Bull. Nov. Comp. Center, Math. Model. in Geoph., 22 (2020), 1–10

The Laguerre spectral method for solving dynamic problems in porous media*

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Abstract. The paper illustrates the applicability of the Laguerre spectral method to solving a one-dimensional problem of the dynamics of a saturated porous medium. A one-dimensional non-stationary problem of a pulsed action on a saturated porous medium is investigated taking into account the effects of electromagnetoacoustics. The features of the acoustic response to a pulsed electromagnetic action in such a system are numerically discovered. The dependence of the amplitudes of transverse acoustic waves arriving at a boundary of the porous medium on the external magnetic field for various parameters of the porous medium is revealed.

Keywords: porous saturated medium, Laguerre transform, interfacial friction, magnetosonic oscillations

1. Introduction

Models of the dynamics of heterophase saturated porous media are widely used in solving applied geophysical and geological problems. Of particular interest is the problem of the influence of the presence of an external field, in particular, a magnetic field, on the nonlinear dynamics of such media. One of the effective methods for solving nonlinear heterophase models is such a numerical-analytical method as the method of integrating the integral Laguerre transform with respect to time and the finite-difference method with respect to space.

This method is analogous to the spectral-difference method based on the Fourier transform, in which the analogue of frequency is the degree of the Laguerre polynomials. An advantage of the Laguerre transform is the ability to reduce the original system of equations to a system with the separation parameter only on the right side. This approach makes it possible to reduce the computation time when solving non-stationary problems of the dynamics of heterophase media. This method for solving dynamic problems is successfully applied to the problems of the theory of elasticity and viscoelasticity [1, 2]. When applied to two-velocity media, this method was used to describe the acoustics of elastic porous media [3].

*The work was carried out according to the state order of the IGM SB RAS.

2. Formulation of the problem

This paper discusses the axially-symmetric problem of the impulse action of a quasi-stationary magnetic field on a saturated porous medium surrounding a channel with a saturating liquid [4]. The system that is homogeneous along the channel is considered. In the cylindrical coordinates (r, φ, z) for the half-space (R_1, ∞), the velocity of the porous matrix $\boldsymbol{u} = (0, u_\varphi, u_z)$, the velocity of the saturating fluid $\boldsymbol{v} = (0, v_\varphi, v_z)$ and the magnetic field $\boldsymbol{B} = (0, B_\varphi, B_z)$ satisfy the following initial-boundary value problem. The system of governing equations obtained within the framework of the combined theory of electromagnetism in porous fluid-saturated media [5, 6] has the form:

$$\frac{\partial^2 u_\varphi}{\partial t^2} = c_t^2 \frac{\partial}{\partial r}(L u_\varphi) - \varepsilon \varpi \left(\frac{\partial u_\varphi}{\partial t} - \frac{\partial v_\varphi}{\partial t} \right) + \frac{\alpha \varepsilon c_e}{4\pi\sigma} \frac{\partial}{\partial r} \left(\frac{\partial B_z}{\partial t} \right), \tag{1}$$

$$\frac{\partial^2 u_z}{\partial t^2} = c_t^2 L \left(\frac{\partial u_z}{\partial r} \right) - \varepsilon \varpi \left(\frac{\partial u_z}{\partial t} - \frac{\partial v_z}{\partial t} \right) - \frac{\alpha \varepsilon c_e}{4\pi\sigma} L \left(\frac{\partial B_\varphi}{\partial t} \right), \tag{2}$$

$$\frac{\partial v_\varphi}{\partial t} = \varpi (u_\varphi - v_\varphi) + \frac{B_0}{4\pi\rho_l r} L B_\varphi - \frac{\alpha c_e}{4\pi\sigma} \frac{\partial B_z}{\partial r}, \tag{3}$$

$$\frac{\partial v_z}{\partial t} = \varpi (u_z - v_z) + \frac{B_0}{4\pi\rho_l r} \frac{\partial B_z}{\partial r} + \frac{\alpha c_e}{4\pi\sigma} L B_\varphi, \tag{4}$$

$$\frac{\partial B_\varphi}{\partial t} = \frac{\partial}{\partial r} \left(\frac{c_e^2}{4\pi\sigma} L B_\varphi - \frac{\alpha c_e \rho_l}{\sigma} (u_z - v_z) + v_\varphi \frac{B_0}{r} \right), \tag{5}$$

$$\frac{\partial B_z}{\partial t} = L \left(\frac{c_e^2}{4\pi\sigma} \frac{\partial B_z}{\partial r} + \frac{\alpha c_e \rho_l}{\sigma} (u_\varphi - v_\varphi) + v_z \frac{B_0}{r} \right). \tag{6}$$

The boundary $r = R_1$ is considered to be free from the stress and magnetic field

$$\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} = 0, \quad \frac{\partial u_z}{\partial r} = 0, \quad B_\varphi = 0, \quad B_z = B_{z0}(t), \tag{7}$$

and the initial Cauchy data are taken to be zero.

Here $\rho = \rho_l + \rho_s$, ρ_s and ρ_l are the partial densities of the matrix and liquid, u and v are the matrix and fluid velocities, B is the magnetic induction, σ is the electrical conductivity, $\varpi = \rho_l \chi$, χ is the coefficient of friction, α is the electroacoustic parameter, A_t is the sound velocity, c_e is the velocity of light. A constant magnetic field $(B_0/r, 0, 0)$ is directed along the axis Or. The operator L is defined by the relation $Lu = \dfrac{\partial u}{\partial r} + \dfrac{u}{r}$.

3. Algorithm for solving the problem

To solve problem (1)–(7), the integral Laguerre transform with respect to time is applied

$$\vec{W}_m(x_1,x_2) = \int_0^\infty \vec{W}(x_1,x_2,t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht),$$

with the inversion formulas

$$\vec{W}(x_1,x_2,t) = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} \vec{W}_m(x_1,x_2) l_m^\alpha(ht),$$

where $l_m^\alpha(ht)$ are the Laguerre functions, which are expressed in terms of the classical orthonormal Laguerre polynomials $L_m^\alpha(ht)$ [7].

In this paper, the parameter α is chosen to be integer and positive. Then

$$l_m^\alpha(ht) = (ht)^{\alpha/2} e^{-ht/2} L_m^\alpha(ht).$$

For the first derivative of the Laguerre polynomials with respect to the variable t, the following expression is valid:

$$\frac{\partial}{\partial t} L_m^\alpha(ht) = -h \sum_{k=0}^{m-1} L_k^\alpha(ht).$$

It can be noted that in order to satisfy the initial conditions of problem (1)–(7), it is sufficient to choose the value $\alpha \geq 1$. In addition, the shift parameter $h > 0$ is introduced in the formulas, whose meaning and efficiency is discussed in detail in [8, 9].

As a result of this transformation, the original problem is reduced to a boundary value problem for a system of ordinary differential equations in the spectral domain

$$\frac{h^2}{4} u_\varphi^m - c_t^2 \frac{d}{dr} L u_\varphi^m + \varepsilon\varpi \frac{h}{2}(u_\varphi^m - v_\varphi^m) - \frac{\varepsilon\alpha A_e}{4\pi\sigma} \frac{h}{2} \frac{dB_z^m}{dr} = f_1^m, \tag{8}$$

$$\frac{h^2}{4} u_z^m - c_t^2 L \frac{du_z^m}{dr} u_\varphi^m + \varepsilon\varpi \frac{h}{2}(u_z^m - v_z^m) + \frac{\varepsilon\alpha A_e}{4\pi\sigma} \frac{h}{2} L B_\varphi^m = f_2^m, \tag{9}$$

$$\frac{h}{2} v_\varphi^m - \varpi(u_\varphi^m - v_\varphi^m) - \frac{B_0}{4\pi\sigma r} L B_\varphi^m + \frac{\alpha A_e}{4\pi\sigma} \frac{dB_z^m}{dr} = -h \sum_{n=0}^{m-1} v_\varphi^n, \tag{10}$$

$$\frac{h}{2} v_z^m - \varpi(u_z^m - v_z^m) - \frac{B_0}{4\pi\rho_l r} \frac{dB_z^m}{dr} - \frac{\alpha A_e}{4\pi\sigma} L B_\varphi^m = -h \sum_{n=0}^{m-1} v_z^n, \tag{11}$$

$$\frac{h}{2} B_\varphi^m - \frac{d}{dr}\left(\frac{A_e^2}{4\pi\sigma} L B_\varphi^m - \frac{\alpha A_e \rho_l}{\sigma}(u_z^m - v_z^m) + v_\varphi^m \frac{B_0}{r} \right) = -h \sum_{n=0}^{m-1} B_\varphi^n, \tag{12}$$

$$\frac{h}{2} B_z^m - L\left(\frac{A_e^2}{4\pi\sigma} \frac{dB_z^m}{dr} + \frac{\alpha A_e \rho_l}{\sigma}(u_\varphi^m - v_\varphi^m) + v_z^m \frac{B_0}{r} \right) = -h \sum_{n=0}^{m-1} B_z^n \tag{13}$$

with the boundary conditions at $r = R_1$

$$\frac{\partial u_\varphi^m}{\partial r} - \frac{u_\varphi^m}{r} = 0, \quad \frac{\partial u_z^m}{\partial r} = 0, \quad B_\varphi^m = 0, \quad B_z^m = B_{z0}(t). \tag{14}$$

Here f_1^m, f_2^m are the coefficients of the Laguerre expansion of the source function $f(t)$:

$$f_1^m = -h^2 \sum_{n=0}^{m-1} (m-n) u_\varphi^n - h\varepsilon\varpi \sum_{n=0}^{m-1} (u_\varphi^n - v_\varphi^n) + \frac{h\varepsilon\alpha A_e}{4\pi\sigma} \sum_{n=0}^{m-1} \frac{dB_z^n}{dr},$$
$$f_2^m = -h^2 \sum_{n=0}^{m-1} (m-n) u_z^n - h\varepsilon\varpi \sum_{n=0}^{m-1} (u_z^n - v_z^n) - \frac{h\varepsilon\alpha A_e}{4\pi\sigma} \sum_{n=0}^{m-1} L B_\varphi^n,$$

u_i^m, v_i^m, B_i^m, $i \in \{\varphi, z\}$, are the expansion coefficients of the corresponding field components in a series of the Laguerre functions. The superscript m denotes the number of the coefficient in the Laguerre expansion. Note that the value of m explicitly presents only in the right-hand side of the equations in the form of a recurrent dependence for all the field components.

The finite difference approximation of derivatives with respect to the spatial coordinates for solving this problem is carried out on staggered grids with the fourth order of accuracy [10]. For this, in the computational domain in the direction of the coordinate r, the grids ωr and $\omega r_{1/2}$ are introduced with a discretization step Δr that is shifted relative to each other by $\Delta r/2$:

$$\omega r = (j\Delta r, t), \quad \omega r_{1/2} = ((j+1/2)\Delta r, t), \quad j = 0, \dots, M.$$

Let us define the unknown components of the solution vector at the following grid nodes:

$$u_i^m(r), v_i^m(r), B_i^m(r) \in \omega r, \quad u_r^m(r), v_r^m(r), B_r^m(r) \in \omega r_{1/2}.$$

The choice of the location of the components at the integer and half-integer grid nodes is based on the difference approximation of the equations of system (8)–(13) and satisfying the boundary conditions (14), for which the second order of accuracy is used.

As a result of the finite difference approximation of problem (8)–(14), we obtain a system of linear algebraic equations. Representing the desired solution vector $\vec{W}$ in the form

$$\vec{W}(m) = \big(V_0, V_1, \dots, V_m\big)^T,$$
$$V_j = (u_\varphi^j, u_z^j, u_r^{j+1/2}, v_\varphi^j, v_z^j, v_r^{j+1/2}, B_\varphi^j, B_z^j, B_r^{j+1/2}),$$

this system of linear algebraic equations can be written down as

$$\Big(A_\Delta + \frac{h}{2}E\Big)\vec{W}(m) = \vec{F}_\Delta(m-1). \tag{15}$$

A sequence of the wave field components in the solution vector $\vec{W}$ is selected taking into account the minimization of the number of diagonals in the matrix A_Δ. In this case, on the main diagonal of the matrix, the components included in the equations of the system as terms with the parameter h as a factor are specially located. It should be noted that by choosing the parameter h, it is possible to significantly improve the conditionality of the system matrix. Having solved the system of linear algebraic equations (15), we can determine the spectral values for all the components of the wave field $\vec{W}(m)$. Then, using the Laguerre transformation inversion formula, we can obtain a solution to the original problem (1)–(7).

To solve the system of linear algebraic equations (15), the most effective was the use of the iterative conjugate gradient method. The advantages of this method is that there is no need to store the entire matrix in a machine memory for large-scale systems, as well as fast convergence to the solution of the problem provided that the matrix of the system is well conditioned. The matrix A_Δ due to the parameter h has this property. The choice of h can significantly accelerate the convergence of the iterative process. The optimal value of h is chosen based on minimizing the number of Laguerre harmonics in the inversion formula, as well as reducing the number of iterations when finding a solution for each harmonic. The analysis of test calculations shows the stability of the algorithm proposed for the studied class of the model.

4. Simulation results

The solution to the problem in question provides a basis for the development of technological methods for measuring the permeability and electrical conductivity of the rock [4, 11, 12]. In the model problem, a medium with the following physical parameters (CGS system) is considered: $\alpha = 10^7$ $(\mathrm{cm}^3/\mathrm{gc}^2)^{1/2}$, $R_1 = 10$ cm, $\sigma = 10^9$ c^{-1}, $\varpi = 10^7$ c^{-1}, $B_0 = 10^3$, $\rho_l = 0.1$ g/cm^3, $c_e = 3 \cdot 10^{10}$ cm/c, $c_t = 1 \cdot 10^5$ cm/c, $\varepsilon = 0.1$. At the channel boundary, an impulse action as a function of time is set. The configurable source signal

$$B_{z0}(t) = 10^3 \exp(-\pi f_0 (t - t_0)^2/8) \sin(2\pi f_0 (t - t_0)),$$

with $f_0 = 100$ Hz, $t_0 = 0.015$ s is shown in Figure 1.

Figure 1. The source signal

Figures 2–5 show the calculated seismic traces of the components of the displacement velocities of the porous matrix for two values of the electrical conductivity σ. The distance along the coordinate r between the points of the calculated paths is $\Delta r = 50$ m. The calculation results for the component u_φ and for the time derivative of the component u_z are presented.

Let us also present the graphs illustrating a change in the displacement rates with a simultaneous proportional change in the coefficient α and in the coefficient of the interfacial friction χ. The corresponding calculated seismic traces are presented in Figures 6, 7. The distance along the coordinate r between the points of the calculated paths is $\Delta r = 50$ m.

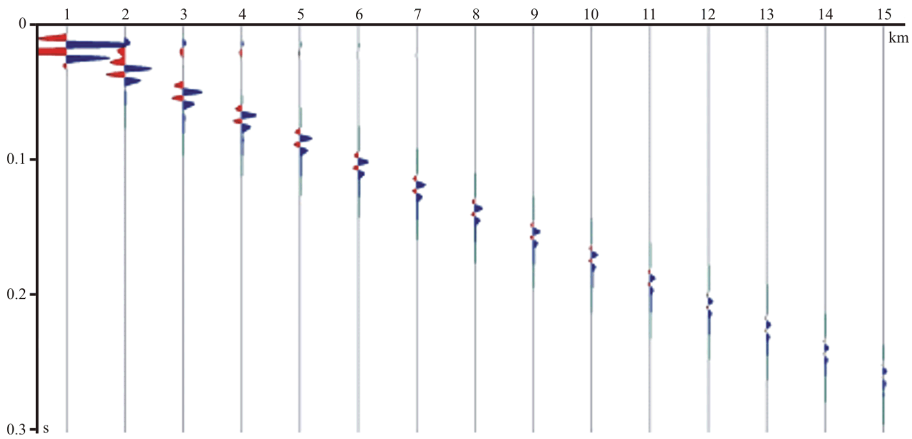

Figure 2. Calculated seismic traces for the component u_φ at $\sigma = 10^9$ c^{-1}

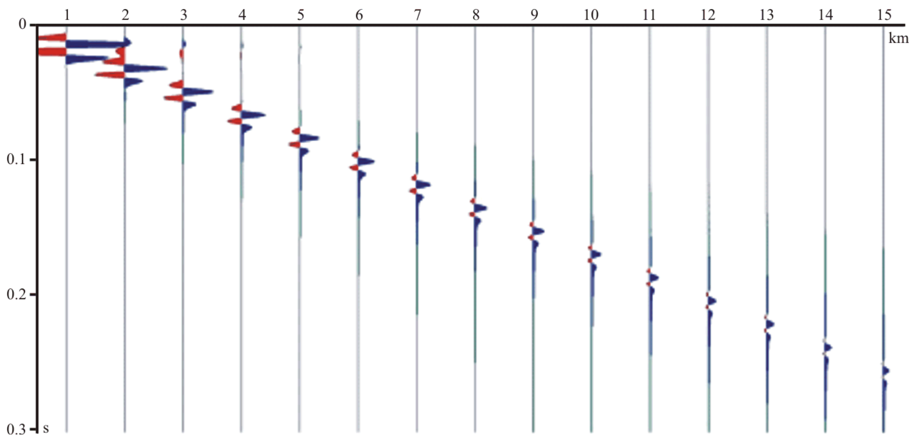

Figure 3. Calculated seismic traces for the component u_φ at $\sigma = 5 \cdot 10^9$ c^{-1}

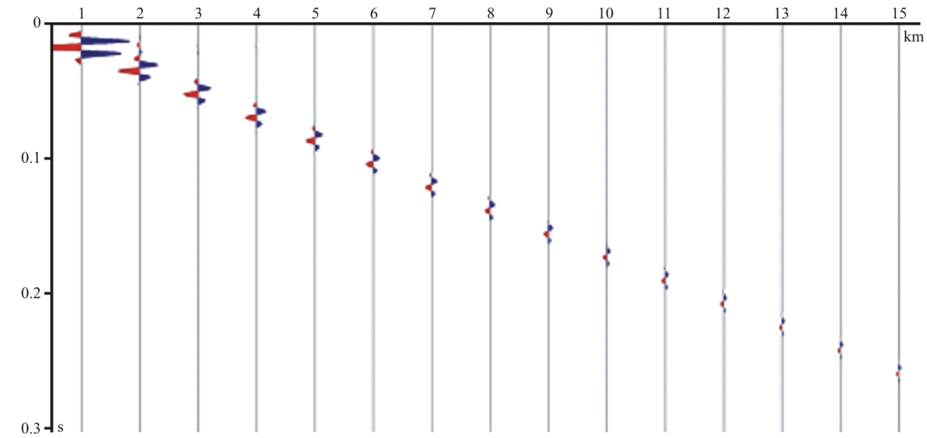

Figure 4. Calculated seismic traces for the component $\dfrac{\partial u_z}{\partial t}$ at $\sigma = 10^9$ c^{-1}

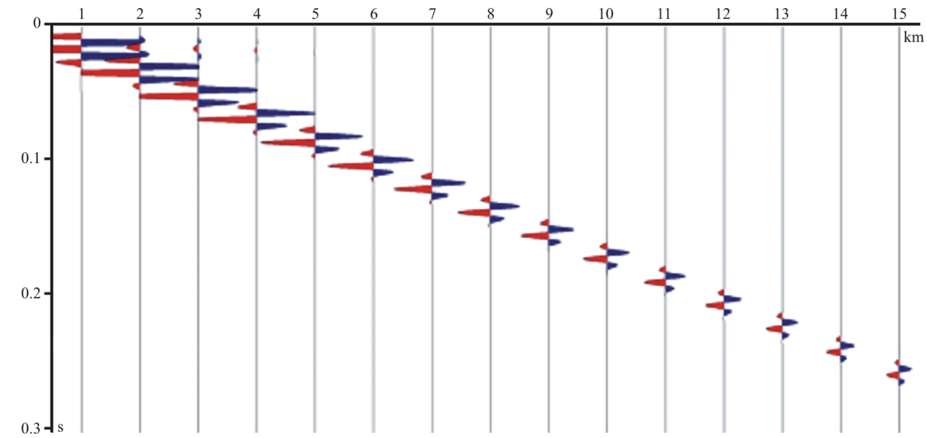

Figure 5. Calculated seismic traces for the component $\dfrac{\partial u_z}{\partial t}$ at $\sigma = 5 \cdot 10^9$ c^{-1}

The numerical analysis of the calculations performed has shown that at the inner boundary of the porous space, the maximum values of acoustic amplitudes depend on the parameters of the saturated porous medium and the amplitude of the magnetic field

$$\frac{\partial u_z}{\partial t} = \frac{\sigma}{c_e} \frac{B_0 B}{4\pi\rho}, \qquad u_\varphi = \frac{\alpha}{\chi} \frac{B}{4\pi\rho} \frac{1}{k(R_1)},$$

where $k(R_1)$ is the calculated function of the inner radius of the well. The fulfilment of the formula $u_\varphi \sim \partial u_z / \partial t$ makes possible to generalize the earlier obtained formula in the frequency representation $u_z = i\omega_* \omega^{-1} u_y$ [13] (ω_* is the characteristic frequency at which the moduli of the acoustic response of the shear waves $|u_y|$, $|u_z|$ are equal). The implementation of this formula

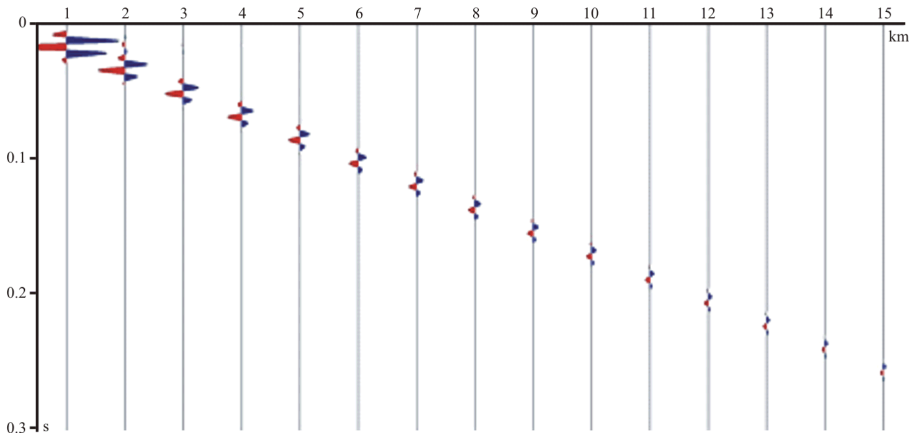

Figure 6. Calculated seismic traces for the component $\dfrac{\partial u_z}{\partial t}$ at $\varpi = 10^7$ c^{-1}, $\alpha = 10^7$ (cm^3/gc^2)$^{1/2}$

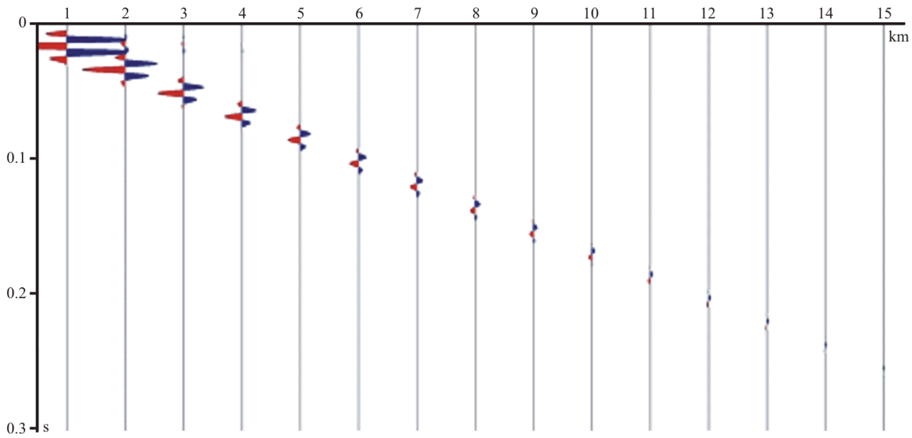

Figure 7. Calculated seismic traces for the component $\dfrac{\partial u_z}{\partial t}$ at $\varpi = 3 \cdot 10^7$ c^{-1}, $\alpha = 3 \cdot 10^7$ (cm^3/gc^2)$^{1/2}$

makes it possible to determine the electrical conductivity of a rock and the ratio of its permeability and electroacoustic constant from the measured values of the amplitudes of acoustic and electromagnetic fields at a known rock density.

5. Conclusion

A numerical-analytical algorithm based on the spectral Laguerre method as applied to the problem of analyzing the acoustic response of a porous saturated medium to an external induction action in an external magnetic field is considered. This algorithm is an analogue of the known spectral methods for solving dynamic problems in the continuum mechanics, however, unlike the Fourier and the Laplace transforms, the use of the Laguerre transform leads to a system of equations in which the harmonic separation parameter is included only in the right-hand side in a recurrent form. As a result, the matrix of the reduced problem system is well conditioned. This fact allows the use of effective methods for solving systems of linear algebraic equations.

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