Bull. Nov. Comp. Center, Num. Anal., 13 (2005), 33–42

On the finite volume solution of the 1D parabolic nonlinear equation*

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Abstract. The mixed finite volume approach is described for solving the one-dimensional nonlinear parabolic equation on the non-uniform grid. The estimates of the truncation errors for the approximate solution as well as for the flux are investigated. The results of numerical experiments for the two model problems are presented.

1. Introduction

In this paper, we consider the finite volume approach for the numerical solution of the mixed boundary value problem (BVP) for the 1D nonlinear parabolic equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{1}{x^\alpha}\frac{\partial}{\partial x}\Big(x^\alpha a\frac{\partial u}{\partial x} + bu\Big) + cu + f, \\ &0 \le L_0 < x < L_1, \quad 0 < t \le T < \infty, \end{aligned} \tag{1}$$

where the coefficients $a > 0$, $c \le 0$, b and the function f depend on the variables x, t and, possibly, on the unknown function u, also. In (1), the values $\alpha = 0$ and $\alpha = 1$ correspond to the Cartesian and the cylindrical spatial coordinates.

The initial value $u(x, t = 0) = u^0(x)$ is supposed to be given, and the boundary conditions are written down formally as

$$\begin{aligned} &\Big(\alpha_0 u + \beta_0\frac{\partial u}{\partial x}\Big)\Big|_{x=L_0} = \gamma_0, \quad \Big(\alpha_1 u + \beta_1\frac{\partial u}{\partial x}\Big)\Big|_{x=L_1} = \gamma_1, \\ &\alpha_0\beta_0 \le 0, \quad \alpha_1\beta_1 \ge 0, \quad |\alpha_0| + |\beta_0| \ne 0, \quad |\alpha_1| + |\beta_1| \ne 0. \end{aligned} \tag{2}$$

So, different kinds of the BVPs (Dirichlet, Neumann, Robin, and mixed type) are included into the statement. Let us remark that in the physical sense, a more natural boundary condition presents the given flux

$$v(x) = \Big(-x^\alpha a\frac{\partial u}{\partial x} - b\,u\Big)\Big|_{L_k} = \gamma_k, \quad k = 0, 1,$$

*Supported by the Russian Foundation for Basic Research under Grant 05-01-00487 and NWO-RFBR under Grant 047.016.008.

and, in this case if $\alpha > 0$ and $L_0 = 0$, the condition of boundedness of the solution is $\lim\limits_{x\to 0} v(x) = 0$ [1].

Investigations of similar problems by the finite element and other methods were carried out, especially, for the linear case, by many authors (see e.g., [1–3] and the references therein). Application of the mixed FVM for a similar stationary diffusion problem was considered in [4]. The aim of this paper is to extend the results considered in [4] to a nonstationary nonlinear problem with an additional convection term.

In Section 2, we analyze the features of the mixed FVM for problems (1), (2), under the assumption that the desired solution is piecewise smooth. In Section 3, the results of numerical experiments for the Burger equation [5] and the linearized filtration problem [6] are discussed.

2. The mixed finite volume algorithms

Let us rewrite equations (1) in the following mixed formulation:

$$\frac{\partial v}{\partial x} = x^\alpha\Big(cu - \frac{\partial u}{\partial t} + f\Big), \tag{3}$$

$$v = -x^\alpha a\frac{\partial u}{\partial x} - bu. \tag{4}$$

Here, the flux function v can be represented as

$$v = -\frac{x^\alpha a}{\mu}\frac{\partial \mu u}{\partial x}, \quad \mu = e^\eta, \quad \eta = \int\limits_0^x \frac{b\,d\zeta}{\zeta^\alpha a}. \tag{5}$$

For approximation of system (3), (4), we introduce a non-uniform grid

$$\begin{aligned}&x_{i+1} = x_i + h_i, \quad x_0 = L_0, \quad x_{I+1} = L_1, \quad i = 0,1,\dots,I,\\ &t_{n+1} = t_n + \tau_n, \quad n = 0,1,\dots\end{aligned}$$

After integrating equations (3) and (5) over the intervals $[x_{i-1/2}, x_{i+1/2}]$, $x_{i\pm1/2} = (x_i + x_{i\pm1})/2$, and $[x_i, x_{i+1}]$, respectively, we obtain the exact relations

$$v_{i+1/2} - v_{i-1/2} = \int\limits_{x_{i-1/2}}^{x_{i+1/2}} g\,dx, \quad g = x^\alpha\Big(cu - \frac{\partial u}{\partial t} + f\Big), \tag{6}$$

$$(\mu u)_i - (\mu u)_{i+1} = \int\limits_{x_i}^{x_{i+1}} \frac{v\mu}{x^\alpha a}\,dx. \tag{7}$$

Using the piecewise linear interpolation of the function g in (6), we have

$$\begin{aligned}&v_{i+1/2} - v_{i-1/2} = \hat g_i + O(h^3), \quad h = \max_i\{h_i\},\\ &\hat g_i = [3(h_i + h_{i-1})g_i + h_{i-1}g_{i-1} + h_i g_{i+1}]/8.\end{aligned} \tag{8}$$

Approximation of the integral in (7) provides the relation

$$
\begin{aligned}
(\mu u)_i - (\mu u)_{i+1} &= v_{i+1/2} \int\limits_{x_i}^{x_{i+1}} \frac{\mu dx}{x^\alpha a} + O(h^3) \\
&= v_{i+1/2} \int\limits_{x_i}^{x_{i+1}} e^{\chi_{i+1/2} + (x - x_{i+1/2})(\chi_{i+1} - \chi_i)/h_i} dx + O(h^3) \\
&= v_{i+1/2} e^{\chi_{i+1/2}} \frac{h_i}{\chi_{i+1} - \chi_i} \Big(e^{\frac{\chi_{i+1} - \chi_i}{2}} - e^{-\frac{\chi_{i+1} - \chi_i}{2}} \Big) + O(h^3),
\end{aligned}
$$

where the function χ and its differences are defined as

$$
\begin{aligned}
&\chi = \eta - \xi, \quad \xi = \ln(x^\alpha a) \\
&\chi_{i+1} - \chi_i = \eta_{i+1} - \eta_i - \xi_{i+1} + \xi_i \\
&\qquad\qquad = h_i \frac{b_i + b_{i+1}}{x^\alpha_{i+1/2}(a_i + a_{i+1})} + \ln \frac{x_i^\alpha a_i}{x^\alpha_{i+1} a_{i+1}} + O(h^3).
\end{aligned}
$$

From the latter, we can find the flux

$$
v_{i+1/2} = \Big(e^{\eta_i - \eta_{i+1/2}} u_i - e^{\eta_{i+1} - \eta_{i+1/2}} u_{i+1} \Big) \frac{(\chi_{i+1} - \chi_i) x^\alpha_{i+1/2} a_{i+1/2}}{h_i (e^{\frac{\chi_{i+1} - \chi_i}{2}} - e^{-\frac{\chi_{i+1} - \chi_i}{2}})} + O(h^3). \tag{9}
$$

If we write a similar relation for $v_{i-1/2}$ and substitute it into (8), we will obtain the following equations:

$$
(\hat{A}u)_i \equiv \hat{a}_{i,i-1} u_{i-1} + \hat{a}_{i,i} u_i + \hat{a}_{i,i+1} u_{i+1} = \hat{g}_i + O(h^3), \ \ i = 1, 2, \ldots, I, \tag{10}
$$

where the values $\tilde{g}_i$ depend on u and $\dfrac{\partial u}{\partial t}$, and the entries of the matrix $\tilde{A}$ are defined as

$$
\hat{a}_{i,i-1} = -\bar{a}_{i,i-1}\mu_{i-1}, \ \ \hat{a}_{i,i+1} = -\bar{a}_{i,i+1}\mu_{i+1}, \ \ \hat{a}_{i,i} = (\bar{a}_{i,i-1} + \bar{a}_{i,i+1})\mu_i. \tag{11}
$$

Here we use the notations

$$
\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = -\frac{(\chi_{i+1} - \chi_i) x^\alpha_{i+1/2} a_{i+1/2} \mu_{i+1/2}}{h_i (e^{\frac{\chi_{i+1} - \chi_i}{2}} - e^{-\frac{\chi_{i+1} - \chi_i}{2}})}.
$$

If we use the definition of $\tilde{g}_i$ and the time discretization of (10) at the moment $t_{n+\theta} = t_n + \theta\tau_n$, $0 \le \theta \le 1$, the following relations are obtained:

$$
\frac{1}{\tau_n} B^h (u^{n+1} - u^n) + (A^h u)^{n+\theta} = f_h^{n+\theta} + \psi_u^{n+\theta}, \quad A^h = \hat{A}^h + \check{A}^h, \tag{12}
$$

where

$$u^n = \{u(x_i, t_n)\}, \quad u^{n+\theta} = \theta u^{n+1} + (1-\theta)u^n,$$
$$\psi_u^{n+\theta} = O(h^3 + h\tau^2 + h(1-2\theta)\tau), \quad f_h^{n+\theta} = \{\tilde{f}_i^{n+\theta} = (B^h f)_i^{n+\theta}\};$$

$B^h = \{b_{i,j}\}$ and $\breve{A}^h = \{\breve{a}_{i,j}\}$ are tridiagonal matrices with the entries

$$b_{i,i-1} = h_{i-1}x_{i-1}^{\alpha}, \quad b_{i,i+1} = h_i x_{i+1}^{\alpha}, \qquad b_{i,i} = (h_{i-1} + h_i)x_i^{\alpha},$$
$$\breve{a}_{i,i-1} = b_{i,i-1}c_{i-1}, \quad \breve{a}_{i,i+1} = b_{i,i+1}c_{i+1}, \quad \breve{a}_{i,i} = (h_{i-1}c_{i-1} + h_i c_i)x_i^{\alpha}.$$

If the matrix elements in (12) depend on time, the matrix-vector product is defined as

$$(A^h u)^{n+\theta} = \theta A_{n+1}^h u^{n+1} + (1-\theta)A_n^h u^n,$$

and the entries of A_n^h are used at n-th time step. Due to the properties of the coefficients a, c from (1) and those of the matrix entries from (10), (11), it follows that $A = \{a_{i,j}\}$ is M-matrix (see [1, 2]), i.e., $A^{-1} \geq 0$ because of the column diagonal dominance and non-positiveness of the off-diagonal entries:

$$a_{i,i\pm 1} \leq 0, \quad a_{i,i} + a_{i,i-1} + a_{i,i+1} \geq 0, \tag{13}$$

with, at least, one strong inequality in the last relation. The only exclusion is a pure Neumann BVP with $\alpha_0 = \alpha_1 = c = 0$ in (1), (2).

The boundary conditions (2) are taken into account without any additional error. If $\beta_0 = 0$ and/or $\beta_1 = 0$, then the equations

$$u_0 = \gamma_0/\alpha_0, \quad u_{I+1} = \gamma_1/\alpha_1, \tag{14}$$

present an "exact approximation" of the Dirichlet conditions. And if $\beta_0 \neq 0$, for example, then, in addition to equations (6) for $i = 1, 2, \ldots$, the boundary equation

$$v_{1/2} - v_0 = \int_{x_0}^{x_{1/2}} g\,dx \tag{15}$$

is introduced. The term $v_{1/2}$ in (14) is approximated according to (9), as usual, but v_0 is defined, with the help of (4), as

$$v_0 = \Big(\frac{\alpha_0}{\beta_0}x_0^{\alpha}a_0 - b_0\Big)u_0 - \frac{\gamma_0}{\beta_0}x_0^{\alpha}a_0. \tag{16}$$

Therefore, using the spatial and time approximation of the right-hand side in (15) and substituting v_0 from (16) and $v_{1/2}$ from (9) into (15), we obtain the equation for $i = 0$, similar to (12). Also, if $\beta_1 \neq 0$, we define an additional flux equation

$$v_{I+1} - v_{I+1/2} = \int\limits_{x_{I+1/2}}^{x_{I+1}} g\,dx$$

and the exact boundary condition

$$v_{I+1} = \Big(\frac{\alpha_1}{\beta_1} x_{I+1}^{\alpha} a_{I+1} - b_{I+1}\Big) u_{I+1} - \frac{\gamma_1}{\beta_1} x_{I+1}^{\alpha} a_{I+1}, \tag{17}$$

which provides the $(I+1)$-th grid equation, similar to (12).

For a more natural Robin type boundary condition, we have a given flux, for example, at the left boundary

$$-\Big(x^{\alpha} a \frac{\partial u}{\partial x} + bu\Big)\Big|_{x=L_0} = \gamma_0.$$

In this case, equation (16) is reduced to a simpler form $v_0 = \gamma_0$.

After exclusion of the fluxes in the near boundary points, the dimension of the vector u^n is $N = I+2$ in a general case of equation (12), and $N = I$ or $N = I+1$ if we have the Dirichlet or a mixed boundary value problem, respectively.

If we delete the approximation term $\psi^{n+\theta}$ in (12), we will obtain the equation for the grid solution $u_h^n = \{u_i^{h,n}\}$:

$$\frac{1}{\tau_n} B^h (u_h^{n+1} - u_h^n) + \theta A_{n+1}^h u_h^{n+1} + (1-\theta) A_n^h u_h^n = f_h^{n+\theta}. \tag{18}$$

In a similar way, from (9), we define the grid flux

$$v_{i+1/2} = \Big(e^{\eta_i - \eta_{i+1/2}} u_i^n - e^{\eta_{i+1} - \eta_{i+1/2}} u_{i+1}^n\Big) \frac{(\chi_{i+1} - \chi_i) x_{i+1/2}^{\alpha} a_{i+1/2}}{h_i (e^{\frac{\chi_{i+1}-\chi_i}{2}} - e^{-\frac{\chi_{i+1}-\chi_i}{2}})}.$$

It is important to remark that the finite volume scheme obtained is fully conservative, in the sense, that the following exact grid balance holds:

$$\tau_n[\theta(v_{i''+1/2}^{n+1} - v_{i'-1/2}^{n+1}) + (1-\theta)(v_{i''+1/2}^{n} - v_{i'-1/2}^{n})] = \\ \sum_{i=i'}^{i''} \big\{ B^h (u_h^{n+1} - u_h^n)_i + \tau_n \theta [\tilde{g}_i^{n+1} + (\check{A} u_h^{n+1})_i] + \tau_n (1-\theta)[\tilde{g}_i^n + (\check{A} u_h^n)_i] \big\},$$

for any pair of the indices i', i''. It is evident that each side of this equation presents the approximation of the corresponding term of the conservative law, or the balance equation:

$$\int\limits_{t_n}^{t_{n+1}} (v_{i''+1/2} - v_{i'-1/2})\,dt = \int\limits_{x_{i'-1/2}}^{x_{i''+1/2}} x^\alpha (u^{n+1} - u^n)\,dx + \int\limits_{t_n}^{t_{n+1}} dt \int\limits_{x_{i'-1/2}}^{x_{i''+1/2}} (cu - f) x^\alpha\,dx.$$

It can be mentioned, that in [1], a similar exponential type scheme was considered on the uniform grid, but it was a nonconservative one.

Now, by means of the term-by-term subtraction of equations (12) and (18) we obtain the following linearized equation for the grid solution error vectors $z_h^n = \{u(x_i, t_n) - u_i^{h,n}\}$:

$$\frac{1}{\tau_n} B^h (z_h^{n+1} - z_h^n) + \theta (A_{n+1}^h + C_{n+1}^h) z_h^{n+1} + (1-\theta)(A_n^h + C_n^h) z_h^n = \psi_u^{n+\theta}. \quad (19)$$

Here $C_n^h = \{c_{i,j}\}$ is a tridiagonal matrix, nonsymmetric and indefinite, in general, whose entries consist of derivatives of the matrix elements $a_{i,j}$ and can be understood from the following example:

$$\begin{aligned} & a_{i,i}(u_i, u_{i-1}, u_{i+1}) u_i - a_{i,i}(u_i^h, u_{i-1}^h, u_{i+1}^h) u_i^h + \\ & \quad a_{i,i}(u_i, u_{i-1}, u_{i+1}) u_i^h - a_{i,i}(u_i, u_{i-1}, u_{i+1}) u_i^h \\ & \qquad = a_{i,i}(u_i, u_{i-1}, u_{i+1}) z_i^h + \frac{\partial a_{i,i}}{\partial u_i}(u_i^*) u_i^h z_i^h + \\ & \qquad\quad \frac{\partial a_{i,i}}{\partial u_{i-1}}(u_{i-1}^*) u_i^h z_{i-1}^h + \frac{\partial a_{i,i}}{\partial u_{i+1}}(u_{i+1}^*) u_i^h z_{i+1}^h \\ & \qquad = a_{i,i} z_i^h + c_{i,i} z_i^h + c_{i,i-1} z_{i-1}^h + c_{i,i+1} z_{i+1}^h, \end{aligned}$$

where the values of derivatives are defined for some arguments $u_{i\pm1}^* \in [u_{i\pm1}, u_{i\pm1}^h]$. In other words, the matrix C_n^h is defined from the vector-matrix relation

$$A^h(u^n) u^n - A^h(u^{h,n}) u^{h,n} = (A^h(u^n) + C^h(u^*)) z_h^n.$$

The error equation can be written down in the form (we omit here and further the index "h" for matrices and vectors)

$$(B + \tau_n \theta M_{n+1}) z^{n+1} = [B - \tau_n (1-\theta) M_n] z^n + \tau_n \psi_u^{n+\theta}, \quad (20)$$

where $M_{n+1} = A_{n+1} + C_{n+1} = \{m_{i,j}^{n+1} = a_{i,j}^{n+1} + c_{i,j}^{n+1}\}$.

Let us now, for simplicity, suppose that nonlinearity of the function a, c, and f in (1) is not too strong, so that $B + \tau_n \theta M_{n+1}$ is M-matrix and conditions the similar to (13) are valid for all the rows:

$$\begin{aligned} b_{i,i\pm1} + \tau_n \theta m_{i,i\pm1} &\le 0, \\ b_{i,i} + b_{i-1,i} + b_{i+1,i} + \tau_n \theta (m_{i,i} + m_{i-1,i} + m_{i+1,i}) &\ge 0. \end{aligned} \quad (21)$$

Moreover, we assume, that for some positive vector $w = \{w_i > 0\}$ with the norm $\|w\|_\infty = \max_i |w_i| = 1$ the following inequality holds:

$$(B + \tau_n \theta M_{n+1}) w \geq (1 + \tau_n \varkappa_0)^{-1} [B - \tau_n (1-\theta) M_n] e, \tag{22}$$

where e is the vector with unit entries. Here and below the constant $\varkappa_k$, $k = 0, 1, \ldots$, means the value which does not depend on $\tau = \max_n\{\tau_n\}$, h, and $\check{h} = \min_i\{h_i\}$. Conditions (21), (22) provide the matrix norm inequality, see [2]:

$$\|(B + \tau_n \theta M_{n+1})^{-1} [B - \tau_n (1-\theta) M_n]\|_\infty \leq 1 + \tau_n \varkappa_0. \tag{23}$$

One more matrix property is assumed: $\|(B + \tau_n \theta M_{n+1})^{-1}\| \leq \varkappa_1/\check{h}$, from which, together with (23) and (20), we obtain the inequality

$$\|z^{n+1}\|_\infty \leq (1 + \tau_n \varkappa_0) \|z^n\|_\infty + \tau_n \|\psi^{n+\theta}\|_\infty / \check{h}.$$

Now we can formulate the main result of the previous consideration.

Theorem 1. *Let conditions* (21)*,* (22) *be satisfied for all* $n = 0, 1, \ldots$ *Then the grid solution error has the norm* $\|z^n\|_\infty = O(\tau^2 + h^2)$ *for* $\theta = 1/2$ (*the Crank–Nicolson scheme*) *and* $\|z^n\|_\infty = O(\tau + h^2)$ *for* $1/2 < \theta \leq 1$.

We can remark that for $\theta = 1$ (a pure implicit scheme) this means an unconditionable convergence of the grid solution, but for $\theta < 1$, the convergence in the infinite norm demands the condition $\tau = O(h^2)$.

Let us now consider the grid flux error vector

$$y_h^n = \{y_{i+1/2}^{h,n} = v(x_{i+1/2}, t_n) - v_{i+1/2}^{h,n}\},$$

where the grid flux vector $v_h^n = \{v_{i+1/2}^{h,n}\}$ satisfies for $\theta > 0$ the equations

$$\begin{aligned} v_{i+1/2}^{h,n+\theta} - v_{i-1/2}^{h,n+\theta} &= \Big[\frac{1}{\tau_n} B^h (u_h^{n+1} - u_h^n) + \theta \bar{A}_{n+1} u_h^{n+1} + (1-\theta) \bar{A}_n u_h^n\Big]_i, \\ v^{h,n+\theta} &= \theta v^{h,n+1} + (1-\theta) v^{h,n}. \end{aligned} \tag{24}$$

After subtraction of equations (8) (after their approximation in the variable t) and (24) we obtain

$$\begin{aligned} y_{i+1/2}^{h,n+\theta} - y_{i-1/2}^{h,n+\theta} = {} & \Big[\frac{1}{\tau_n} B^h (z_h^{n+1} - z_h^n) + \theta \bar{A}_{n+1} z_h^{n+1} + (1-\theta) \bar{A}_n z_h^n\Big]_i + \\ & O(h^3 + h\tau^2 + h(1-2\theta)\tau). \end{aligned} \tag{25}$$

If, for $x = L_0$, $\beta_0 \neq 0$ and/or, for $x = L_1$, $\beta_1 \neq 0$, then at any time step for $i = 0$ and/or $i = I + 1$ the boundary grid fluxes are defined by equations (16) and/or (17), and the corresponding flux errors are defined by similar equations.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then the following estimates for the flux error are valid:*

$$\|y^{h,n+\theta}\|_\infty = \begin{cases} O(\tau^2 + h^2), & \text{for } \theta = 1/2, \\ O(\tau + h^2), & \text{for } 1/2 < \theta \leq 1. \end{cases}$$

The proof of these results follows from the recurrent relation, obtained from (25) and (19):

$$y_{i+1/2}^{h,n+\theta} = y_{i-1/2}^{h,n+\theta} - [\theta(\tilde{A}_{n+1} + C_{n+1})z_h^{n+1} + (1-\theta)(\tilde{A}_n + C_n)z_h^n]_i + O(h^3 + h\tau^2 + h(1-\theta)\tau).$$

Remark. If for $x = L_0$ and/or $x = L_1$ the Dirichlet boundary condition is given, then it can be stated from system (20) that z_1^n and/or z_I^n has the order $O(h^3 + h^2\tau^2 + h^2(1-2\tau)\tau)$ for all n, see [3]. In such a case, the estimates $y_{1/2}^n = O(h^2 + h\tau^2 + h(1-2\tau)\tau)$ and/or $y_{I+1/2}^n = O(h^2 + h\tau^2 + h(1-2\tau)\tau)$ can be obtained with the help of equation (9).

3. Numerical experiments

In this section, we consider the results of numerical experiments for the two problems. The first one presents the nonlinear Burger equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \delta\frac{\partial^2 u}{\partial x^2}, \quad \delta = \text{const} > 0, \quad 0 < x < 1,$$

which has an analytical solution

$$u(x,t) = \delta\Big[1 - \tanh\frac{1}{2}(x - \delta t)\Big]. \tag{26}$$

This problem was solved by the Crank–Nicolson scheme under the Dirichlet boundary conditions and the initial value, defined by formula (26), at the time interval $0 < t \le T = 1.28$. In Tables 1, 2 we give the values of the errors of solution

$$\Delta_u = \max_i |u(x_i, T) - u^h(x_i, T)|,$$

and the errors of the fluxes

$$\Delta_v = \max_i |v(x_{i+1/2}, T) - v^h(x_{i+1/2}, T)|,$$

obtained on different uniform grids with the meshsteps τ and $h = 1/(I+1)$.

An analysis of these data demonstrates sufficiently well the second order of accuracy in terms of h and τ both (it is evident for the small time steps and small space steps, respectively). In this experiments about 5 nonlinear iterations provide a high accuracy ($\approx 10^{-9}$) at each time step.

Table 1. Grid solution error for Burger equation

I	$\tau = 0.04$	$\tau = 0.02$	$\tau = 0.01$	$\tau = 0.005$
8	$4.76 \cdot 10^{-5}$	$4.57 \cdot 10^{-5}$	$4.52 \cdot 10^{-5}$	$4.51 \cdot 10^{-5}$
16	$1.37 \cdot 10^{-5}$	$1.18 \cdot 10^{-5}$	$1.13 \cdot 10^{-5}$	$1.12 \cdot 10^{-5}$
32	$5.29 \cdot 10^{-6}$	$3.43 \cdot 10^{-6}$	$2.96 \cdot 10^{-6}$	$2.80 \cdot 10^{-6}$
64	$3.19 \cdot 10^{-6}$	$1.32 \cdot 10^{-6}$	$8.57 \cdot 10^{-7}$	$7.40 \cdot 10^{-7}$
128	$2.66 \cdot 10^{-6}$	$7.98 \cdot 10^{-7}$	$3.31 \cdot 10^{-7}$	$2.14 \cdot 10^{-7}$
256	$2.53 \cdot 10^{-6}$	$6.67 \cdot 10^{-7}$	$2.00 \cdot 10^{-7}$	$8.28 \cdot 10^{-8}$
512	$2.51 \cdot 10^{-6}$	$6.35 \cdot 10^{-7}$	$1.67 \cdot 10^{-7}$	$5.00 \cdot 10^{-8}$
1024	$2.50 \cdot 10^{-6}$	$6.26 \cdot 10^{-7}$	$1.58 \cdot 10^{-7}$	$4.17 \cdot 10^{-8}$

Table 2. Grid flux error for Burger equation

I	$\tau = 0.04$	$\tau = 0.02$	$\tau = 0.01$	$\tau = 0.005$
8	$3.46 \cdot 10^{-4}$	$3.37 \cdot 10^{-4}$	$3.34 \cdot 10^{-4}$	$3.34 \cdot 10^{-4}$
16	$9.80 \cdot 10^{-5}$	$8.80 \cdot 10^{-5}$	$8.55 \cdot 10^{-5}$	$8.49 \cdot 10^{-5}$
32	$3.51 \cdot 10^{-5}$	$2.47 \cdot 10^{-5}$	$2.21 \cdot 10^{-5}$	$2.15 \cdot 10^{-5}$
64	$1.93 \cdot 10^{-5}$	$8.86 \cdot 10^{-6}$	$6.23 \cdot 10^{-6}$	$5.56 \cdot 10^{-6}$
128	$1.54 \cdot 10^{-5}$	$4.88 \cdot 10^{-6}$	$2.22 \cdot 10^{-6}$	$1.56 \cdot 10^{-6}$
256	$1.45 \cdot 10^{-5}$	$3.89 \cdot 10^{-6}$	$1.22 \cdot 10^{-6}$	$5.57 \cdot 10^{-7}$
512	$1.43 \cdot 10^{-5}$	$3.64 \cdot 10^{-6}$	$9.75 \cdot 10^{-7}$	$3.07 \cdot 10^{-7}$
1024	$1.42 \cdot 10^{-5}$	$3.58 \cdot 10^{-6}$	$9.13 \cdot 10^{-7}$	$2.44 \cdot 10^{-7}$

The second problem in question is a linearized filtration equation [4]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + f(x), \quad 0 < t \le T < \infty,$$
$$u|_{x=0} = u_0, \quad u|_{x=1} = u_{I+1},$$

where the initial data, boundary values, and the function $f(x)$ are defined from the chosen exact solution $u(x,t) = t^2 e^x$.

For this model problem, the Crank–Nicolson scheme provides an exact numerical solution ($\Delta_u = \Delta_v = 0$) on any uniform grid. This phenomenon can be simply explained: for given solution the flux v is constant and truncation error is zero for the proposed approximations.

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