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On hydrodynamics equations for conducting porous media*

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A combined mathematical model of Maxwell's system of equations and a system of equations for porous media taking into account admixtures is constructed.

1. Introduction

A nonlinear mathematical model of a conducting liquid moving through a conducting elastic deformable porous medium was proposed in [1]. An analysis of small oscillations for the linear variant of the obtained combined mathematical model of Maxwell's equations and the equations of continual filtration theory was performed. An analysis of the propagation of monochromatic waves along the x-axis of a Cartesian system of coordinates showed that for small values of a constant external magnetic field $\boldsymbol{H} = (H_x, H_y, 0)$ there exist four types of sonic oscillations: (a) oscillations of the Alfven type, i.e., the velocity of a wave of this type is proportional to the amplitude of the external magnetic field $u_A \sim H_x$; (b) transverse oscillations, i.e., the velocity of a transverse wave has a quadratic addition to the field H_x; (c) and two types of longitudinal oscillations not affected by the longitudinal component of the external magnetic field H_x. The velocity of the wave has a quadratic addition to the field H_y.

For an arbitrarily oriented external constant magnetic field $\boldsymbol{H}$, we have six sonic modes: two transverse modes and four mixed modes which result from interaction of transverse magnetosonic oscillations with two longitudinal oscillations.

The dispersion of an "Alfven wave" caused by the rubbing of components against each other was investigated. It turns out that for weak magnetic fields the velocity of an Alfven wave under conditions of absorption at friction between components is dispersive: the velocity is a monotonically increasing function of the frequency (it is proportional to the square root of the frequency). The velocity varies from zero at $\omega = 0$ to a constant value, i.e., the Alfven velocity u_A at $\omega = \infty$.

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It was shown experimentally [2] that the effect of a constant electric current of small density on a liquid-saturated porous medium in its natural state is expressed by the fact that the time dependencies of the velocities of longitudinal and transverse waves are nonlinear. The important role of water and its mineralization was pointed out in [3, 4]. It was also found (see [5]) that in elastic media, in contrast to seismomagnetic effects, seismoelectric effects occur only in water-saturated samples.

A combined mathematical model of Maxwell's equations and the equations of porous media taking into account admixtures is constructed here.

2. Ideal case

Consider a conducting liquid in a conductive elastic deformable porous medium. In the continual approximation, an element of the continuum is characterized by the unit-volume entropy S, the density ρ, the velocities of the conducting porous body $\boldsymbol{u}$ and the conducting liquid $\boldsymbol{v}$, by the corresponding partial densities (conductivities) $\rho_s(\sigma_s)$, and $\rho_l(\sigma_l)$, and by the metric deformation tensor g_{ik}.

Here the following conservation laws of mass, entropy, momentum (with allowance for electromagnetic forces), energy, and the equations of metric deformation tensor must be satisfied:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0, \quad \rho = \rho_s + \rho_l, \quad \boldsymbol{j} = \rho_s \boldsymbol{u} + \rho_l \boldsymbol{v}, \\
&\frac{\partial \rho_\nu}{\partial t} + \operatorname{div} \boldsymbol{j}_\nu = 0, \quad \nu = 1, 2, 3, \dots, \\
&\frac{\partial j_i}{\partial t} + \partial_k \Pi_{ik} = \frac{1}{c}[\boldsymbol{J}, \boldsymbol{B}]_i, \quad \Pi_{ik} = \rho_s u_i u_k + \rho_l v_i v_k + p\delta_{ik} + h_{ij} g_{jk}, \\
&\frac{\partial S}{\partial t} + \operatorname{div}\Big(\frac{S}{\rho}\boldsymbol{j}\Big) = 0, \\
&\frac{\partial g_{ik}}{\partial t} + g_{kj}\partial_i u_j + g_{ij}\partial_k u_j + u_j \partial_j g_{ik} = 0, \quad \rho_s = \mathrm{const}\sqrt{\det(g_{ik})}, \\
&\frac{\partial e}{\partial t} + \operatorname{div}\Big(\boldsymbol{Q} + \frac{c}{4\pi}[\boldsymbol{E}, \boldsymbol{B}]\Big) = 0, \\
&Q_k = \Big(\mu + \frac{\boldsymbol{v}^2}{2} + \frac{TS}{\rho}\Big) j_k + \rho_s(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}) u_k + u_i h_{km} g_{mi}.
\end{aligned}
\tag{1}
$$

Also, the system of Maxwell's equations

$$
\operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \operatorname{rot} \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \quad \operatorname{div} \boldsymbol{B} = \operatorname{div} \boldsymbol{E} = 0 \tag{2}
$$

must be satisfied.

In equations (1) and (2), e is the energy per unit volume, $\boldsymbol{j}_\nu$ are mass fluxes of admixtures, ρ_ν are the partial densities of the admixtures, $\boldsymbol{B}$ and $\boldsymbol{E}$ are the electric and magnetic field strengths respectively. The following relations guarantee that the conservation laws are invariant with respect to Galilei's transformation [6]:

$$e = e_0(\rho, \rho_1, \dots, \rho_\nu, \dots, S, \boldsymbol{j}_0, g_{ik}, \boldsymbol{B}, \boldsymbol{E}) + \frac{\rho \boldsymbol{v}^2}{2} + (\boldsymbol{v}, \boldsymbol{j}_0), \quad \boldsymbol{j} = \rho \boldsymbol{v} + \boldsymbol{j}_0,$$

where the internal energy includes the energy of admixtures:

$$de_0 = TdS + \mu d\rho + \mu_\nu d\rho_\nu + (\boldsymbol{u} - \boldsymbol{v}, d\boldsymbol{j}_0) + \frac{h_{ik}}{2} dg_{ik} + \left(\frac{\boldsymbol{B}}{4\pi}, d\boldsymbol{B}\right) + \left(\frac{\boldsymbol{E}}{4\pi}, d\boldsymbol{E}\right).$$

Here and hereafter summation is over ν; μ and μ_ν are chemical potentials and T is the temperature.

Let us write the equation of motion of a conducting liquid taking into account the conditions of equilibrium from [7]:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = -\nabla \mu - \frac{S}{\rho} \nabla T + C_\nu \nabla \mu_\nu + \frac{\sigma_l}{c \rho_l \sigma} [\boldsymbol{J}, \boldsymbol{B}].$$

Repeating a procedure of integrating for a reversible model [1], we obtain the total system of equations of nonlinear filtration theory in the presence of admixtures. In the magnetic hydrodynamic approximation, this system of differential equations has the following form:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0, \quad \rho = \rho_s + \rho_l, \quad \boldsymbol{j} = \rho_s \boldsymbol{u} + \rho_l \boldsymbol{v},\\
&\frac{\partial \rho_\nu}{\partial t} + \operatorname{div}\left(\frac{\rho_\nu}{\rho} \boldsymbol{j}\right) = 0, \quad \nu = 1, 2, 3, \dots,\\
&\frac{\partial S}{\partial t} + \operatorname{div}\left(\frac{S}{\rho} \boldsymbol{j}\right) = 0, \quad \frac{\partial j_i}{\partial t} + \partial_k \tilde{\Pi}_{ik} = 0, \quad \tilde{\Pi}_{ik} = \Pi_{ik} - \frac{B_i B_k}{4\pi} + \frac{B^2}{8\pi} \delta_{ik},\\
&\frac{\partial g_{ik}}{\partial t} + g_{kj} \partial_i u_j + g_{ij} \partial_k u_j + u_j \partial_j g_{ik} = 0, \qquad \rho_s = \mathrm{const} \sqrt{\det(g_{ik})},\\
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \frac{\rho_s}{2\rho} \nabla (\boldsymbol{u} - \boldsymbol{v})^2 - \frac{h_{ik}}{2\rho} \nabla g_{ik} + \frac{\sigma_l}{4\pi \rho_l \sigma} [\operatorname{rot} \boldsymbol{B}, \boldsymbol{B}],\\
&\frac{\partial \boldsymbol{B}}{\partial t} - \operatorname{rot}\left[\frac{\sigma_s}{\sigma} \boldsymbol{u} + \frac{\sigma_l}{\sigma} \boldsymbol{v}, \boldsymbol{B}\right] = 0, \qquad \operatorname{div} \boldsymbol{B} = 0,\\
&e_0 = e_0(\rho, \rho_1, \dots, \rho_\nu, \dots, S, \boldsymbol{j}_0, g_{ik}, \boldsymbol{B}).
\end{aligned} \tag{3}$$

Here we introduced the pressure as follows:

$$p = -e_0 + TS + \mu \rho + \mu_\nu \rho_\nu + (\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{j}_0).$$

This coincides with the traditional thermodynamic definition. The system of equations (3) does not contain explicitly the conservation laws of mass and energy for the conducting elastic porous body:

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div} \rho_s \boldsymbol{u} = 0, \quad \frac{\partial e}{\partial t} + \operatorname{div}\Big(\boldsymbol{Q} - \frac{\sigma_s}{4\pi\sigma}\big[[\boldsymbol{u}, \boldsymbol{B}], \boldsymbol{B}\big] - \frac{\sigma_l}{4\pi\sigma}\big[[\boldsymbol{v}, \boldsymbol{B}], \boldsymbol{B}\big]\Big) = 0,$$

since these equations follow from the other equations.

3. Dissipative case

Having a formal scheme of construction of a reversible approximation, it is possible to construct full equations of motion taking into account coefficients of viscosity, friction, heat conductivity, and other kinetic quantities. The irreversible approximation is constructed by adding irreversible terms which are invariant with respect to Galilei's transformations to the corresponding fluxes:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0, \quad \rho = \rho_s + \rho_l, \quad \boldsymbol{j} = \rho_s \boldsymbol{u} + \rho_l \boldsymbol{v}, \\
&\frac{\partial \rho_\nu}{\partial t} + \operatorname{div}\Big(\frac{\rho_\nu}{\rho}\boldsymbol{j} - \frac{\lambda_\nu}{T}(\boldsymbol{j} - \rho \boldsymbol{u}) + \boldsymbol{L}_\nu\Big) = 0, \qquad \nu = 1, 2, 3, \dots, \\
&\frac{\partial S}{\partial t} + \operatorname{div}\Big(\frac{S}{\rho}\boldsymbol{j} - \frac{\lambda}{T}(\boldsymbol{j} - \rho\boldsymbol{u}) + \frac{\lambda_\nu \mu_\nu}{T^2}(\boldsymbol{j} - \rho\boldsymbol{u}) - \frac{\mu_\nu \boldsymbol{L}_\nu}{T} + \frac{\boldsymbol{q}}{T}\Big) = \frac{R}{T}, \qquad (4)\\
&\frac{\partial j_i}{\partial t} + \partial_k\Big(\tilde{\Pi}_{ik} + \pi_{ik}\Big) = 0, \qquad \frac{\partial e}{\partial t} + \operatorname{div}\Big(\boldsymbol{Q} + \frac{c}{4\pi}[\boldsymbol{E}, \boldsymbol{B}] + \boldsymbol{W}\Big) = 0, \\
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = -\nabla(\mu + h) - \frac{\rho_\nu}{\rho}\nabla \mu_\nu - \frac{S}{\rho}\nabla T + \frac{\sigma_l}{c\rho_l \sigma}[\boldsymbol{J}, \boldsymbol{B}] + \boldsymbol{f}^\partial.
\end{aligned}$$

Here the fluxes $\boldsymbol{L}_\nu$, π_{ik}, $\boldsymbol{q}$, $\boldsymbol{W}$, the functions of dissipative nature $\boldsymbol{f}^\partial$, h, and the dissipative function R are to be determined. The addition to the mass $\boldsymbol{j}$ does not agree with the invariance of the irreversible flow with respect to Galilei's transformations [8].

The combination procedure results in the fluxes:

$$W_i = q_i + h(j_i - \rho u_i) - \lambda(j_i - \rho u_i) + v_k \pi_{ik}, \qquad \pi_{ik} = A_{ik} + a\delta_{ik}, \qquad (5)$$

and makes it possible to calculate the dissipative function:

$$\begin{aligned}
-R = {} & \Big(\frac{\boldsymbol{q}}{T}, \nabla T\Big) + \Big(f_i^\partial + \frac{1}{\rho_l}\partial_k \pi_{ik}\Big)(j_i - \rho u_i) - \\
& \Big(\boldsymbol{J}, \boldsymbol{E} + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}] + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}]\Big) + T\boldsymbol{L}_\nu \nabla\Big(\frac{\mu_\nu}{T}\Big) + \\
& a \operatorname{div} \boldsymbol{v} + h \operatorname{div}(\boldsymbol{j} - \rho\boldsymbol{u}) + \frac{1}{2}A_{ik}\Big(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v}\Big). \qquad (6)
\end{aligned}$$

In accordance with the Curie theorem for isotropic bodies, the thermodynamic fluxes

$$h, \quad \boldsymbol{a}, \quad \boldsymbol{J}, \quad \boldsymbol{q}, \quad f_i^{\partial} + \frac{1}{\rho_l}\partial_k \pi_{ik}, \quad \boldsymbol{L}_\nu, \quad \text{and} \quad A_{ik}$$

are proportional to the thermodynamic forces

$$\operatorname{div}(\boldsymbol{j} - \rho \boldsymbol{u}), \quad \operatorname{div} \boldsymbol{v}, \quad \boldsymbol{E} + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}] + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}], \quad \nabla T,$$
$$\boldsymbol{j} - \rho \boldsymbol{u}, \quad \nabla\left(\frac{\mu_\nu}{T}\right), \quad \text{and} \quad \partial_i v_k + \partial_k v_i - \frac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v},$$

taking into account the respective tensor dimensionalities.

For scalar fluxes we have

$$-a = \zeta_{11} \operatorname{div} \boldsymbol{v} + \zeta_{12} \operatorname{div}(\boldsymbol{j} - \rho \boldsymbol{u}), \quad -h = \zeta_{12} \operatorname{div} \boldsymbol{v} + \zeta_{22} \operatorname{div}(\boldsymbol{j} - \rho \boldsymbol{u}). \tag{7}$$

For vector fluxes we have

$$\begin{aligned}
-f_i^{\partial} - \frac{1}{\rho_l}\partial_k \pi_{ik} &= \alpha_{11}(j_i - \rho u_i) + \alpha_{12}\left(E_i + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}]_i + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}]_i\right) + \\
&\quad \frac{\lambda}{T}\partial_i T + \lambda_\gamma \partial_i\left(\frac{\mu_\nu}{T}\right), \\
J_i &= \alpha_{12}(j_i - \rho u_i) + \alpha_{22}\left(E_i + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}]_i + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}]_i\right) + \\
&\quad \frac{\alpha_{23}}{T}\partial_i T + \alpha_{2\gamma}\partial_i\left(\frac{\mu_\nu}{T}\right), \\
-q_i &= \lambda(j_i - \rho u_i) + \alpha_{23}\left(E_i + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}]_i + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}]_i\right) + \\
&\quad \frac{\alpha_{33}}{T}\partial_i T + \alpha_{3\gamma}\partial_i\left(\frac{\mu_\nu}{T}\right), \\
-T\boldsymbol{L}_{\nu,i} &= \lambda_\nu(j_i - \rho u_i) + \alpha_{2\nu}\left(E_i + \frac{\sigma_s}{c\sigma}[\boldsymbol{u}, \boldsymbol{B}]_i + \frac{\sigma_l}{c\sigma}[\boldsymbol{v}, \boldsymbol{B}]_i\right) + \\
&\quad \frac{\alpha_{3\nu}}{T}\partial_i T + \alpha_{\nu\gamma}\partial_i\left(\frac{\mu_\nu}{T}\right).
\end{aligned} \tag{8}$$

The tensor flux is expressed in terms of

$$A_{ik} = -\eta\left(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v}\right). \tag{9}$$

System (4) with formulas (5) to (9) represents a full system of differential equations for a conducting liquid moving through a conducting elastically deformable porous medium with admixtures. The equation of state $e_0(\rho, \rho_1, \ldots, \rho_\nu, \ldots, \boldsymbol{j}_0, g_{ik}, \boldsymbol{B}, \boldsymbol{E})$ is specified.

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