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Three-dimensional vortex flows of incompressible two-velocity media at constant saturation of substances

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Abstract. A flow of incompressible viscous two-velocity fluids for the case of pressure equilibrium of phases at constant saturation of substances is described with the help of scalar functions. A system of differential equations for these functions is obtained. An example illustrating this method is presented.

Keywords: Two-velocity hydrodynamics, hyperbolic system, viscous fluid, vortex flows, quasi potential, Bernoulli integral.

1. Introduction

A fundamental goal of modern hydrodynamics originating from the classical fluid mechanics, is to study the dynamics and interaction of vortex structures. According to P.G. Saffman [1], vortices are "the sinews and muscles" of hydrodynamics. Construction of dynamic models of turbulence, solving problems of flow stability and control, improvement of wing aerodynamics, and development of many other areas of hydrodynamic theory and technical applications are impossible without understanding the mechanisms that determine the behavior of vortices.

In the long history of development of hydrodynamics, several formulations of the initial equations to be used as a basis for investigating various aspects of the dynamics of vortices have been proposed. For instance, one can consider velocity as a function of coordinates in space, or the current coordinates of fluid particles, as functions of their initial locations. In some cases, one can take the velocity components, the complex potential, and the stream function as independent variables, and use the Clebsch potentials, Hamiltonian formalism, etc. The well-known theoretical methods are subdivided into the three large groups depending on the principles used to describe the fluid motion: Eulerian, Lagrangian, or mixed (combining the elements of the first two ones) [1].

Numerous theoretical and experimental papers are devoted to studying vortices, specific forms of fluid flows with almost closed streamlines, and nonzero vorticity in a bounded space domain. The interest in this form of flows is caused by several mutually complementary factors: first of all, it is widespread in natural conditions, and the parameters of such flows have a wide dynamic range of variation. In terrestrial conditions, one can observe vortices with scales of several hundreds kilometers; in the atmosphere spiral cloud systems and their extreme forms (typhoons or hurricanes) [2]; and in the ocean mesoscale vortices and annular flows [3]. Compact vortices in the atmosphere (tornadoes) cause a great economic damage and loss of life. The vortex form of flows is often used in industrial devices to stabilize physico-chemical processes (for instance, combustion) and for environmental protection by separating the components and extracting solid or liquid impurities [4].

As noted in [5], investigations of incompressible fluid flows are most successful when the vector equations of motion for the velocity field of flows are reduced to one scalar equation for some function. In this case, the corresponding scalar function may be either a hydrodynamic potential or a stream function. In the former case, the range of problems to be solved is limited to potential flows, whereas in the latter case flows may also be vortices but efficiently two-dimensional.

In [5], the third version for describing flows of incompressible viscous fluid by one scalar function is proposed. This approach can also be applied to three-dimensional flows without symmetry and depending on all the three spatial coordinates. In this case, it is only assumed that one of the vorticity components is zero. This approach does not depend on the choice of a coordinates system. The purpose of this paper is to describe flows of incompressible viscous two-velocity fluids for the case of the pressure equilibrium of phases with the help of two scalar functions based on the method proposed in [5].

2. Equations of two-velocity hydrodynamics with one pressure

In [6, 7], a nonlinear two-velocity model of fluid motion through a deformable porous medium was constructed based on conservation laws, invariance of the equations with respect to Galilei transformations, and the thermodynamic consistency condition. A two-velocity two-fluid hydrodynamic theory with the condition of pressure equilibrium of the subsystems was developed in [8]. The equations of motion of a two-velocity medium in the dissipative case with one pressure in the system in the isothermal case have the following form [8]:

$$\frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\tilde{\rho}\tilde{\boldsymbol{v}} + \rho \boldsymbol{v}) = 0, \qquad \frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\tilde{\rho}\tilde{\boldsymbol{v}}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = -\frac{\nabla p}{\bar{\rho}} + \nu \Delta \boldsymbol{v} + \frac{\tilde{\rho}}{2\bar{\rho}} \nabla(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2 + \boldsymbol{f}, \tag{2}$$

$$\frac{\partial\tilde{\boldsymbol{v}}}{\partial t}+(\tilde{\boldsymbol{v}},\nabla)\tilde{\boldsymbol{v}}=-\frac{\nabla p}{\bar\rho}+\tilde\nu\Delta\tilde{\boldsymbol{v}}-\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2+\boldsymbol{f}, \tag{3}$$

where $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ are the velocity vectors of the subsystems forming a two-velocity continuum with the corresponding partial densities $\tilde\rho$ and ρ, ν and $\tilde\nu$ are the corresponding kinematic viscosities, $\bar\rho=\tilde\rho+\rho$ is the total density of the two-velocity continuum; $p=p(\bar\rho,(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2)$ is the equation of state of the two-velocity continuum; and $\boldsymbol{f}$ is the mass force vector per unit mass.

Let us rewrite equations (2) and (3) in the equivalent form:

$$\frac{\partial\boldsymbol{v}}{\partial t}+\frac{1}{2}\nabla(v^2)-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}=-\frac{\nabla p}{\bar\rho}+\nu\Delta\boldsymbol{v}+\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2+\boldsymbol{f}, \tag{4}$$

$$\frac{\partial\tilde{\boldsymbol{v}}}{\partial t}+\frac{1}{2}\nabla(\tilde v^2)-\tilde{\boldsymbol{v}}\times\operatorname{rot}\tilde{\boldsymbol{v}}=-\frac{\nabla p}{\bar\rho}+\tilde\nu\Delta\tilde{\boldsymbol{v}}-\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2+\boldsymbol{f}. \tag{5}$$

From these equations one can derive other equations for the time variation of vortices. Denote $\boldsymbol{\Omega}=\operatorname{rot}\boldsymbol{v}$ and $\tilde{\boldsymbol{\Omega}}=\operatorname{rot}\tilde{\boldsymbol{v}}$. Then we apply the operator rot to both sides of (4) and (5). As a result we have

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}-\operatorname{rot}(\boldsymbol{v}\times\boldsymbol{\Omega})=-\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big)+\nu\Delta\boldsymbol{\Omega}+\operatorname{rot}\Big(\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\Big)+\operatorname{rot}\boldsymbol{f},$$

$$\frac{\partial\tilde{\boldsymbol{\Omega}}}{\partial t}-\operatorname{rot}(\tilde{\boldsymbol{v}}\times\tilde{\boldsymbol{\Omega}})=-\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big)+\tilde\nu\Delta\tilde{\boldsymbol{\Omega}}-\operatorname{rot}\Big(\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\Big)+\operatorname{rot}\boldsymbol{f}.$$

Hence, using the vector analysis formula, we obtain

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}-\operatorname{rot}(\boldsymbol{v}\times\boldsymbol{\Omega})=\frac{1}{\bar\rho^2}(\nabla\bar\rho\times\nabla p)+\nu\Delta\boldsymbol{\Omega}+\frac{1}{2}\bigg(\nabla\Big(\frac{\tilde\rho}{\bar\rho}\Big)\times\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\bigg)+\operatorname{rot}\boldsymbol{f},$$

$$\frac{\partial\tilde{\boldsymbol{\Omega}}}{\partial t}-\operatorname{rot}(\tilde{\boldsymbol{v}}\times\tilde{\boldsymbol{\Omega}})=\frac{1}{\bar\rho^2}(\nabla\bar\rho\times\nabla p)+\tilde\nu\Delta\tilde{\boldsymbol{\Omega}}-\frac{1}{2}\bigg(\nabla\Big(\frac{\rho}{\bar\rho}\Big)\times\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\bigg)+\operatorname{rot}\boldsymbol{f}.$$

3. A scalar description of three-dimensional vortex flows of two-velocity hydrodynamics of incompressible media with constant saturation

In the absence of mass forces $\boldsymbol{f}=0$, the system of equations (1)–(3) has a solution: $\boldsymbol{v}=0$, $\tilde{\boldsymbol{v}}=0$, $\rho=\rho^0$, $\tilde\rho=\tilde\rho^0$, $p=p^0$ for a mixture of fluids at rest with a uniform pressure $p=p^0$, partial densities ρ^0, $\tilde\rho^0$, and a temperature T. In the case of homogeneous incompressible media, that is, provided $\rho^f=\text{const}$, $\tilde\rho^f=\text{const}$, where ρ^f, $\tilde\rho^f$ are the physical densities of phases with constant saturation of substances forming a two-phase continuum: $\Rightarrow\rho=\text{const}$, $\tilde\rho=\text{const}\Rightarrow\operatorname{div}\boldsymbol{v}=0$, $\operatorname{div}\tilde{\boldsymbol{v}}=0\Leftrightarrow\boldsymbol{v}=\operatorname{rot}\boldsymbol{A}$, $\tilde{\boldsymbol{v}}=\operatorname{rot}\tilde{\boldsymbol{A}}$. Here $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ are the corresponding vector potentials of the velocities $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$, and ρ^0 and $\tilde\rho^0$ are the physical phase densities. In other words, the vectors

$\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are solenoidal. Since the vector-potential has a gradient invariance, without loss of generality one of its components can be made zero.

According to [5], assume that the vector-potentials are non-divergent: $\operatorname{div}\boldsymbol{A} = 0$, $\operatorname{div}\tilde{\boldsymbol{A}} = 0$. This assumption limits the class of flows to be considered. Hence, the two-component vector-potentials are expressed in terms of the scalar functions $\sigma(x,y,z,t)$ and $\tilde{\sigma}(x,y,z,t)$ as follows: $\boldsymbol{A} = \frac{\partial\sigma}{\partial y}\mathbf{i} - \frac{\partial\sigma}{\partial x}\mathbf{j}$, $\tilde{\boldsymbol{A}} = \frac{\partial\tilde{\sigma}}{\partial y}\mathbf{i} - \frac{\partial\tilde{\sigma}}{\partial x}\mathbf{j}$. These limitations lead to the fact that the vorticity fields are also two-component. In fact, since $\boldsymbol{\Omega} = \operatorname{rot}\boldsymbol{v} = \operatorname{rot}\operatorname{rot}\boldsymbol{A} = -\Delta\boldsymbol{A} + \nabla\operatorname{div}\boldsymbol{A}$, $\tilde{\boldsymbol{\Omega}} = \operatorname{rot}\tilde{\boldsymbol{v}} = \operatorname{rot}\operatorname{rot}\tilde{\boldsymbol{A}} = -\Delta\tilde{\boldsymbol{A}} + \nabla\operatorname{div}\tilde{\boldsymbol{A}}$, we have $\boldsymbol{\Omega} = -\frac{\partial\Delta\sigma}{\partial y}\mathbf{i} + \frac{\partial\Delta\sigma}{\partial x}\mathbf{j}$, $\tilde{\boldsymbol{\Omega}} = -\frac{\partial\Delta\tilde{\sigma}}{\partial y}\mathbf{i} + \frac{\partial\Delta\tilde{\sigma}}{\partial x}\mathbf{j}$. In this case the velocity fields remain three-dimensional:

$$\boldsymbol{v} = \frac{\partial^2\sigma}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\sigma}{\partial y\partial z}\mathbf{j} - \left(\frac{\partial^2\sigma}{\partial x^2} + \frac{\partial^2\sigma}{\partial y^2}\right)\mathbf{k} = \frac{\partial^2\sigma}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\sigma}{\partial y\partial z}\mathbf{j} + \left(\frac{\partial^2\sigma}{\partial z^2} - \Delta\sigma\right)\mathbf{k},$$

$$\tilde{\boldsymbol{v}} = \frac{\partial^2\tilde{\sigma}}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\tilde{\sigma}}{\partial y\partial z}\mathbf{j} - \left(\frac{\partial^2\tilde{\sigma}}{\partial x^2} + \frac{\partial^2\tilde{\sigma}}{\partial y^2}\right)\mathbf{k} = \frac{\partial^2\tilde{\sigma}}{\partial x\partial z}\mathbf{i} + \frac{\partial^2\tilde{\sigma}}{\partial y\partial z}\mathbf{j} + \left(\frac{\partial^2\tilde{\sigma}}{\partial z^2} - \Delta\tilde{\sigma}\right)\mathbf{k}.$$

Since the third vorticity component is absent, it follows from (4), (5) when projecting onto the axis z that $\operatorname{rot}(\boldsymbol{v}\times\boldsymbol{\Omega}) = \operatorname{rot}(\tilde{\boldsymbol{v}}\times\tilde{\boldsymbol{\Omega}}) = 0$. These equalities mean that $J\left(\Delta\sigma, \frac{\partial^2\sigma}{\partial z^2}\right) = J\left(\Delta\tilde{\sigma}, \frac{\partial^2\tilde{\sigma}}{\partial z^2}\right) = 0$ ($J(f,g) \equiv f_x g_y - f_y g_x$). It follows from these relations that $\Delta\sigma = -H\left(\frac{\partial^2\sigma}{\partial z^2}\right)$, $\Delta\tilde{\sigma} = -\tilde{H}\left(\frac{\partial^2\tilde{\sigma}}{\partial z^2}\right)$, where H and $\tilde{H}$ are arbitrary functions of their arguments.

Now it is convenient to introduce functions

$$\Phi(x,y,z,t) = \frac{\partial\sigma}{\partial z}, \qquad \tilde{\Phi}(x,y,z,t) = \frac{\partial\tilde{\sigma}}{\partial z}.$$

Then the velocity fields can be represented in the form

$$\boldsymbol{v} = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \left[\frac{\partial\Phi}{\partial z} + H\left(\frac{\partial\Phi}{\partial z}\right)\right]\mathbf{k} = \nabla\Phi + H\left(\frac{\partial\Phi}{\partial z}\right)\mathbf{k}, \tag{6}$$

$$\tilde{\boldsymbol{v}} = \frac{\partial\tilde{\Phi}}{\partial x}\mathbf{i} + \frac{\partial\tilde{\Phi}}{\partial y}\mathbf{j} + \left[\frac{\partial\tilde{\Phi}}{\partial z} + \tilde{H}\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)\right]\mathbf{k} = \nabla\tilde{\Phi} + \tilde{H}\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)\mathbf{k}. \tag{7}$$

Note that in a particular case, where $H\left(\frac{\partial\Phi}{\partial z}\right) \equiv 0$, $\tilde{H}\left(\frac{\partial\tilde{\Phi}}{\partial z}\right) \equiv 0$, the velocity fields are potential, and the functions $\Phi(x,y,z,t)$ and $\tilde{\Phi}(x,y,z,t)$ are hydrodynamic potentials. According to [5], such functions will be called quasipotentials.

The vorticity fields are expressed in terms of the quasipotentials as follows:

$$\mathbf{\Omega} = \frac{\partial H}{\partial x}\mathbf{i} - \frac{\partial H}{\partial y}\mathbf{j} = H' \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial x}\mathbf{i} - \frac{\partial \Phi}{\partial y}\mathbf{j}\right), \tag{8}$$

$$\tilde{\mathbf{\Omega}} = \frac{\partial \tilde{H}}{\partial x}\mathbf{i} - \frac{\partial \tilde{H}}{\partial y}\mathbf{j} = \tilde{H}' \frac{\partial}{\partial z}\left(\frac{\partial \tilde{\Phi}}{\partial x}\mathbf{i} - \frac{\partial \tilde{\Phi}}{\partial y}\mathbf{j}\right), \tag{9}$$

where the prime denotes differentiation of the functions H and $\tilde{H}$ with respect to the corresponding arguments. In this case the continuity equations are written down as

$$\Delta\Phi + \frac{\partial H}{\partial z} = 0, \tag{10}$$

$$\Delta\tilde{\Phi} + \frac{\partial \tilde{H}}{\partial z} = 0, \tag{11}$$

or

$$\Delta\Phi + H'\frac{\partial^2 \Phi}{\partial z^2} = 0, \tag{12}$$

$$\Delta\tilde{\Phi} + \tilde{H}'\frac{\partial^2 \tilde{\Phi}}{\partial z^2} = 0. \tag{13}$$

Substituting the velocity fields from (6), (7) into equations (4), (5) in the case of homogeneous media, we obtain the following integrals of motion for the first two components:

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \int H\, d\Phi_z = -\frac{p}{\bar{\rho}} + \nu\, \Delta\Phi - F + R(z,t) + \frac{\tilde{\rho}}{2\bar{\rho}}\Big[(\Phi_x - \tilde{\Phi}_x)^2 + (\Phi_y - \tilde{\Phi}_y)^2 + (\Phi_z - \tilde{\Phi}_z + H - \tilde{H})^2\Big], \tag{14}$$

$$\frac{\partial \tilde{\Phi}}{\partial t} + \frac{(\nabla\tilde{\Phi})^2}{2} + \int \tilde{H}\, d\tilde{\Phi}_z = -\frac{p}{\bar{\rho}} + \tilde{\nu}\, \Delta\tilde{\Phi} - F + \tilde{R}(z,t) - \frac{\rho}{2\bar{\rho}}\Big[(\Phi_x - \tilde{\Phi}_x)^2 + (\Phi_y - \tilde{\Phi}_y)^2 + (\Phi_z - \tilde{\Phi}_z + H - \tilde{H})^2\Big], \tag{15}$$

where $F(x, y, z, t)$ is the potential of mass forces, and $R(z, t)$ and $\tilde{R}(z, t)$ are arbitrary functions of their arguments defined by the boundary conditions.

As a consequence of these equations, we have

$$\frac{\partial(\rho\Phi + \tilde{\rho}\tilde{\Phi})}{\partial t} + \frac{\rho(\nabla\Phi)^2 + \tilde{\rho}(\nabla\tilde{\Phi})^2}{2} + \rho\int H\, d\Phi_z + \tilde{\rho}\int \tilde{H}\, d\tilde{\Phi}_z + p + \bar{\rho}F = \nu\rho\Delta\Phi + \tilde{\nu}\tilde{\rho}\Delta\tilde{\Phi} + \rho R(z,t) + \tilde{\rho}\tilde{R}(z,t). \tag{16}$$

For the third velocity component, from (14), (15) there follow two more relations with the quasipotentials satisfying them:

$$H' \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \int H\, d\Phi_z\right) = \nu \Delta H - \frac{\partial R}{\partial z}, \tag{17}$$

$$\tilde{H}' \frac{\partial}{\partial z}\left(\frac{\partial \tilde{\Phi}}{\partial t} + \frac{(\nabla\tilde{\Phi})^2}{2} + \int \tilde{H}\, d\tilde{\Phi}_z\right) = \tilde{\nu} \Delta \tilde{H} - \frac{\partial \tilde{R}}{\partial z}. \tag{18}$$

System (14), (15) is a generalization of the Bernoulli equation for the two-velocity hydrodynamics. It naturally transforms to the well-known Bernoulli equation for potential flows [5] at the same velocities and physical phase densities, provided that the functions R and $\tilde{R}$ depend only on time and $H\left(\frac{\partial \Phi}{\partial z}\right) \equiv 0$, $\tilde{H}\left(\frac{\partial \tilde{\Phi}}{\partial z}\right) \equiv 0$. Note that the found integral (16) makes possible to determine the pressure field if the quasipotentials at the given functions $H\left(\frac{\partial \Phi}{\partial z}\right)$ and $\tilde{H}\left(\frac{\partial \tilde{\Phi}}{\partial z}\right)$ are known. Thus, to construct velocity fields and find the corresponding vorticity and pressure fields, it is necessary to solve the systems of equations (10), (11) and (17), (18) for the quasipotentials, and then use equations (6)–(9) and (16).

To illustrate this approach, let us consider the case when the functions H and $\tilde{H}$ linearly depend on their arguments: $H(\Phi_z) = \lambda\Phi_z$ and $\tilde{H}(\tilde{\Phi}_z) = \tilde{\lambda}\tilde{\Phi}_z$. Then equations (12), (13), (17), (18) take the following form:

$$\Delta\Phi + \lambda\Phi_{zz} = 0, \tag{19}$$

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{\lambda}{2}(\Phi_z)^2 + \lambda\nu\frac{\partial^2\Phi}{\partial z^2} = -\frac{R(z,t)}{\lambda} + Q(x,y,t), \tag{20}$$

$$\Delta\tilde{\Phi} + \tilde{\lambda}\tilde{\Phi}_{zz} = 0, \tag{21}$$

$$\frac{\partial \tilde{\Phi}}{\partial t} + \frac{(\nabla\tilde{\Phi})^2}{2} + \frac{\tilde{\lambda}}{2}(\tilde{\Phi}_z)^2 + \tilde{\lambda}\,\tilde{\nu}\frac{\partial^2\tilde{\Phi}}{\partial z^2} = -\frac{\tilde{R}(z,t)}{\tilde{\lambda}} + \tilde{Q}(x,y,t), \tag{22}$$

where $Q(x,y,t)$ and $\tilde{Q}(x,y,t)$ are arbitrary functions.

In [5], three types of solutions to system (19)–(22) are constructed:

Solution I:

$$\Phi(x,y,z,t) = e^{-\nu\lambda k^2 t}\big(Ae^{-kz} + Be^{kz}\big)\sin(\alpha x + \beta y),$$
$$\tilde{\Phi}(x,y,z,t) = e^{-\tilde{\nu}\tilde{\lambda}\tilde{k}^2 t}\big(\tilde{A}e^{-\tilde{k}z} + \tilde{B}e^{\tilde{k}z}\big)\sin(\tilde{\alpha} x + \tilde{\beta} y).$$

This solution is valid under the condition $\lambda > 0$, $\tilde{\lambda} > 0$, $\alpha^2 + \beta^2 = (1+\mu)k^2$, $\tilde{\alpha}^2 + \tilde{\beta}^2 = (1+\tilde{\lambda})\tilde{k}^2$, and

$$R(z,t) = -\frac{\lambda}{2}(\alpha^2+\beta^2)e^{-2\nu\lambda k^2 t}\big(Ae^{-kz} + Be^{kz}\big)^2,$$
$$Q(x,y,t) = -2AB(\alpha^2+\beta^2)e^{-2\nu\lambda k^2 t}\sin^2(\alpha x + \beta y),$$

$$\tilde{R}(z,t) = -\frac{\tilde{\lambda}}{2}(\tilde{\alpha}^2+\tilde{\beta}^2)e^{-2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\big(\tilde{A}e^{-\tilde{k}z}+\tilde{B}e^{\tilde{k}z}\big)^2,$$
$$\tilde{Q}(x,y,t) = -2\tilde{A}\tilde{B}(\tilde{\alpha}^2+\tilde{\beta}^2)e^{-2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\sin^2(\tilde{\alpha}x+\tilde{\beta}y).$$

Solution II:

$$\Phi(x,y,z,t) = e^{\nu\lambda k^2t}\big(Ae^{-(\alpha x+\beta y)}+Be^{\alpha x+\beta y}\big)\sin kz,$$
$$\tilde{\Phi}(x,y,z,t) = e^{\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\big(\tilde{A}e^{-(\tilde{\alpha}x+\tilde{\beta}y)}+\tilde{B}e^{\tilde{\alpha}x+\tilde{\beta}y}\big)\sin\tilde{k}z.$$

This solution is valid if $-1<\lambda<0$, $-1<\tilde{\lambda}<0$, $\alpha^2+\beta^2=(1+\lambda)k^2$, $\tilde{\alpha}^2+\tilde{\beta}^2=(1+\tilde{\lambda})\tilde{k}^2$, and

$$R(z,t) = -2\lambda AB(\alpha^2+\beta^2)e^{2\nu\lambda k^2t}\cos^2 kz,$$
$$Q(x,y,t) = \frac{1}{2}(\alpha^2+\beta^2)e^{2\nu\lambda k^2t}\big(Ae^{-(\alpha x+\beta y)}+Be^{\alpha x+\beta y}\big)^2,$$
$$\tilde{R}(z,t) = -2\tilde{\lambda}\tilde{A}\tilde{B}(\tilde{\alpha}^2+\tilde{\beta}^2)e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\cos^2\tilde{k}z,$$
$$\tilde{Q}(x,y,t) = \frac{1}{2}(\tilde{\alpha}^2+\tilde{\beta}^2)e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\big(\tilde{A}e^{-(\tilde{\alpha}x+\tilde{\beta}y)}+\tilde{B}e^{\tilde{\alpha}x+\tilde{\beta}y}\big)^2,$$

Solution III:

$$\Phi(x,y,z,t) = Ae^{\nu\lambda k^2t}\sin(\alpha x+\beta y)\sin kz,$$
$$\tilde{\Phi}(x,y,z,t) = \tilde{A}e^{\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\sin(\tilde{\alpha}x+\tilde{\beta}y)\sin\tilde{k}z.$$

This solution is valid if $\lambda<-1$, $\tilde{\lambda}<-1$, $\alpha^2+\beta^2=-(1+\lambda)k^2$, $\tilde{\alpha}^2+\tilde{\beta}^2=-(1+\tilde{\lambda})\tilde{k}^2$, and

$$R(z,t) = -\frac{\lambda}{2}A^2(\alpha^2+\beta^2)e^{2\nu\lambda k^2t}\sin^2 kz,$$
$$Q(x,y,t) = -\frac{A^2}{2}(\alpha^2+\beta^2)e^{2\nu\lambda k^2t}\sin^2(\alpha x+\beta y),$$
$$\tilde{R}(z,t) = -\frac{\tilde{\lambda}}{2}\tilde{A}^2(\tilde{\alpha}^2+\tilde{\beta}^2)e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\sin^2\tilde{k}z,$$
$$\tilde{Q}(x,y,t) = -\frac{\tilde{A}^2}{2}(\tilde{\alpha}^2+\tilde{\beta}^2)e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2t}\sin^2(\tilde{\alpha}x+\tilde{\beta}y).$$

Solution III can be considered a doubly periodic Kolmogorov flow [9]. By turning around the system of coordinates about the axis z in the plane x, y, the solution can be made dependent only on x. Therefore, without loss of generality it can be assumed that $\beta=0$. In this case, the velocity fields contain the two components, x and z:

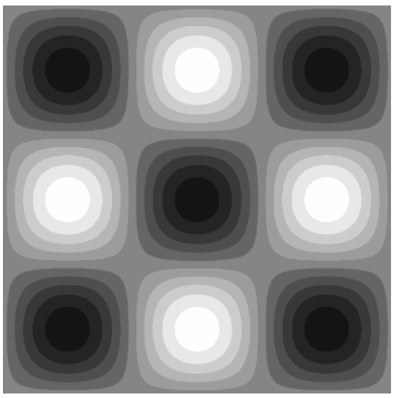
$$\boldsymbol{v} = \alpha A e^{2\nu\lambda k^2 t}\left[\cos\alpha x \sin kz\,\mathbf{i} + \frac{1}{\sqrt{-(1+\lambda)}}\sin\alpha x\cos kz\,\mathbf{k}\right],$$
$$\tilde{\boldsymbol{v}} = \tilde{\alpha}\tilde{A} e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2 t}\left[\cos\tilde{\alpha} x \sin \tilde{k}z\,\mathbf{i} + \frac{1}{\sqrt{-(1+\tilde{\lambda})}}\sin\tilde{\alpha} x\cos \tilde{k}z\,\mathbf{k}\right],$$

and the vorticity fields contain only one component y:

$$\boldsymbol{\Omega} = -\lambda\alpha A e^{2\nu\lambda k^2 t}\cos\alpha x\sin kz\,\mathbf{j},$$
$$\tilde{\boldsymbol{\Omega}} = -\tilde{\lambda}\tilde{\alpha}\tilde{A} e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2 t}\cos\tilde{\alpha} x\sin \tilde{k}z\,\mathbf{j}.$$

Then we find pressure from equation (16) ignoring the external forces and assuming that $F(x, y, z, t) \equiv 0$:

$$p = \rho\frac{(\alpha A)^2}{2}e^{2\nu\lambda k^2 t}\left[\sin^2\alpha x - (1+\lambda)\sin^2 kz\right] +$$
$$\tilde{\rho}\frac{(\tilde{\alpha}\tilde{A})^2}{2}e^{2\tilde{\nu}\tilde{\lambda}\tilde{k}^2 t}\left[\sin^2\tilde{\alpha} x - (1+\tilde{\lambda})\sin^2 \tilde{k}z\right].$$



The figure presents the flow fields for the viscous case at $\nu = 0$, $\lambda = -1.25$. The opposite flow fields are shown black and white in color. Since the solution is periodic in the plane x, z, one can identify an elementary cell whose lateral boundaries can be considered to be solid walls. In the absence of viscosity on these walls, only normal velocity components must be zero, which is satisfied by virtue of Solution III.

More complicated examples of flows can be constructed from Solutions I–III, for example, by combining doubly periodic Solutions III with Solutions I and II, which are periodic with respect to one coordinate and exponentially decaying with respect to the other coordinate. Even more complex classes of flows can be constructed with an appropriate choice of the nonlinear functions $H(\Phi_z)$ and $\tilde{H}(\tilde{\Phi}_z)$.

4. Conclusion

Two scalar functions were used to describe the three-dimensional vortex flows of an incompressible two-velocity continuum with pressure equilibrium of phases at constant saturation of substances. A system of differential equations for these functions has been derived. Systems of differential equations for the quasipotentials and generalized Bernoulli integrals have been obtained.

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