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Laguerre spectral method as applied to numerical solution of a two-dimensional linear dynamic seismic problem for porous media*

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Introduction

Seismic methods based on the seismic waves propagation in an acoustic or an ideally elastic medium, were successfully applied to various geophysical problems to identify geological structures. In such studies, properties of a pore liquid such as density, the module of volumetric deformation, fluidsaturation and viscosity were generally ignored. A porous medium of consisting it, is an elastic, i.e., deformable matrix filled with a viscous liquid, being a realistic model which allows us to explain observable effects of seismic research of properties of rocks in the presence of a pore liquid. Recently the numerical simulation of seismic wave propagation in fluidsaturated liquid porous media, has received a special attention because of its practical application in various areas of problems of geophysics, biomechanics and oil reservoir characterization. In the case, the Frenkel–Biot model [1, 2] is generally used as a basis. A characteristic feature of models of this type alongside with distribution of transverse and longitudinal seismic waves, is the presence of an additional second longitudinal wave. The speeds of propagation of such waves are functions of four elastic parameters in the Frenkel–Biot theory for preset values of physical density of a solid matrix, a saturating liquid and porosity [1, 2]. In 1989, V.N. Dorovsky [3] based on the common first physical principles, constructed nonlinear mathematical model for porous media. Just as in the Frenkel–Biot theory, in the Dorovsky model there are three types of the sound oscillations: transverse and two types of longitudinal. As opposed to models of the Frenkel–Biot type, in the linearized Dorovsky models a medium is described by three elastic parameters [4, 5]. These elastic parameters in one-to-one correspondence are expressed by three speeds of seismic wave propagations. This circumstance is important for the numerical modeling of elastic waves in a porous medium when distributions of speeds of acoustic waves and physical density of the matrix of saturating liquids and porosity are known.

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Finite difference methods of solving of problems for the Biot equation system have been formulated in several ways, these are: the central difference finite difference method in terms of displacement [6, 7], the predictor-corrector finite difference method for the velocity-stress system of the equations [8]. Semi-analytic method for the Biot equation system in terms of displacement is offered in [9, 10].

In this paper, the system of linearized equations of porous media [4, 5] in the absence of dissipation of energy in 2D heterogeneous is case numerically solved. The initial system of equations as first order hyperbolic system in terms of velocity of a solid matrix, velocity of a saturating liquid, solid stress, and fluid pressure. For the numerical solution of the task in question, the method of combination of analytical Laguerre transformation and a finite difference method is used. The above-considered method of solving dynamic problems of the elasticity theory was first proposed in [11, 12], and then developed for viscoelasticity problems in [13, 14].

The proposed method of the solution can be considered as analogue to the known spectral-difference method on the basis of Fourier transform, only instead of frequency we have a parameter m — the degree of the Laguerre polynomials. However, unlike Fourier transform, application of integral Laguerre transform with respect time allows us to reduce the initial problem to solving a system of equations in which the parameter of division is present only on the right-hand side of equations and has a recurrent dependence.

As compared to finite difference methods, with the help of an analytical transformation in the spectral-difference method it is possible to reduce an original problem to solving the system of differential equations, in which there are only derivatives with respect to spatial coordinates. This allows us to apply a known stable difference scheme for recurrent solutions to similar systems. Such an approach is effective when solving non-stationary dynamic problems for porous media. Thus, because of the presence of the second longitudinal wave with a low velocity, the use of difference schemes in all coordinates for stable solutions requires a consistent small step both in time and in space, which inevitably results in an increase of computer costs.

1. Statement of problem

Let us assume that the half-plane $x_2 > 0$ is filled with a porous medium of a saturated liquid. Then propagation of seismic waves to the given medium in the absence of dissipation of energy is described by the following initial boundary value problem [4, 5, 15]:

$$\frac{\partial u_i}{\partial t} + \frac{1}{\rho_{0,s}} \frac{\partial h_{ik}}{\partial x_k} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} = F_i, \qquad \frac{\partial v_i}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} = F_i,$$

$$\frac{\partial h_{ik}}{\partial t} + \mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) + \left(\frac{\rho_{0,l}}{\rho_0}K - \frac{2}{3}\mu\right)\delta_{ik}\,\mathrm{div}\,\vec{u} - \frac{\rho_{0,s}}{\rho_0}K\delta_{ik}\,\mathrm{div}\,\vec{v} = 0, \quad (1)$$

$$\frac{\partial p}{\partial t} - (K - \alpha\rho_0\rho_{0,s})\,\mathrm{div}\,\vec{u} + \alpha\rho_0\rho_{0,l}\,\mathrm{div}\,\vec{v} = 0,$$

$$u_i\big|_{t=0} = v_i\big|_{t=0} = h_{ik}\big|_{t=0} = p\big|_{t=0} = 0, \quad (2)$$

$$h_{22} + p\big|_{x_2=0} = h_{12}\big|_{x_2=0} = \frac{\rho_{0,l}}{\rho_0}p\Big|_{x_2=0} = 0, \quad (3)$$

where $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ are the vector of the velocity of the solid matrix with the partial density $\rho_{0,s}$ and a liquid with partial density $\rho_{0,l}$, respectively, p is pore pressure, h_{ik} is the stress tensor, $\vec{F} = (F_1, F_2)$ is the vector of body forces, $\rho_0 = \rho_{0,l} + \rho_{0,s}$, $\rho_{0,s} = \rho^f_{0,s}(1 - d_0)$, $\rho_{0,l} = \rho^f_{0,l}d_0$, $\rho^f_{0,s}$ and $\rho^f_{0,l}$ are the physical densities of the solid matrix and the liquid respectively, d_0 is porosity, δ_{ik} is the Kronecker delta, $K = \lambda + 2\mu/3$, $\lambda > 0$, $\mu > 0$ are elastic modules $\alpha = \rho_0\alpha_3 + K/\rho_0^2$, $\rho_0^3\alpha_3 > 0$ is the module of volumetric compression of a liquid component of a heterophase medium. Elastic modules K, μ, and α_3 are expressed through the speed of the transverse wave c_s and two speeds of longitudinal waves c_{p_1} and c_{p_2} by the following formulas [16, 17]:

$$\mu = \rho_{0,s}c_s^2,$$

$$K = \frac{\rho_0}{2}\frac{\rho_{0,s}}{\rho_{0,l}}\left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3}\frac{\rho_{0,l}}{\rho_0}c_s^2 - \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9}\frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2}c_s^4}\right),$$

$$\alpha_3 = \frac{1}{2\rho_0^2}\left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3}\frac{\rho_{0,s}}{\rho_0}c_s^2 + \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9}\frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2}c_s^4}\right).$$

2. Algorithm of solution

For solving problem (1)–(3) let us apply the integral Laguerre transform with respect to time:

$$\overrightarrow{W}_m(x_1, x_2) = \int_0^\infty \overrightarrow{W}(x_1, x_2, t)(ht)^{-\alpha/2}l_m^\alpha(ht)\,d(ht),$$

with the inverse formula

$$\overrightarrow{W}(x_1, x_2, t) = (ht)^{\alpha/2}\sum_{m=0}^{\infty}\frac{m!}{(m+\alpha)!}\overrightarrow{W}_m(x_1, x_2)l_m^\alpha(ht).$$

Here $l_m^\alpha(ht)$ are the Laguerre functions.

As a result of the given transformation, initial problem (1)–(3) is reduced to a 2D spatial differential problem in the spectral domain:

$$
\begin{gathered}
\frac{h}{2}u_i^m + \frac{1}{\rho_s}\frac{\partial \sigma_{ik}^m}{\partial x_k} + \frac{1}{\rho_0}\frac{\partial P^m}{\partial x_i} = f_i^m - h\sum_{n=0}^{m-1} u_i^n, \\
\frac{h}{2}v_i^m + \frac{1}{\rho_0}\frac{\partial P^m}{\partial x_i} = f_i^m - h\sum_{n=0}^{m-1} v_i^n, \\
\frac{h}{2}\sigma_{ik}^m + \mu\left(\frac{\partial u_k^m}{\partial x_i} + \frac{\partial u_i^m}{\partial x_k}\right) + \left(\lambda - \frac{\rho_s}{\rho_0}K\right)\delta_{ik}\operatorname{div}\vec{u}^m - \\
\frac{\rho_s}{\rho_0}K\delta_{ik}\operatorname{div}\vec{v}^m = -h\sum_{n=0}^{m-1}\sigma_{ik}^n, \\
\frac{h}{2}P^m - (K - \alpha\rho_0\rho_s)\operatorname{div}\vec{u}^m + \alpha\rho_0\rho_l\operatorname{div}\vec{v}^m = -h\sum_{n=0}^{m-1} P^n, \\
\sigma_{22}^m + P^m\big|_{x_2=0} = \sigma_{12}^m\big|_{x_2=0} = \frac{\rho_l}{\rho_0}P^m\Big|_{x_2=0} = 0.
\end{gathered}
\tag{4}
$$

For the solution of the reduced problem, we use a finite difference approximation of derivatives along the spatial coordinates on the staggered grids with fourth order accuracy [18]. For this purpose, let us introduce in the calculation domain towards the coordinate $z = x_2$ the grids ω_z and $\overset{\circ}{\omega}_z$ with a quantization step Δz. These grids are $\Delta z/2$ staggered relative to each other:

$$
\begin{aligned}
\omega_z &= \Big\{(x, j\Delta z, t),\ \ j = 0, \ldots, M\Big\}, \\
\overset{\circ}{\omega}_z &= \Big\{\Big(x, j\Delta z + \frac{\Delta z}{2}, t\Big),\ \ j = 0, \ldots, M-1\Big\}.
\end{aligned}
$$

Similarly, we introduce in the direction of the coordinate $x = x_1$ the grids ω_x and $\overset{\circ}{\omega}_x$ with a quantization step Δx, which are staggered relative to each other by $\Delta x/2$:

$$
\begin{aligned}
\omega_x &= \Big\{(i\Delta x, z, t),\ \ i = 0, \ldots, N\Big\}, \\
\overset{\circ}{\omega}_x &= \Big\{\Big(i\Delta x + \frac{\Delta x}{2}, z, t\Big),\ \ i = 0, \ldots, N-1\Big\}.
\end{aligned}
$$

On the given grids, we introduce differentiation operators D_x and D_z, approximating the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ with fourth order of accuracy along the coordinates $z = x_2$ and $x = x_1$:

$$
\begin{aligned}
D_x u(x,z) = \frac{9}{8\Delta x}\Big[u(x + \Delta x/2, z) - u(x - \Delta x/2, z)\Big] - \\
\frac{1}{24\Delta x}\Big[u(x + 3\Delta x/2, z) - u(x - 3\Delta x/2, z)\Big],
\end{aligned}
$$

$$D_z u(x,z) = \frac{9}{8\Delta x}\Big[u(x,z+\Delta z/2) - u(x,z-\Delta z/2)\Big] - \frac{1}{24\Delta x}\Big[u(x,z+3\Delta z/2) - u(x,z-3\Delta z/2)\Big].$$

Let us determine the required components of the vector of solution in the following nodes of the grids:

$$u_1^m,\ v_1^m \text{ at } \omega_x \times \omega_z, \qquad u_2^m,\ v_2^m \text{ at } \overset{\circ}{\omega}_x \times \overset{\circ}{\omega}_z,$$

$$\sigma_{11}^m,\ \sigma_{22}^m,\ P^m \text{ at } \overset{\circ}{\omega}_x \times \omega_z, \qquad \sigma_{12}^m \text{ at } \omega_x \times \overset{\circ}{\omega}_z.$$

As a result of the finite difference approximation of problem (4), we obtain a system of the linear algebraic equations. Represent a required vector of the solution $\vec{W}$ in the following form:

$$\vec{W}(m) = (\vec{V}_0(m), \vec{V}_1(m), ..., \vec{V}_{M+N}(m))^T,$$

$$\vec{V}_{i+j} = \Big(u_1^{i,j}, u_2^{i+\frac{1}{2},j+\frac{1}{2}}, v_1^{i,j}, v_2^{i+\frac{1}{2},j+\frac{1}{2}}, \sigma_{11}^{i+\frac{1}{2},j}, \sigma_{22}^{i+\frac{1}{2},j}, \sigma_{12}^{i,j+\frac{1}{2}}, P^{i+\frac{1}{2},j}\Big)^T.$$

Then, the given system of the linear algebraic equations can be written down in the vector form as:

$$\Big(A_\Delta + \frac{h}{2}E\Big)\vec{W}(m) = \vec{F}_\Delta(m-1).$$

As a result, the matrix of the system of the reduced problem has good conditionality that allows us to use fast methods for solving systems of linear algebraic equations on the basis of iterative methods, such as the conjugate gradients, converging to the solution with desired accuracy of all for some iterations. At this stage of carrying out calculations a version of the conjugate gradients method has been parallelized. In terms of the input data, when setting a medium model, it is equivalent to decomposition of the initial domain to a set of subdomains equal to the number of processors. This enables to distribute memory, both in setting input parameters of the model, and in further numerical realization of the algorithm in subdomains.

3. Numerical results

The results of numerical modeling of seismic wave fields for a test model of a medium are represented. This model consists of two homogeneous layers: the upper layer is an elastic medium, the lower one is porous. Physical characteristics of the layers are the following:

The upper layer — $\rho = 1.2$ g/cm^3, $c_p = 1.5$ km/s, $c_s = 1$ km/s;

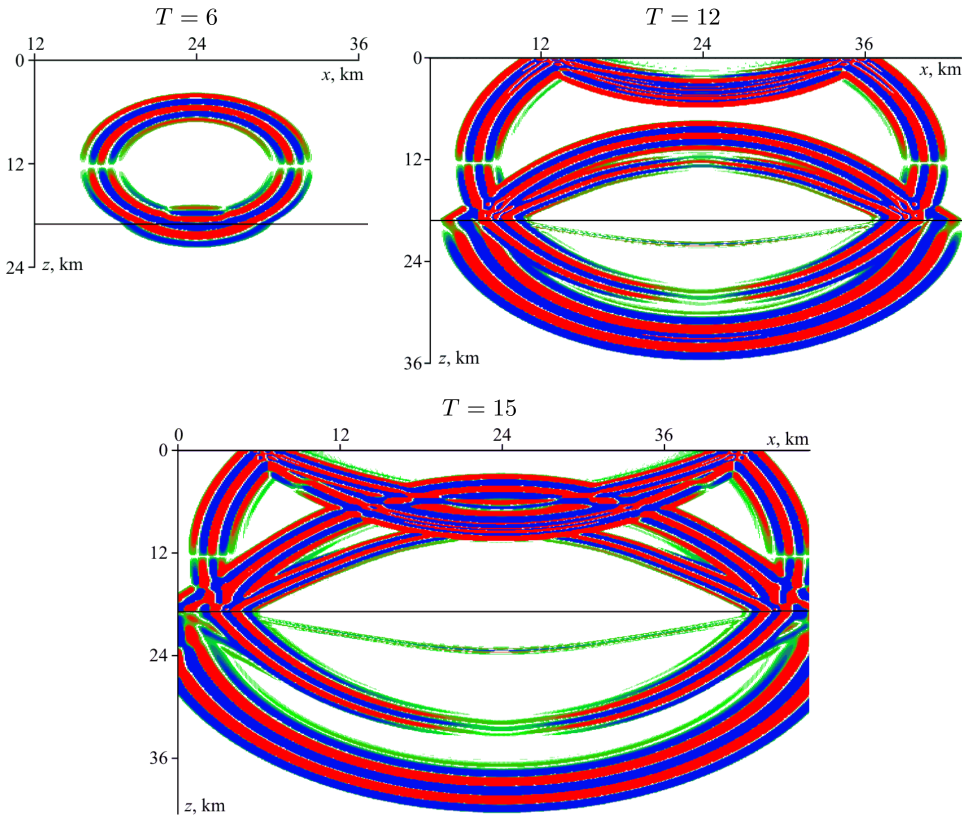
The lower layer — $\rho_{0,s}^f = 1.5$ g/cm^3, $\rho_{0,l}^f = 1$ g/cm^3, $c_{p_1} = 2$ km/s, $c_{p_2} = 0.45$ km/s, $c_s = 1.3$ km/s, $d = 0.1$.

The upper layer is 18 km thick. The wave field was simulated from a point source such as the center of expansion with coordinates $x_0 = 24$ km, $z_0 = 12$ km, being place in the upper elastic layer. The time signal in the source was set as:

$$f(t) = \exp\left(-\frac{2\pi f_o(t-t_0)^2}{\gamma^2}\right)\sin(2\pi f_0(t-t_0)),$$

where $\gamma = 4$, $f_0 = 1$ Hz, $t_0 = 1.5$ s.

The results of numerical calculations of the wave field for the given medium model are presented in the figure at T of 6, 12 and 15 seconds. From the represented figures it is seen, that at falling a longitudinal wave radiated by a source such as center of expansion on to the layers interface, the corresponding types of waves for the elastic and porous media are formed. In the upper layer, there are longitudinal and transverse waves, and in the lower porous layer, there are two longitudinal and transverse waves.



Snapshots of the wave field for $u_z(x, z)$ component of displacement velocity

Conclusion

The proposed algorithm is an analog to known spectral methods of the solution of dynamic problems. However, as opposed to classical Fourier and Laplace transformations, the application of the Laguerre transformations reduces to a system of equations, in which the transformation parameter is recurrently present only in the right hand side. As a result, the matrix of the system of the problem has a good conditioning number, allowing the use of effective numerical methods of the solution to systems of linear algebraic equations.

References

[1] Frenkel Ya.I. On the theory of seismic and seismoelectric phenomena in a moist soil // J. Phys. USSR. — 1944. — Vol. 8. — P. 230–241.

[2] Biot M.A. Theory of propagation of elastic waves in fluid-saturated porous solid. I. low-frequency range // J. Acoustical Society of America. — 1956. — Vol. 28. — P. 168–178.

[3] Dorovsky V.N. Continual theory of filtration // Sov. Geology and Geophysics. — 1989. — Vol. 30, No. 7. — P. 34–39.

[4] Dorovsky V.N., Perepechko Yu.V., Romensky E.I. Wave processes in saturated porous elastically deformed media // Combustion, Explosion and Shock Waves. — 1993. — Vol. 29, No. 1. — P. 93–103.

[5] Blokhin A.M., Dorovsky V.N. Mathematical Modelling in the Theory of Multivelocity Continuum. — New York: Nova Science, 1995.

[6] Zhu X., McMechan G.A. Numerical simulation of seismic responses of poroelastic reservoirs using Biot theory // Geophysics. — 1991. — Vol. 56. — P. 328–339.

[7] Zeng Y.Q., He J.Q., Liu Q.H. The application of the perfectly matched layer in numerical modeling of wave propagation in poroelastic media // Geophysics. — 2001. — Vol. 66. — P. 1258–1266.

[8] Dai N., Vafidis A., Kanasewich E.R. Wave propagation in heterogeneous, porous media: a velocity-stress, finite-difference method // Geophysics. — 1995. — Vol. 60. — P. 327–340.

[9] Philippacopoulos A.J. Lamb's problem for fluid-saturated porous media // Bull. Seism. Soc. Am. — 1988. — Vol. 78. — P. 908–923.

[10] Miroshnikov V.V., Fatyanov A.G. Semi-analytical method for computational of wave fields in layered porous mediums // Proceedings of Computing Center of the Siberian Branch of the Russian Academy of Science. Mathematical Modeling in Geophysics. — Novosibirsk, 1993. — P. 27–57 (In Russian).

[11] Konyukh G.V., Mikhailenko B.G. Application of integral Laguerre transformation for solving dynamic seismic problem // Bull. Of the Novosibirsk Computing Center. Ser. Mathematical Modeling in Geophysics. — Novosibirsk, 1998. — Iss. 4. — P. 79–91.

[12] Mikhailenko B.G. Spectral Laguerre method for the approximate solution of time dependent problems // Applied Mathematics Letters. — 1999. — No. 12. — P. 105–110.

[13] Mikhailenko B.G., Mikhailov A.A., Reshetova G.V. Numerical modeling of transient seismic fields in viscoelastic media based on the Laguerre spectral method // Pure Apll. Geophys. — 2003. — No. 160. — P. 1207–1224.

[14] Mikhailenko B.G., Mikhailov A.A., Reshetova G.V. Numerical viscoelastic modeling by the spectral Laguerre method // Geophysical Prospecting. — 2003. — No. 51. — P. 37–48.

[15] Imomnazarov Kh.Kh. A mathematical model for the movement of a conducting liquid through a conducting porous medium: I. Excitation of oscillations of the magnetic field by the surface rayleigh wave // Math. Comput. Modelling. — 1996. — Vol. 24, No. 1. — P. 79–84.

[16] Imomnazarov Kh.Kh. Some remarks on the Biot system of equations // Doklady RAS. — 2000. — Vol. 373, No. 4. — P. 536–537 (In Russian).

[17] Imomnazarov Kh.Kh. Some remarks on the Biot system of equations describing wave propagation in a porous medium // Appl. Math. Lett. — 2000. — Vol. 13, No. 3. — P. 33–35.

[18] Levander A.R. Fourth order velocity-stress finite-difference scheme // Proc. 57th SEG Annual Meeting. — New Orleans, 1987. — P. 234–245.