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Concentrated force in a porous half-space*

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The formulas for calculation the displacement of an elastic porous body, strains and pressure for simple forces using the Galerkin vector are obtained.

In [1] R. Mindlin and D. Cheng considered in an elastic half-space a problem of a concentrated force. They obtained the formulas for calculation the displacements and pressure for various simple forces using the Galerkin vector.

The similar problem for porous half-space is considered in the given work: to calculate a displacement of an elastic porous body, strains and pressure for simple forces. Thus the surface $z = 0$ is considered free from strains and pressure.

The equations of motions in a porous medium are obtained in terms of displacements [2], and in terms of velocities of displacements [3, 4]. Thus the Biot model contains four elastic constants. As was shown in [4, 5], the equations of motions of a porous medium contain three elastic constants.

We consider a boundary value problem in a porous half-space $z > 0$:

$$\mathrm{N}\Delta \boldsymbol{U} + (\mathrm{D}+\mathrm{N})\nabla \operatorname{div} \boldsymbol{U} + \mathrm{Q}\nabla \operatorname{div} \boldsymbol{V} + \rho_{0,s}\boldsymbol{f} = 0, \tag{1}$$

$$\mathrm{Q}\nabla \operatorname{div} \boldsymbol{U} + \mathrm{R}\nabla \operatorname{div} \boldsymbol{V} + \rho_{0,l}\boldsymbol{f} = 0, \tag{2}$$

$$\sigma_{xz}|_{z=0} = \sigma_{yz}|_{z=0} = \sigma_{zz}|_{z=0} = \frac{\rho_{0,l}}{\rho_0} P\Big|_{z=0} = 0, \tag{3}$$

where Q, R, N, D are the elastic constants of the Biot theory, $\boldsymbol{U}$ and $\boldsymbol{V}$ are the vectors of displacements of an elastic porous body and fluids with partial densities $\rho_{0,s}$ and $\rho_{0,l}$ correspondingly, σ_{jz} is a strain tensor, $j = x, y, z$, P is a pressure, $\boldsymbol{f}$ is a mass force, $\rho_0 = \rho_{0,s} + \rho_{0,l}$.

After differentiating on time system (4.2) from [2] and comparing with system (7.4), (7.5) from [5], we obtain the expression for the coefficients Q, R, N, D.

Then the system of equations (1), (2) has the following form:

$$\mu\Delta \boldsymbol{U} + \left(\lambda + \mu + \alpha\rho_{0,s}^2 - 2K\frac{\rho_{0,s}}{\rho_0}\right)\nabla \operatorname{div} \boldsymbol{U} - \rho_{0,l}(K/\rho_0 - \rho_{0,s}\alpha)\nabla \operatorname{div} \boldsymbol{V} + \rho_{0,s}\boldsymbol{f} = 0, \tag{4}$$

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$$-\rho_{0,l}(K/\rho_0 - \rho_{0,s}\alpha)\nabla \operatorname{div} \boldsymbol{U} + \rho_{0,l}^2\alpha\nabla \operatorname{div} \boldsymbol{V} + \rho_{0,l}\boldsymbol{f} = 0. \tag{5}$$

Here λ, μ, $\alpha = \rho_0\alpha_3 + K/\rho_0^2$ are the constants from the equation of state [5], $K = \lambda + 2\mu/3$.

We consider a porous medium, i.e., $\rho_{0,l} > 0$.

Let us exclude $\nabla \operatorname{div} \boldsymbol{V}$ from (4). Using equation (5), we obtain

$$\mu\Delta\boldsymbol{U} + (\tilde{\lambda} + \mu)\nabla \operatorname{div} \boldsymbol{U} + (\alpha\rho_0)^{-1}K\boldsymbol{f} = 0, \tag{6}$$

where $\tilde{\lambda} = \lambda - (\rho_0^2\alpha)^{-1}K^2$.

From boundary conditions (3) we exclude $\operatorname{div} \boldsymbol{V}$, then using the definitions of a tensor strains and pressure, we obtain

$$\mu\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right)\Big|_{z=0} = 0, \qquad \mu\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right)\Big|_{z=0} = 0,$$
$$2\mu\frac{\partial U_z}{\partial z} + \tilde{\lambda} \operatorname{div} \boldsymbol{U}\Big|_{z=0} = 0. \tag{7}$$

Now we exclude a mass force $\boldsymbol{f}$ from equation (5) and using equation (4), we obtain $\operatorname{div} \boldsymbol{V}$ from the equation

$$\nabla \operatorname{div} \boldsymbol{V} = \frac{\rho_0}{\rho_{0,l}}\frac{\mu}{K}\Delta\boldsymbol{U} + \left(\frac{\lambda+\mu}{K}\frac{\rho_0}{\rho_{0,l}} - \frac{\rho_{0,s}}{\rho_{0,l}}\right)\nabla \operatorname{div} \boldsymbol{U} \tag{8}$$

with a boundary condition

$$\operatorname{div} \boldsymbol{V}|_{z=0} = (\alpha\rho_{0,l})^{-1}\left(K/\rho_0 - \rho_{0,s}\alpha\right) \operatorname{div} \boldsymbol{U}|_{z=0}. \tag{9}$$

Thus, boundary value problem (1)–(3) was divided in two sequentially soluble problems. With the help of a Galerkin vector $\boldsymbol{F} = iX + jY + kZ$, we find the vector of displacements and strains by the following formulas [1]:

$$\begin{aligned}
U_x &= \frac{1}{2\mu}\left[2(1-\tilde{\nu})\Delta X - \frac{\partial}{\partial x}\operatorname{div} \boldsymbol{F}\right],\\
\sigma_{xx} &= 2(1-\tilde{\nu})\frac{\partial}{\partial x}\Delta X + \left(\tilde{\nu}\Delta - \frac{\partial^2}{\partial x^2}\right)\operatorname{div} \boldsymbol{F},\\
U_y &= \frac{1}{2\mu}\left[2(1-\tilde{\nu})\Delta Y - \frac{\partial}{\partial y}\operatorname{div} \boldsymbol{F}\right],\\
\sigma_{yy} &= 2(1-\tilde{\nu})\frac{\partial}{\partial y}\Delta Y + \left(\tilde{\nu}\Delta - \frac{\partial^2}{\partial y^2}\right)\operatorname{div} \boldsymbol{F},\\
U_z &= \frac{1}{2\mu}\left[2(1-\tilde{\nu})\Delta Z - \frac{\partial}{\partial z}\operatorname{div} \boldsymbol{F}\right],\\
\sigma_{zz} &= 2(1-\tilde{\nu})\frac{\partial}{\partial z}\Delta Z + \left(\tilde{\nu}\Delta - \frac{\partial^2}{\partial z^2}\right)\operatorname{div} \boldsymbol{F},
\end{aligned}$$

$$\sigma_{xy} = (1-\tilde{\nu})\left(\frac{\partial}{\partial y}\Delta X + \frac{\partial}{\partial x}\Delta Y\right) - \frac{\partial^2}{\partial x \partial y}\operatorname{div}\boldsymbol{F},$$

$$\sigma_{xz} = (1-\tilde{\nu})\left(\frac{\partial}{\partial x}\Delta Z + \frac{\partial}{\partial z}\Delta X\right) - \frac{\partial^2}{\partial x \partial z}\operatorname{div}\boldsymbol{F},$$

$$\sigma_{yz} = (1-\tilde{\nu})\left(\frac{\partial}{\partial z}\Delta Y + \frac{\partial}{\partial y}\Delta Z\right) - \frac{\partial^2}{\partial y \partial z}\operatorname{div}\boldsymbol{F},$$

$$\Delta^2 \boldsymbol{F} = \frac{1}{\tilde{\nu}-1}(\alpha\rho_0)^{-1}K\boldsymbol{f}.$$

Here i, j, k are the unit vectors of an orthogonal system, $\tilde{\nu} = \tilde{\lambda}/[2(\tilde{\lambda}+\mu)]$. We note that $\tilde{\nu}$ has a sign.

Solving the boundary value problem (8), (9) we obtain the expressions for the pressure

$$P = \left(K - \frac{\lambda+\mu}{K}\rho_0^2\alpha\right)\operatorname{div}\boldsymbol{U} - \left(K - \frac{\lambda+\mu+0.5\tilde{\lambda}}{K}\rho_0^2\alpha\right)\operatorname{div}\boldsymbol{U}|_{z=0} -$$

$$\frac{\mu}{K}\rho_0^2\alpha\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\int_0^z U_3(x,y,\xi)d\xi + \frac{\partial U_3}{\partial z}\right).$$

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