On the solvability of the inverse problem for the parabolic equation

Kh.Kh. Imomnazarov

1. Introduction

The given paper considers the problem of determination of the right hand side of the high-order parabolic equation with the variable coefficients

\[ L(x', \partial_t, \partial_x) \ u(t, x) = f(x_n) \cdot \lambda(t, x'), \quad x = (x', x_n) \in R_n, \quad t > 0, \]

using the information given on the hyperplane \( x_n = 0 \):

\[ u|_{x_n=0} = \psi(t, x'), \]

where \( f(x_n) \) is the known function.

Solvability in the Sobolev weight space is proved.

A sufficiently complete bibliography on the theory of inverse problems can be found in [1–5].

2. Statement of the problem

Let, us consider, in the half-space \( R^{n+1}_n = \{(t, x) \mid t > 0, x \in R_n \} \), the parabolic high-order equation:

\[ L(x', \partial_t, \partial_x) \ u(t, x) = f(x_n) \cdot \lambda(t, x'), \]

(1)

where \( u = u(t, x), \ x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n), \)

\[ L(x', \partial_t, \partial_x) = \partial_t + \sum_{|\alpha|=2m} a_\alpha(x') \partial_x^\alpha, \]

\[ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j}, \]

\[ \alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_j \geq 0, \quad j = 1, 2, \ldots, n. \]

We introduce the following notations: \( r = (1, 2m, \ldots, 2m, 2p) \) — \((n + 1)\)-dimensional vector; \( s = (1, 2m, \ldots, 2m) \) — \(n\)-dimensional vector; \( 2p > 4m + 1 \).
Problem 1. It is necessary to find the functions

\[ (u(t, x), \lambda(t, x')) \in W_{2, \gamma}^p(R_{n+1}^+) \times L_{2, \gamma}^p(R_n^+) \]

(the rest functions are known) from equation (1), if the functions

\[ u |_{t=0} = 0, \quad x \in R_n, \tag{2} \]
\[ u |_{z_n=0} = \psi(t, x), \quad t > 0, \quad x' \in R_{n-1}, \tag{3} \]

are known.

Here \( W_{2, \gamma}^p(R_{n+1}^+) \) denotes the weight Sobolev spaces [6].

Let \( \gamma_0 \) be such a positive number that the inequality

\[ \frac{C}{\gamma_0} \left| \int f \cdot W_{2, \gamma}^{2p}(R_1) \right| \leq \frac{1}{2}, \quad 2p > 4m + 1, \tag{4} \]

\[ C = \frac{|a_{m, n, 0, 2m}(z_0')|}{|f(0)|} \int_{R_1} \frac{\eta^{2m}}{1 + \eta^{2p}} d\eta, \]

\[ \psi(0, x') = 0, \quad x' \in R_{n-1}, \tag{5} \]

and the condition A

\[ \psi \in W_{2, \gamma}^p(R_n^+), \quad f \in W_{2, \gamma}^{2p}(R_1), \quad f(0) \neq 0, \quad \gamma > \gamma_0 \]

are fulfilled.

Theorem 1. Let the coefficients of operator \( L(x', \partial_t, \partial_x) \) be constant. If the conditions A, (4), (5) are fulfilled, then, with \( \gamma > \gamma_0 \), Problem 1 has the unique solution \( u(t, x) \in W_{2, \gamma}^p(R_{n+1}^+), \lambda(t, x') \in L_{2, \gamma}^p(R_n^+) \)

Proof. Let \( x_0' \in R_{n-1} \) be fixed. Consider equation (1) when the coefficients at the point \( x_0' \) are frozen. We reduce Problem 1 to the linear integrodifferential equation by the method of work [1]. We obtain the Cauchy problem from (1)–(3), in the Fourier transform (with respect to the variable \( x \)), of the function \( u(t, x) \) after excluding of the unknown \( \lambda(t, x') \). The solution is given by the formula

\[ u(t, \xi) = \int_0^t e^{-(t-s) L_{2, \gamma}^p(i\xi)} g(t, \xi) ds, \tag{6} \]

where
\[ v(t, \xi) = F_{x \to \xi} [u(t, x)] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(t, x) dx, \]
\[ g(t, \xi) = G(t, \xi) + \frac{\hat{f}(\xi)}{f(0)} \sum_{|\alpha|=2m} a_\alpha x_\alpha e^{i\xi \cdot x} \int_{\mathbb{R}^n} (-i\xi)^{2m} v d\xi, \]
\[ G(t, \xi) = \partial_t \tilde{\psi} + \sum_{|\alpha'|=2m} a_\alpha x_\alpha' e^{i\xi \cdot x'} \frac{\hat{f}(\xi)}{f(0)}, \]
\[ \hat{f}(\xi) = F_{x \to \xi} [f(x)], \quad \tilde{\psi}(t, \xi) = F_{x' \to \xi} [\psi(t, x')], \]
\[ L_{2m}(i\xi) = \sum_{|\alpha|=2m} a_\alpha x_\alpha e^{i\xi \cdot x}. \]

Write formula (6) in the equivalent form using the function \( \theta(t) \),
\[ v(t, \xi) = \int_{R^1} \theta(t-s) e^{-(t-s)} L_{2m}(i\xi) \theta(s) g(s, \xi) ds. \]  

(7)

Estimate the norm of the function \( v(t, \xi) \) in \( L_{2, \gamma}(R^1_+) \). We have
\[ \| v, L_{2, \gamma}(R^1_+) \| \leq \| v, L_{2, \gamma}(R_1) \| \leq \frac{C_1}{\gamma + |\xi|^{2m}} \| g, L_{2, \gamma}(R^1_+) \|. \]  

(8)

Here we have used the Young inequality and
\[ C_1 (\gamma + |\xi|^{2m}) \leq |\gamma + L_{2m}(i\xi)| \leq C_2 (\gamma + |\xi|^{2m}). \]

Estimate the norm \( \frac{1 + |\xi|^{2p}}{\gamma + |\xi|^{2m}} \| g, L_{2, \gamma}(R^1_+) \| \), using the explicit form of the function \( g(t, \xi) \):
\[ \frac{1 + |\xi|^{2p}}{\gamma + |\xi|^{2m}} \| g, L_{2, \gamma}(R^1_+) \| \leq \frac{1}{\gamma + |\xi|^{2m}} \| (1 + |\xi|^{2p}) G, L_{2, \gamma}(R^1_+) \| + \frac{C}{\gamma_0} \| (1 + |\xi|^{2p}) v, L_{2, \gamma}(R^1_+) \| \cdot \| (1 + |\xi|^{2p}) \hat{f}, L_{2, \gamma}(R_1) \|. \]  

(9)

From estimates (8), (9) with allowance for (4), we obtain
\[ \| (1 + |\xi|^{2p} + |\xi|^2) \| v, L_{2, \gamma}(R^1_+) \| \leq 2 \| (1 + |\xi|^{2p}) G, L_{2, \gamma}(R^1_+) \|. \]

Hence, by the Plancherel theorem it follows that
\[ u(t, x) = F_{\xi \to x}^{-1} [v(t, \xi)] \in W_{2, \gamma}(R^1_+). \]

The uniqueness is proved in a standard way.

\[ \square \]

**Corollary 1.** If \( \psi \in W_{2, \gamma}^2(R^1_+), f \in W_{2}^{2p}(R_1), \gamma > \gamma_0 \), then the solution to Problem 1 is given by the formulas
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\[ u(t, x) = T_1(f, G_1), \quad G_1(t, x') = \partial_t \psi + \sum_{|\alpha'|=2m} a_{\alpha'}(x'_0) \partial_{x'}^{\alpha'} \psi, \]

\[ \lambda(t, x') = \left( G_1(t, x') + a_{0,\ldots,0,2m}(x'_0) \int_{R_1} (-iz^i_n)^{2m} F_{z_n \rightarrow \xi_n}[u(t, x)] \, d\xi_n \right) / f(0), \]

where \( T_1 : W^{2p}_2(R_1) \times W^{2\gamma}_2(R_1^+) \to W^{2\gamma}_2(R_1^+) \).

**Remark 1.** In fulfilling the conditions of Theorem 1 we estimate the norm

\[ \| \partial_x^{\alpha} T_1(f, G_1), L_{2, \gamma}(R_1^+) \| \leq \frac{C \| f, W^{2p}_2(R_1) \|}{\gamma(2m-|\alpha|)/2m} \| G_1, L_{2, \gamma}(R_1^+) \|. \]  

Choose \( \varepsilon > 0 \) such that the following inequality is fulfilled

\[ \frac{\varepsilon}{|f(0)|} \| f, W^{2p}_2(R_1) \| (C + \frac{C_1}{\gamma_0}) = \eta < 1. \]

**Theorem 2.** Suppose the conditions \( A, (4), (5), (11) \) be fulfilled, and the coefficients \( a_{\alpha}(x') \) satisfy the conditions

\[ a_{\alpha}(x') \equiv a_\alpha, \quad |x'| > r_1 > 0, \quad \sum_{|\alpha|=2m} \sup_{x'} |a_\alpha(x') - a_\alpha(x'_0)| \leq \varepsilon. \]

Then the statement of Theorem 1 is true.

**Proof.** After the exclusion of the unknown function \( \lambda(t, x') \) from (1)–(3) we obtain the Cauchy problem for the integrodifferential equation

\[ L^0(x', \partial_t, \partial_x) u(t, x) \]

\[ \begin{align*}
&= \partial_t u(t, x) + \sum_{|\alpha|=2m} a_\alpha(x') \partial_x^{\alpha} u(t, x) + \\
&= \frac{f(x_n)}{f(0)} a_{0,\ldots,0,2m}(x'_0) \int_{R_1} (-iz^i_n)^{2m} F_{z_n \rightarrow \xi_n}[u(t, x)] \, d\xi_n \\
&= \frac{f(x_n)}{f(0)} a_{0,\ldots,0,2m}(x'_0) \int_{R_1} (-iz^i_n)^{2m} F_{z_n \rightarrow \xi_n}[u(t, x)] \, d\xi_n \\
&= \mu(t, x') f(x_n),
\end{align*} \]

(12)

\[ u|_{t=0} = 0, \quad x \in R_n. \]

(13)

Here \( \mu(t, x') = (\partial_t \psi + \sum_{|\alpha|=2m} a_{\alpha}(x') \partial_{x'}^{\alpha} \psi) / f(0). \)

The solution to the Cauchy problem (12), (13) is sought for in the form

\[ u(t, x) = T_1(f, G_1), \]

(14)

where \( \mu_1(t, x') \in L_{2, \gamma}(R_1^+) \) - the unknown function, the operator \( T_1 \) is defined in Corollary 1.

Apply the operator \( L^0(x'_0, \partial_t, \partial_x) - L^0(x', \partial_t, \partial_x) \) to the function \( u(t, x) \) from (14) and after simple transformations we obtain for \( \mu_1(t, x') \) the equation
\[ \mu_1(t, z') = S_1(\mu_1) + \mu(t, z'). \] (15)

Here \( S_1(\mu_1) = f^{-1}(0) (L^0(z'_0, \partial_t, \partial_x) - L^0(z', \partial_t, \partial_x)) T_1(f, \mu_1) |_{z_n=0}. \)

We shall show that the norm of the operator \( S_1(\mu_1) \) is small in the norm \( L_{2, \gamma}(R^+_n) \), if the coefficients little differ from constants. Really

\[
\|S_1(\mu_1), L_{2, \gamma}(R^+_n)\| \leq \sum_{|\alpha|=2m} \| (a_\alpha(z') - a_\alpha(z'_0)) \partial_\alpha T_1(f, \mu_1, L_{2, \gamma}(R^+_n)) \| + \\
\| (a_{0,0,2m}(z') - a_{0,0,2m}(z'_0)) \times \\
\int_{R^+_1} \left( -i \xi_n \right)^{2m} F_{\xi_n \rightarrow 0} \|T_1(f, \mu_1)\| d\xi_n, L_{2, \gamma}(R^+_n) \| \\
\leq \varepsilon (C + \frac{C_1}{\gamma_0}) \| f, W^{2p}_2(\mathbb{R}_1) \| \cdot \| \mu_1, L_{2, \gamma}(R^+_n) \|.
\]

Here we have used inequality (10). Hence, with allowance for (11), from equation (15), we obtain the estimate

\[
\| \mu_1, L_{2, \gamma}(R^+_n) \| \leq \frac{1}{1 - q_1} \| \mu, L_{2, \gamma}(R^+_n) \|. \] (16)

By the method of successive approximations from equation (15), we find \( \mu_1(t, z') \). Substituting \( \mu_1(t, z') \) into (14), we obtain the solutions of the Cauchy problem (12), (13). The function \( \lambda(t, z') \) is calculated by the formula

\[
\lambda(t, z') = \left( \frac{\partial}{\partial t} \psi + \sum_{|\alpha'|=2m} a_{\alpha'}(z'_0) \partial_{\alpha'} \psi + \\
a_{0,0,2m}(z') \int_{R^+_1} \left( -iz\eta_n \right)^{2m} F_{\eta_n \rightarrow 0} [u(t, z)] d\eta_n \right) / f(0). \] (17)

The uniqueness is proved in a standard way. \( \square \)

**Corollary 2.** The solutions to the Cauchy problem (12), (13) can be written in the operator form

\[ u(t, x) = T_2(f, \mu), \]

where \( T_2 : W^{2p}_2(\mathbb{R}_1) \times W^{2p}_2(\mathbb{R}^+_n) \rightarrow W^{2p}_2(\mathbb{R}^+_n+1) \).

**Remark 2.** \( \forall \mu \in L_{2, \gamma}(R^+_n), f \in W^{2p}_2(\mathbb{R}_1), \gamma > \gamma_0 \) the following estimate is valid

\[
\| \partial_{z_2} T_2(f, \mu), L_{2, \gamma}(R^+_n+1) \| \leq \frac{C \| f, W^{2p}_2(\mathbb{R}_1) \|}{(1 - q_1) \gamma(2m - |\alpha'|)/2m \| \mu, L_{2, \gamma}(R^+_n) \|}. \] (18)
Lemma. Let $G \equiv B_{R+1}(0) = \{x' | x' | < R + 1\}$ be a ball in $\mathbb{R}^{n-1}$. Then \( \forall \delta > 0, \exists y^1, \ldots, y^N \in G, \exists \varphi_0(y), \ldots, \varphi_N(y), \) such that $\varphi_0(y) \in C_0^\infty(R_{n-1})$, $\varphi_0(y) \geq 0$, $\varphi_j(y) \in C_0^\infty(B_{\delta/2}(y^j))$, $j = 1, 2, \ldots, N$. These functions satisfy the conditions

$$
\varphi_j(y) \geq 0, \quad j = 1, 2, \ldots, N,
$$

$$
\sum_{j=0}^{N} \varphi_j(y) = 1 \quad \text{everywhere in } R_{n-1}.
$$

The proof of the lemma follows from the theorem on unit partitions [7]. Choose $\gamma_1 \geq \gamma_0$, such that the following inequality is fulfilled

$$
\frac{CC_4 \|f, W_2^{2p}(R_1)\|}{(1 - q_1)\gamma_1^{1/2m}} = q_2 < 1. \quad (19)
$$

Here $C_4$ is a positive number.

Theorem 3. Under assumptions about infinitely differentiability of the coefficients of the operator $L(x', \partial_t, \partial_x)$, and if they are constants out of the ball $B_R(0)$ and, moreover, conditions $A$, (4), (5), (19) hold, then for $\gamma \geq \gamma_1$ Theorem 1 is valid.

Proof. Let $\delta > 0$. Using the functions $\varphi_0(x'), \ldots, \varphi_N(x')$ from the lemma, we construct the following functions $\tilde{\varphi}_0(y), \ldots, \tilde{\varphi}_N(y), y = x'$:

(a) $\tilde{\varphi}_k(y) \in C_0^\infty(R_{n-1}), \ 0 \leq \tilde{\varphi}_k(y) \leq 1$;
(b) $\tilde{\varphi}_0(y) \equiv 1, y \in \text{supp } \varphi_0(y); \tilde{\varphi}_0(y) \equiv 0, |y| \leq R + 1 - \delta$;
(c) $\tilde{\varphi}_k(y) \equiv 1, |y - y^k| \leq \delta; \tilde{\varphi}_k(y) \equiv 0, |y - y^k| \geq 2\delta; k = 1, 2, \ldots, N$.

Also introduce the functions $\beta_1(y), \ldots, \beta_N(y)$:

$$
\beta_j(y) \in C_0^\infty(R_{n-1}), \quad 0 \leq \beta_j(y) \leq 1,
$$

$$
\beta_j(y) = \begin{cases} 
1, & |y - y^j| \leq 2\delta, \\
0, & |y - y^j| \geq 3\delta,
\end{cases}
$$

and the integrodifferential operators

$$
L_0(x', \partial_t, \partial_x)u = \partial_t u + \sum_{|\alpha|=2m} a_\alpha \partial_\alpha^2 u + \frac{f(x_n)}{f(0)} a_{0, \ldots, 0, 2m} \int_{R_1} (-iz_n)^{2m} F_{x_n \rightarrow \xi_n}[u(t, x)] d\xi_n,
$$

where $a_\alpha = a_\alpha(x'), |x'| \geq R$. 


\[ L_k(x', \partial_t, \partial_x) u \equiv \partial_t u + \sum_{|\alpha|=2m} a^\alpha_k(x') \partial^\alpha u + \]
\[ \frac{f(x_n)}{f(0)} a^0_{0,...,0,2m}(x') \int_{R_l} (-iz_i) F_{n\rightarrow \xi_n} [u(t, x)] d\xi_n, \]

where
\[ a^\alpha_k(x') = \beta_k(x') a^\alpha_0(x') + (1 - \beta_k(x')) a^\alpha_k(y^k), \]
\[ a^0_k(x') \equiv a^0_0(x'), \quad x' \in B_{2\delta}(y^k), \]
\[ a^k_k(x') \equiv a^0_k(y^k), \quad x' \notin B_{2\delta}(y^k), \quad k = 1, 2, \ldots, N. \]

By the choice of \( \delta \) we can assume that coefficients of the operator \( L_k \) are almost constants.

We consider a series of the Cauchy problem:
\[ L_k(x', \partial_t, \partial_x) u_k = \mu_k(t, x') f(x_n), \quad t > 0, \quad x \in R_n, \]
\[ u_k|_{t=0} = 0, \quad k = 0, 1, \ldots, N. \]  \hspace{1cm} (20)

The solution to the Cauchy problem (20) is written in the operator form
\[ u_k(t, x) = T^k_2(f, \mu_k), \quad k = 0, 1, \ldots, N, \]  \hspace{1cm} (21)

where \( T^0_2(f, \mu) = T_1(f, \mu), \quad T^k_2(f, \mu_k) = T_2(f, \mu_k), \quad k = 1, 2, \ldots, N, \) operators \( T_1, T_2 \) are defined in Remarks 1, 2. The solution to the Cauchy problem (12), (13) will be found as the sum
\[ u(t, x) = \sum_{k=0}^{N} \varphi_k(x^k) T^k_2(f, \varphi_k g). \]  \hspace{1cm} (22)

Here the function \( g(t, x') \in L^2_2(R^+_n) \) is an unknown. Apply the operator \( L^0(x', \partial_t, \partial_x) \) to the function \( u(t, x) \) from (22) and after simple transformations, we obtain for \( g(t, x') \) the Fredholm equation of the second kind
\[ g(t, x') = S_2(g(t, x')) + \mu(t, x'). \]  \hspace{1cm} (23)

Here
\[ S_2(g(t, x')) = f^{-1}(0) \sum_{i,k=0}^{N} \varphi_i \sum_{|\alpha|+|\beta|=2m} b_{\alpha\beta}(x') \partial^\alpha_x \varphi_k \partial^\beta_x T^k_2(f, \varphi_k g)|_{x_n=0}, \]

\( b_{\alpha\beta}(x') \) are known functions, which are determined by the coefficients \( a^\alpha_k(x') \).

Estimate the norm of the operator \( S_2(g(t, x')) \):
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\[
\|S_2(g) \leq C_4 \sum_{k=0}^{N} \sum_{|\sigma| \leq 2m-1} \|\partial_{\sigma}^2 T_2^k (f, \varphi_k g) \|_{x_n=0, L_{2,\gamma}(R^n_+)} \|
\]

\[
\leq q_2 \|g \leq L_{2,\gamma}(R^n_+)\|,
\]

(24)

\[C_4 = (2m - 1)N \sum_{k=0}^{N} \varphi_i \sum_{|\sigma| \leq 2m} \sup_{x'} |b_{\sigma}(x')| \cdot |\partial_{\sigma}^2 \varphi_k|.
\]

Here we have used the inequality (18). From equation (23) with allowance for estiamtes (19), (24), we obtain

\[
\|g \leq L_{2,\gamma}(R^n_+)\| \leq (1 - q_2)^{-1} \|\mu \leq L_{2,\gamma}(R^n_+)\|.
\]

By applying the method of successive approximations to equation (23), we find \(g(t, x')\). Substituting it into (22) we obtain the solution to the Cauchy problem (12), (13). The function \(\lambda(t, x')\) is calculated by formulae (17).

The uniqueness is proved in a standard way.

\[\square\]

References


