Bull. Nov. Comp. Center, Num. Anal., 9 (2000), 45–51

Multivariant partial solution of linear systems of 5-point equations by the diagonal transfer method*

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Using the diagonal transfer method, a partial solution of a linear system of equations is found. The description of corresponding algorithm is given. Computational costs of the algorithm are discussed.

1. Introduction

The simulation of a certain field on the basis of the classical equivalent circuit transformations in electrotechnics was introduced in [1]. Later, this algorithm was algebraically elaborated and investigated in [2–4]. Such a direct method is used to solve systems of the 5-point grid equations by special matrix graph transformations of systems similar to the "triangle–star" and the "star–triangle" transformations of the electrical circuits. The method provides intermediate subsystems of the same matrix density and has the arithmetical cost $O(L^3)$ for the two-dimensional $L \times L$ grid systems similar to the well-known nested dissections algorithm [5].

In this paper, the method is applied to a multivariant partial solution of the grid boundary value problems (BVPs). This means that for the new variant of the problem only the right-hand side of equations for one grid line, where the solution is looked for, is changed. Such a problem corresponds to the Neumann to Dirichlet map (the current to voltage map) in geophysics and arises in modelling the electromagnetic logging [6] when the field sources and receivers are placed near the axis of a well.

A partial solution to the linear algebraic systems was considered in [7] for the problems with separable variables.

The algorithm being presented is formally applicable to any symmetrizable 5-point grid systems of the positive type [8]. Its number of arithmetical operations for the solution of N variants is $P = 7L^3 + 4L^2N$.

*Supported by the NWO-NL-RF under Grant 047.008.007 and by the Russian Foundation for Basic Research under Grants 99-01-00579 and 99-05-64430.

2. Mathematical statement

Let $u(x,y)$ be the solution to the equation

$$-\frac{1}{x^\alpha}\frac{\partial}{\partial x}\left(x^\alpha\lambda(x,y)\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(\lambda(x,y)\frac{\partial u}{\partial y}\right)=0,\quad (x,y)\in\Omega=\bigcup_{k=1}^{m}\Omega_k,\quad (1)$$

in the bounded domain Ω composed of m rectangular subdomains Ω_k, where λ is a given positive piece-wise constant and $\alpha = 0,\ 1$ for the Cartesian and cylindrical coordinates, respectively. On the surfaces Γ_i of possible of discontinuity of λ (on the common sides of different Ω_k) the conjugate conditions hold.

The multivariant partial mixed BVP is defined as follows. Let the computational domain be a rectangle $\Omega = \{a < x < b,\ c < y < d\}$ with the boundary $\Gamma = \Gamma_a \cup \Gamma_b$, $\Gamma_a = \{x = a,\ c < y < d\}$, $\Gamma_a = \Gamma_d \cup \Gamma_n$. On different parts of Γ, boundary conditions of different types are given

$$u_k|_{\Gamma_d} = g_k,\qquad \left.\frac{\partial u_k}{\partial n}\right|_{\Gamma_n} = \gamma_k(y),\qquad lu_k|_{\Gamma_b} = g. \quad (2)$$

Here $u_k(x,y)$, $k = 1,\dots,N \gg 1$ are the solutions to the partial BVP (1), (2), where l is a linear operator of the boundary conditions at Γ_b. The end goal is to find the functions $v_k(y) = u_k(a,y)$ on Γ_n and $v_k(y) = \partial u_k/\partial n$ on Γ_d.

To obtain the solution to the problem, the domain Ω is discretized by a non-uniform $(L+1)\times(M+1)$ rectangular mesh with the property that the boundary crosses the grid lines in the nodes only. The finite volume approach is used for the approximation of the BVP. The resulting 5-point grid equations (for the $k-$th problem)

$$p_{00}u_{i,j} - p_{03}u_{i+1,j} - p_{04}u_{i,j+1} - p_{01}u_{i-1,j} - p_{02}u_{i,j-1} = f_{i,j}, \quad (3)$$
$$i = 1,\dots,L+1,\quad j = 1,\dots,M+1,\quad f_{i,j} = 0,\quad i > 1$$

are symmetrizable, positive definite and monotone. For the sake of brevity, the indices (i,j) of the coefficients are omitted here, and the stencil nodes (i,j), $(i-1,j)$, $(i,j-1)$, $(i+1,j)$, $(i,j+1)$ will be referred to later as 0, 1, 2, 3, 4, respectively.

The boundary conditions in system (3) are supposed to be already taken into account.

3. Diagonal transfer method

The direct method of the diagonal bond transfer [3] consists of three main steps:

I. Transformation of the original 2D system (3) into the auxiliary 1D subsystems for the left grid column

$$Bv_k = g_k \tag{4}$$

where B is a tridiagonal matrix of order $M+1$ and

$$v_k = Ru_k, \quad u_k = \{u_{1,j}^{(k)}, \quad j = 1, \ldots, M+1\}, \tag{5}$$

$$g_k = Sf_k, \quad f_k = \{f_{1,j}^{(k)}, \quad j = 1, \ldots, M+1\} \tag{6}$$

are intermediate unknowns and the right-hand side vectors. The matrices R and S are the triangular ones whose entries are defined via the original coefficients in (3).

II. Solving (4) by the usual sweeping method, i.e., factorization of the matrix B into the product $B = LU$ of the two-diagonal triangular matrices and solution of the auxiliary systems

$$L\overline{z}_k = \overline{g}_k, \qquad U\overline{v}_k = \overline{z}_k, \qquad k = 1, \ldots, N. \tag{7}$$

III. Reconstruction of the original solution on the line $x = a$:

$$u_k = R^{-1}v_k, \quad k = 1, \ldots, N. \tag{8}$$

Briefly, the algorithm can be described as a sequence of the diagonal sweeps (in detail, see [2–4]). If we refer to the diagonal sweep by its first (bottom right) node, a sequence of sweeps can be represented by the sequence of index pairs

$$\begin{array}{cccc} (L,1), & \ldots, & (L,M+1), \\ (L-1,1), & \ldots, & (L-1,M+1), \\ & \ldots, & \\ (2,1), & \ldots, & (2,M+1). \end{array}$$

Let us consider transformation of the equations for one grid 5-point stencil with the central node (i,j). Initially, all five equations of the stencil are of the form (3). The sequence of the shifts for one diagonal consists of the Gauss elimination of the first (bottom right) node $(i+1, j-1)$ which results in to the appearance of a new diagonal link 2–3 $((i,j-1)$–$(i+1,j))$ and the subsequent shifts of this link. The diagonal shift brings about recalculation of the initial coefficients and the change of the unknown variables in the equations for the stencil under consideration. The resulting five equations can be described by the following formulas:

$$\frac{q}{\gamma\alpha}p_{00}p_{03}\overline{u}_0 - \frac{p_{02}}{\alpha}u_2 - \frac{p_{03}}{\alpha}u_3 - \frac{q}{\gamma}p_{03}p_{01}u_1 - \frac{q}{\gamma}p_{03}p_{04}u_4 - = \frac{q}{\gamma}p_{03}f_0, \quad (9)$$

$$-\frac{p_{02}}{\alpha}\overline{u}_0 + \Big(\frac{\beta}{\alpha}p_{02} + p_{22}\Big)u_2 = \tilde{f}_2, \quad (10)$$

$$-\frac{p_{03}}{\alpha}\overline{u}_0 + \Big(\frac{\gamma}{\alpha}p_{03} + p_{33}\Big)u_3 = \tilde{f}_3, \quad (11)$$

$$-\frac{q}{\gamma}p_{04}p_{03}\overline{u}_0 + (p_{44} - qp_{04}^2)u_4 - qp_{04}p_{01}u_1 = \tilde{f}_4 + qp_{04}f_0, \quad (12)$$

$$-\frac{q}{\gamma}p_{01}p_{03}\overline{u}_0 + (p_{11} - qp_{01}^2)u_1 - qp_{04}p_{01}u_4 = \tilde{f}_1 + qp_{01}f_0, \quad (13)$$

$$q = \Big(p_{00} + \frac{\alpha}{\gamma}p_{03}\Big)^{-1}.$$

Here $\tilde{f}_n$ is the sum of f_n and the links of the node n with the nodes different from these with the local numbers 0, 2, 3. Also, the new variable with some parameters α, β, γ

$$\overline{u}_0 = \alpha u_0 + \beta u_2 + \gamma u_3, \quad (14)$$

was introduced in the triangle 0–2–3. One can see that the diagonal link 2–3 has moved in these equations to the new link 1–4. The coefficients of the new equations are symmetric. The cutting-off condition of the link 2–3 results in the equalities $\gamma p_{02} = \alpha p_{23}$ or $\beta p_{03} = \alpha p_{23}$. To obtain the new system of the positive type to provide the correctness and numerical stability of the algorithm, the parameter α is taken as follows:

$$\alpha = 1 + p_{23}\frac{p_{02} + p_{03}}{p_{02}p_{03}}. \quad (15)$$

After one diagonal shift, we move to the stencil with the central node $(i-1, j+1)$ and repeat the diagonal shift till its vanishing on the left grid line. Then, the next Gauss elimination for the new diagonal follows, etc.

Stage II of the algorithm is easily implemented by the back sweep. Stage III reconstructs the initial unknowns from the following formulas:

$$u_0 = \alpha_1 + \beta_1 u_4 + \gamma_1 \overline{u}_0, \quad \alpha_1 = qf_0, \quad \beta_1 = qp_{04}, \quad \gamma_1 = \frac{qp_{03}}{\gamma}. \quad (16)$$

Here the coefficients α_1, β_1, γ_1 correspond to the diagonal link $(1, j)$–$(2, j+1)$ exclusion. As the exclusions are multiple for each of such diagonals, the latter coefficients should be saved for each exclusion.

4. Some details of algorithm

Let us consider certain cases which should be especially processed to make the code of the algorithm robust.

The first case arises from the Dirichlet boundary conditions on the left side grid line. From (15) it follows that for the correct implementation of the algorithm, the inequalities $p_{02} \neq 0$, $p_{03} \neq 0$ should be fulfilled. These conditions are violated if the node (i, j) is on the boundary with the Dirichlet condition. To make the diagonal transfer possible, the new unknown corresponding to the mid point (let us give it the local number $0'$) on the diagonal link p_{23} is introduced:

$$u_{0'} = (u_2 + u_3)/2. \tag{17}$$

The equations for nodes 2 and 3 look like

$$\begin{aligned} p_{22}u_2 - p_{23}u_3 &= f_2, \\ p_{33}u_3 - p_{23}u_2 &= f_3. \end{aligned} \tag{18}$$

Let us derive u_3 and u_2 from (17) and substitute them into the latter equations. We arrive at

$$p_{22}u_2 - p_{23}(2u_{0'} - u_2) = f_2, \tag{19}$$

$$p_{33}u_3 - p_{23}(2u_{0'} - u_3) = f_3. \tag{20}$$

Thus, the new system without the diagonal link consists of equations (19), (20) and is scaled by the factor $4p_{23}$ for the sake of symmetry of the coefficients. To reconstruct the solution for such a Dirichlet node, the coefficients α_1, β_1, γ_1 in (16) are set equal to f_0 (the Dirichlet value), 0, 0, respectively.

The second peculiarity is that the value of the diagonal link decreases while the algorithm processes the nodes from left to right. At some step of moving the diagonal, the value can be equal to underflow zero. In this case, we should save the restoring coefficients $\alpha_1 = 0$, $\beta_1 = 0$, $\gamma_1 = 1$ for reconstruction (16) and go to the Gauss elimination of the first node on the next diagonal.

5. Computational complexity

The estimates of the computational complexity of method I–III for $L, M, N \gg 1$ are the following. Let us denote the numbers of arithmetical operations and memory, in the sense of the double precision real values, by P and Q, respectively.

Transformation of system (3) to subsystem (4) needs

$$P_1 = \begin{cases} 7(L^2M - L^3/3), & L \leq M, \\ 7(LM^2 - M^3/3), & L \geq M, \end{cases}$$

(i.e., $P_1 \leq 7LM \min\{L, M\}$) operations and the memory $Q_1 = 3LM$; no additional memory is required to store the coefficients of Stage I which defines the matrix B.

The number of operations to compute vectors g_k, $k = 1, \ldots, N$, is

$$P_2 = \begin{cases} 2NM(M-1), & M \le L, \\ 2N(M - L/2)(L-1), & M \ge L. \end{cases}$$

The complexity of the sweeping method is negligibly small, namely, $P_3 = 3M + 5MN$, and for the reconstruction step (16), $P_4 = P_2$ operations are needed.

So, for $M > L$, the total volumes of the arithmetical operations and the memory are

$$P = \sum_{l=1}^{4} P_l < 4NLM, \qquad Q = 3LM + MN.$$

We can compare these estimates with the computational complexity of the widespread direct and iterative solvers. For example, for the block Gauss elimination method, the volume of arithmetical operations equals $P = 2M^3L/3 + 2M^2N$ [8]. For the fast iterative implicit incomplete factorization method with the conjugate gradients [8] we have $P = N_n LMN_{it}N$, where N_{it} is the number of iterations, $N_n \approx 30$ is the number of arithmetical operations per node per iteration.

Thus, for $N = 500$ and $L \times M = 30 \times 500$, e.g., the proposed code can solve the problem more than 100 times faster than both classical direct or iterative solvers.

The considered approaches provide the robust implementation and high numerical stability of the proposed algorithm. In a wide set of computer model experiments for grids of dimensions of up to 400×400 and for strongly variable mesh steps and coefficients of the equation being solved, the accuracy of the results is about 10^{-10} despite of a very big conditional number of the algebraic system.

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