TP}om_{L_\tau}$ and morphisms, which are the identities and the morphisms with the empty timed pomset as domain.*

Now we define a $\mathcal{TP}_{L_\tau}$-open maps relative to the subcategory $\mathcal{TP}_{L_\tau}$ defined above.

**Definition 6.** *Let $TS$ and $TS'$ be timed event structures. A morphism $\mu : TS \to TS'$ in $\mathcal{CTES}_{weak}$ is called $\mathcal{TP}_{L_\tau}$-open iff for any pomset $TP$ over $L_\tau$ and any morphism $\mu' : TP \to TS'$ there exists a morphism $\mu'' : TP \to TS$ such that $\mu \circ \mu'' = \mu'$.*

Our next aim is to characterize $\mathcal{TP}_{L_\tau}$-openness of morphisms defined prior to that.

**Theorem 1.** *Let $TS$ and $TS'$ be timed event structures. Then a morphism $\mu : TS \to TS'$ in $\mathcal{CTES}_{weak}$ is $\mathcal{TP}_{L_\tau}$-open iff whenever $TC'$ is a timed configuration in $TS'$, there exists a timed configuration $TC$ in $TS$ such that $\mu\ Vis(TC) = Vis(TC')$.*

*Proof Sketch:* Follows from the definition of a $\mathcal{TP}_{L_\tau}$-open map and Proposition 1. ■

Now we produce the following useful property of a span of $\mathcal{TP}_{L_\tau}$-open maps.

**Theorem 2.** *Let $\mu_1 : TS_1 \to TS$ and $\mu_2 : TS_2 \to TS$ be $\mathcal{TP}_{L_\tau}$-open maps. Then there exists a span of $\mathcal{TP}_{L_\tau}$-open maps $\mu_1' : TS_x \to TS_1$, $\mu_2' : TS_x \to TS_2$ with a vertex $TS_x$ and such that $\mu_1 \circ \mu_1' = \mu_2 \circ \mu_2'$.*

*Proof Sketch:* Without loss of generality, let $TS_i = (E_i, \leq_i, \#_i, l_i, Eot_i, Lot_i)$ for $i \in \{1, 2\}$.

For the beginning we construct a timed event structure

$$TS_x = +(TS_{TC_1 \times TC_2} \mid TC_i = (C_i, T_i) \in \mathcal{TC}(TS_i)$$

for all $i \in \{1, 2\}$ and $\mu_1\ Vis(TC_1) = \mu_2\ Vis(TC_2))$, where $TS_{TC_1\times TC_2} = (E_{TC_1\times TC_2}, \leq_{TC_1\times TC_2}, \#_{TC_1\times TC_2}, l_{TC_1\times TC_2}, Eot_{TC_1\times TC_2}, Lot_{TC_1\times TC_2})$ is defined as follows:

- $E_{TC_1 \times TC_2} = \{(e_1, e_2)_{TC_1 \times TC_2} \in Vis(C_1) \times Vis(C_2) \mid \mu_1(e_1) = \mu_2(e_2)\}$;
- $(e_1, e_2)_{TC_1 \times TC_2} \leq_{TC_1 \times TC_2} (e_1', e_2')_{TC_1 \times TC_2} \iff e_i \leq_i e_i'$ for all $i \in \{1, 2\}$;
- $\#_{TC_1 \times TC_2} = \emptyset$;
- $l_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = l_i(e_i)$ for some $i \in \{1, 2\}$;
- $Eot_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = T_i(e_i)$ for some $i \in \{1, 2\}$;
- $Lot_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = T_i(e_i)$ for some $i \in \{1, 2\}$.

It is easy to check that $TS_x$ is really a timed event structure. Then we define maps $\mu_i' : TS_x \longrightarrow TS_i$ $(i = 1, 2)$, as the following functions: $\mu_i'((e_1, e_2)_{TC_1 \times TC_2}) = e_i$. Obviously, these maps are morphisms. The equation $\mu_1 \circ \mu_1' = \mu_2 \circ \mu_2'$ immediately follows from definitions of $TS_x$ and $\mu_i'$ $(i \in \{1, 2\})$.

In order to complete the proof, it is enough to show that $\mu_i'$ is a $\mathcal{TP}_{L_\tau}$-open map $(i \in \{1, 2\})$. To check this fact, we use Theorem 1. Let $TC_i$ be a timed configuration in $TS_i$. Then, since $\mu_i$ is a morphism, we have that $\mu_i\ Vis(TC_i)$ is a visible part of a timed configuration in $TS$. This means that there exists a timed configuration $TC_i'$ such that $Vis(TC_i') = \mu_i\ Vis(TC_i)$. From Proposition 1 we have that $TS\lceil TC_i' \simeq TS_i \lceil TC_i$. Next, since $\mu_{3-i}$ is a $\mathcal{TP}_{L_\tau}$-open morphism, using Theorem 1 we conclude that there exists a timed configuration $TC_{3-i}$ in $TS_{3-i}$ such that $\mu_1\ Vis(TC_1) = \mu_2\ Vis(TC_1)$ and $TS\lceil TC_i' \simeq TS_{3-i}\lceil TC_{3-i}$. Therefore $TS_{TC_1 \times TC_2}$ is a part of $TS_x$. Moreover, we have $TS_{TC_1 \times TC_2} \simeq TS_i \lceil TC_i$. Let $TC_x = (E_{TC_1 \times TC_2}, Eot_{TC_1 \times TC_2})$. Since $TS_{TC_1 \times TC_2}$ is a timed partial order set, $TC_x$ is a timed configuration of $TS_x$. In addition, we get $\mu_i'\ Vis(TC_x) = Vis(TC_i)$ from the definition of $\mu_i'$. Thus, according to Theorem 1, $\mu_i'$ is a $\mathcal{TP}_{L_\tau}$-open morphism $(i \in \{1, 2\})$.■

## 4. The category-theoretic characterization

First, we introduce a timed extension of a weak trace pomset equivalence (*wtp*-equivalence) [6]. This equivalence is the most popular and simplest equivalence in a subclass of weak pomsets equivalences.

**Definition 7.** *Timed event structures $TS$ and $TS'$ are called wtp-equivalent iff $L_{wtp}(TS) = L_{wtp}(TS')$.*

**Example 5.** *Considering the timed event structures $TS_3$ and $TS_4$ shown in Figure 3, we have that they are wtp-equivalent. On the other hand, timed event structures $TS_1$ and $TS_2$ depicted in Figure 1 and 2, respectively, are not wtp-equivalent, because, for instance, the timed pomset $\overset{[0,0]}{a} \parallel \overset{[0,0]}{c}$ belongs to $L_{wtp}(TS_1)$ but not to $L_{wtp}(TS_2)$.*

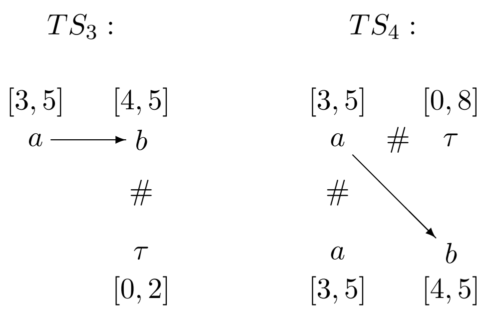


**Figure 3.** Two *wtp*-equivalent timed event structures

Next we define an abstract $\mathcal{TP}_{L_\tau}$-bisimulation based on a span of $\mathcal{TP}_{L_\tau}$-open maps.

**Definition 8.** *Timed event structures $TS_1$ and $TS_2$ are $\mathcal{TP}_{L_\tau}$-bisimular iff there exists a span of $\mathcal{TP}_{L_\tau}$-open maps $TS_1 \xleftarrow{\mu} TS \xrightarrow{\mu'} TS_2$ with a vertex $TS$.*

Due to the property of a span of $\mathcal{TP}_{L_\tau}$-open maps, proved in Theorem 2, we conclude that $\mathcal{TP}_{L_\tau}$-bisimulation defined above is really a relation of equivalence.

Next, the coincidence of *wtp*-equivalence with an abstract $\mathcal{TP}_{L_\tau}$-bisimulation is established.

**Theorem 3.** *Let $TS_1$ and $TS_2$ be timed event structures. Then $TS_1$ and $TS_2$ are $\mathcal{TP}_{L_\tau}$-bisimilar iff they are wtp-equivalent.*

*Proof Sketch:*

($\Rightarrow$) Let $TS_1 \overset{\mu_1}{\leftarrow} TS \overset{\mu_2}{\rightarrow} TS_2$ be a span of $\mathcal{TP}_{L_\tau}$-open maps. We need to show that $L_{wtp}(TS_1) = L_{wtp}(TS_2)$. For the beginning we check that $L_{wtp}(TS_1) \subseteq L_{wtp}(TS_2)$. Let $TP$ be a timed pomset which belongs to $L_{wtp}(TS_1)$. It means that there exists a timed configuration $TC_1$ in $TS_1$ such that $TS_1 \lceil TC_1 \simeq TP$. From Theorem 1, since $\mu_1$ is a $\mathcal{TP}_{L_\tau}$-open map, there exists a timed configuration $TC$ in $TS$ such that $TS \lceil TC \simeq TS_1 \lceil TC_1 \simeq TP$ and $\mu_1\ Vis(TC) = Vis(TC_1)$. Next, because $\mu_2$ is a morphism, we have that $\mu_2\ Vis(TC)$ is a visible part of a timed configuration of $TS_2$. Hence there exists a timed configuration $TC_2$ such that $Vis(TC_2) = \mu_2\ Vis(TC)$. Next, using Proposition 1 we get $TS_2 \lceil TC_2 \simeq TS \lceil TC \simeq TP$. Therefore we have $TP \in L_{wtp}(TS_2)$. Thus $L_{wtp}(TS_1) \subseteq L_{wtp}(TS_2)$. In much the same way we get that $L_{wtp}(TS_2) \subseteq L_{wtp}(TS_1)$. It means that $L_{wtp}(TS_1) = L_{wtp}(TS_2)$.

($\Leftarrow$) Let $L_{wtp}(TS_1) = L_{wtp}(TS_2)$. Without loss of generality, we assume that $TS_i = (E_i, \leq_i, \#_i, l_i, Eot_i, Lot_i)$ for $i \in \{1, 2\}$.

We start defining a timed event structure $TS_x$. Let $TS_x = +(TS_{TC_1 \times TC_2} \mid TC_i = (C_i, T_i) \in \mathcal{TC}(TS_i)$ for $i \in \{1,2\}$; $TS_1 \lceil TC_1 \simeq TS_2 \lceil TC_2$ and $\phi : TS_1 \lceil TC_1 \longrightarrow TS_2 \lceil TC_2$ is an isomorphism), where $TS_{TC_1 \times TC_2} = (E_{TC_1 \times TC_2}, \leq_{TC_1 \times TC_2}, \#_{TC_1 \times TC_2}, l_{TC_1 \times TC_2}, Eot_{TC_1 \times TC_2}, Lot_{TC_1 \times TC_2})$ defined as follows:

- $E_{TC_1 \times TC_2} = \{(e_1, e_2)_{TC_1 \times TC_2} \in Vis(C_1) \times Vis(C_2) \mid \phi(e_1) = e_2\}$;
- $(e_1, e_2)_{TC_1 \times TC_2} \leq_{TC_1 \times TC_2} (e'_1, e'_2)_{TC_1 \times TC_2} \iff e_i \leq_i e'_i$ for all $i \in \{1,2\}$;
- $\#_{TC_1 \times TC_2} = \emptyset$;
- $l_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = l_i(e_i)$ for some $i \in \{1,2\}$;
- $Eot_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = T_i(e_i)$ for some $i \in \{1,2\}$;
- $Lot_{TC_1 \times TC_2}((e_1, e_2)_{TC_1 \times TC_2}) = T_i(e_i)$ for some $i \in \{1,2\}$.

It should be easy to see that $TS_x$ is a timed event structure. Let us define $\mu_i : TS_x \longrightarrow TS_i$ $(i = 1,2)$ as follows: $\mu_i((e_1, e_2)_{TC_1 \times TC_2}) = e_i$. By construction of $TS_x$, it is easy to check that $\mu_1$ and $\mu_2$ are indeed morphisms. In order to complete the proof, we need to show that $\mu_i$ is a $\mathcal{TP}_{L_\tau}$-open map $(i \in \{1,2\})$. Let $TC_i$ be a timed configuration in $TS_i$ and $TS_i \lceil TC_i \simeq TP$. Hence $TP \in L_{wtp}(TS_i)$. Since $L_{wtp}(TS_1) = L_{wtp}(TS_2)$, we conclude that $TP \in L_{wtp}(TS_{3-i})$. By definition of a *wtp*-language, there exists a timed configuration $TS_{3-i}$ in $TS_{3-i}$ such that $TS_{3-i} \lceil TC_{3-i} \simeq TP$. Thus we have $TS_1 \lceil TC_1 \simeq TS_2 \lceil TC_2$. It is obvious that $TS_{TC_1 \times TC_2}$ is a part of $TS_x$. Moreover, we get $TC_x = (E_{TC_1 \times TC_2}, Eot_{TC_1 \times TC_2}) \in \mathcal{TC}(TS_x)$ and $TS_x \lceil TC_x \simeq TS_i \lceil TC_i \simeq TP$. By definition of $\mu_i$ we have $\mu_i \; Vis(TC_x) = Vis(TC_i)$. This shows that $\mu_i$ is a $\mathcal{TP}_{L\tau}$-open map and completes the proof of the theorem. ∎

## 5. Decidability

In this section, we deal only with a special subclass of timed event structures, i.e. structures with a finite set of events and for which all constants referred to in the earliest and latest times of occurrence of events are naturally valued. The subclass of timed event structures is denoted by $\mathcal{TES}_{\mathbf{N}}$.

Due to the category-theoretic characterization of *wtp*-equivalence, showing its decidability amounts to deciding whether there exists a span of $\mathcal{TP}_{L\tau}$-open maps between two finite timed event structures. Our approach is, first, to show that $\mathcal{TP}_{L\tau}$-openness of a morphism between two finite timed event structures is decidable, and, then, to show the upper bound on the size of the vertex of a span for two equivalent timed event structures.

As for many existing results for timed models, including the results in verification of real-time systems, our decision procedure heavily relies on the idea behind regions (equivalence classes of states) of [2], which essentially provides a finite description of the state-space of timed event structures.

Next, we develope a notion of regions for timed event structures.

**Definition 9.** *Given a timed event structure $TS$ and timed configurations $TC = (C, T)$ and $TC' = (C, T')$ from $\mathcal{TC}(TS)$, a region is an equivalence class of timing functions such that $T \approx T'$ iff*

*(i) for each $e \in C$ it holds: $\lfloor T(e) \rfloor = \lfloor T'(e) \rfloor$, and*

*(ii) for every pair of events $e, e' \in C$ we have*

$$fract(T(e)) \ \leq \ fract(T(e')) \Leftrightarrow fract(T'(e)) \ \leq \ fract(T'(e')),$$

*and* $fract(T(e)) = 0 \Leftrightarrow fract(T'(e)) = 0.$

*Here for $d \in \mathbf{R}_0^+$ we use $\lfloor d \rfloor$ for the largest integer smaller than or equal to $d$ and $fract(d)$ for the fractional part of $d$.*

*The region to which $T$ belongs is denoted by $[T]$. For finite timed event structures $TS$, a pair $(C, [T])$, where $TC = (C, T) \in \mathcal{TC}(TS)$, is called an extended timed configuration. We consider $[TC_{TS}] = (\emptyset, [\emptyset])$ as the initial extended timed configuration of $TS$.*

For later use we notice the following facts.

**Proposition 2.** *Consider a finite timed event structure $TS \in \mathcal{TES}_{\mathbf{N}}$.*

*(i) For an event $e$ and a region $[T]$ it holds:*
$Eot(e) \leq T(e) \leq Lot(e) \ \Rightarrow \ \forall T_1 \in [T].Eot(e) \leq T_1(e) \leq Lot(e).$

*(ii) For an extended timed configuration $(C, [T])$, $(C, T')$ is a timed configuration for all $T' \in [T]$.*

We now give a characterization of open maps in terms of extended timed configurations. Before doing so, we introduce some auxiliary notations. A visible part of an extended timed configuration $[(C, T)]$ is called a pair $(Vis(C), [T])$. Let us denote an extended timed configuration $(C, [T])$ as $[(C, T)] = [TC]$ and its visible part as $[Vis(TC)]$. For $[Vis(TC)]$, we shall write $\mu\ [Vis(TC)]$ instead of $(\mu\ Vis(C), [T'])$, where $T' \circ \mu = T \lceil Vis(C)$.

**Theorem 4.** *Let $TS$ and $TS'$ be timed event structures. Then a morphism $\mu : TS \to TS'$ is $\mathcal{TP}'_{L_\tau}$-open iff whenever $[TC']$ is an extended timed configuration in $TS'$, there exists an extended timed configuration $[TC]$ in $TS$ such that $\mu\ [Vis(TC)] = [Vis(TC')]$.*

*Proof Sketch:* It follows from Theorem 1 and Proposition 2. ■

**Corollary 1.** *Given two finite timed event structures* $TS_1$ *and* $TS_2$ *from* $\mathcal{TES}_{\mathbf{N}}$ *and a morphism* $\mu : TS_1 \to TS_2$, $\mathcal{TP}_{L_\tau}$*-openness of* $\mu$ *is decidable.*

*Proof Sketch:* It immediately follows from Theorem 4 and Proposition 2, since the total number of extended timed configurations of $TS_1$ and $TS_2$ is less than or equal to $N \cdot 2^{2N} \cdot (C+1)^N$, where $N = |E_1| * |E_2|$ ($|E_i|$ is the number of events of $TS_i$,($i = 1, 2$)), and $C$ is the largest integer referred to in the earliest and latest times of occurrence for events. ■

**Theorem 5.** *Given two finite timed event structures* $TS_1$ *and* $TS_2$ *from* $\mathcal{TES}_{\mathbf{N}}$*, if there exists a span of* $\mathcal{TP}_{L_\tau}$*-open maps with a vertex* $TS$ *such that* $TS_1 \stackrel{\mu_1}{\longleftarrow} TS \stackrel{\mu_2}{\longrightarrow} TS_2$*, then there exists a span of* $\mathcal{TP}_{L_\tau}$*-open maps with a vertex* $TS'$ *such that* $TS_1 \stackrel{\mu_1'}{\longleftarrow} TS' \stackrel{\mu_2'}{\longrightarrow} TS_2$ *and* $TS' \in \mathcal{TES}_{\mathbf{N}}$.

*Proof Sketch:* Let $TS_1 \stackrel{\mu_1}{\leftarrow} TS \stackrel{\mu_2}{\rightarrow} TS_2$ be a span of $\mathcal{TP}_{L_\tau}$-open maps. Without loss of generality, we assume that $TS_i = (E_i, \leq_i, \#_i, l_i, Eot_i, Lot_i)$ for $i \in \{1,2\}$. For the beginning, we build a timed event structure $TS_x = +(TS_{C_1 \times C_2} \mid \exists TC \in \mathcal{TC}(TS) \diamond Vis(TC_i) = \mu_i\ Vis(TC)$ for $i \in \{1,2\})$, where $TS_{C_1\times C_2} = (E_{C_1\times C_2}, \leq_{C_1\times C_2}, \#_{C_1\times C_2}, l_{C_1\times C_2}, Eot_{C_1\times C_2}, Lot_{C_1\times C_2})$ is defined as follows:

- $E_{C_1\times C_2} = \{(e_1, e_2)_{C_1\times C_2} \in Vis(C_1) \times Vis(C_2) \mid \exists e \in Vis(C) \diamond \mu_i(e) = e_i (i = 1, 2)\}$;
- $(e_1, e_2)_{C_1\times C_2} \leq_{C_1\times C_2} (e_1', e_2')_{C_1\times C_2} \iff e_i \leq_i e_i'$ for all $i = 1, 2$;
- $\#_{C_1\times C_2} = \emptyset$;
- $l_{C_1\times C_2}((e_1, e_2)_{C_1\times C_2}) = l_i(e_i)$ for some $i \in \{1, 2\}$;
- $Eot_{C_1\times C_2}((e_1, e_2)_{C_1\times C_2}) = \max\{Eot_1(e_1), Eot_2(e_2)\}$;
- $Lot_{C_1\times C_2}((e_1, e_2)_{C_1\times C_2}) = \min\{Lot_1(e_1), Lot_2(e_2)\}$.

It is obvious that $TS_x$ defined above is really a finite timed event structure. Note that the set $E_x$ is finite since $|E_x| \leq 2^{|E_1|} \times 2^{|E_2|}$.

Next we need to define morphisms $\mu_i' : TS_x \longrightarrow TS_i$. Let

$$\mu_i'((e_1, e_2)_{C_1\times C_2}) = e_i \text{ for all } (e_1, e_2)_{C_1\times C_2} \in E_x (i = 1, 2).$$

By construction of $TS_x$, we have that $\mu_1'$ and $\mu_2'$ defined above are morphisms in the category $\mathcal{CTES}_{weak}$.

To complete the proof, it is enough to show that $\mu_i'$ is a $\mathcal{TP}_{L_\tau}$-open map ($i \in \{1, 2\}$). Without loss of generality, let $TC_i = (C_i, T_i)$ be a timed configuration in $TS_i$. Since $\mu_i$ is a $\mathcal{TP}_{L_\tau}$-open map, there exists a timed configuration $TC$ in $TS$ such that $\mu_i\ Vis(TC) = Vis(TC_i)$ and $TS\lceil TC \simeq TS_i\lceil TC_i$. Next, because $\mu_{3-i}$ is a morphism, we have $\mu_{3-i}\ Vis(TC)$ is a visible part of some timed configuration $TC_{3-i}$ and $TS\lceil TC \simeq TS_{3-i}\lceil TC_{3-i}$. By definition of $TS_x$, we get that $TS_{C_1\times C_2}$ is a part of $TS_x$. Thus, for all $e \in Vis(C)$, a pair $(\mu_1(e)_1, \mu_2(e))_{C_1\times C_2} \in E_{C_1\times C_2}$. Now we define $C_x$ and $T_x$ as follows: $C_x = \{(\mu_1(e)_1, \mu_2(e))_{C_1\times C_2} \mid e \in \widehat{C}\}$ and $T_x((\mu_1(e)_1, \mu_2(e))_{C_1\times C_2}) = T(e)$. Since $TS\lceil TC \simeq TS_1\lceil TC_1 \simeq TS_2\lceil TC_2$, by definition of $TS_x$ we have that $TC_x = (C_x, T_x)$ is a timed configuration and $TS_x\lceil TC_x \simeq TS_i\lceil TC_i$ for all $i = 1, 2$. Moreover, it is easy to check that $\mu_i'\ Vis(TC_x) = Vis(TC_i)$ for all $i = 1, 2$.

By Theorem 1 we conclude that $\mu_i'$ is indeed a $\mathcal{TP}_{L_\tau}$-open map ($i \in \{1, 2\}$). ■

**Corollary 2.** *For timed event structures from $\mathcal{TES}_{\mathbf{N}}$, wtp-equivalence is decidable.*

*Proof Sketch:* It follows from Corollary 1 and Theorems 3 and 5. ■

## 6. Concluding remarks

In this paper, we tried to investigate in practice the applicability of the theory of open maps by Joyal, Nielsen, and Winskel [16] to the study of a timed variant of a weak trace equivalence based on a partial order in the frame of timed event structures.

In particular, we characterized the mentioned above equivalence on the category and established its decidability for finite timed event structures using the idea behind regions (equivalence classes of states) of [2] which provided a finite description of the state-space.

In our future work, based on the paper [8], we hope to extend the results here obtained to timed generalizations of other weak equivalences, combining the open maps and presheaf approaches.

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