Bull. Nov. Comp. Center, Math. Model. in Geoph., 5 (1999), 59–62

On the diffraction of nonstationary SH-wave on semi-infinite crack in porous elastic medium*

Kh.Kh. Imomnazarov, A.Kh. Matevosyan

The problem of diffraction of a plane SH shock wave on a semi-infinite crack for a model of porous medium for the case of energy loss due to intercomponent friction is considered. Formulas for the scattered wave in terms of the Laplace transforms with respect to time are obtained.

1. Introduction

Elastic oscillations excited in rocks as a result of earthquakes can be of interest not only as a cause of destruction, but also as a creative factor that leads to the formation of deposits of useful minerals. Many physical and biological phenomena are connected with earthquakes. The appearance of cracks resulting from large stresses in plates of lithosphere is observed in areas which may become earthquake origins [1]. Such cracks [2] are characterized by large values of stresses at their edges. The magnitude of these stresses determines the possibilities for stability and further growth of the cracks [3]. During the formation process of a crack, stresses close to a stability limit act at its edges, and minor variations in the stresses can lead to an increase in the size of the crack.

The problem of diffraction of a plane SH-wave on a semi-infinite crack for a model of ideal elasticity theory was considered in [4]. In this paper, we consider a similar problem for a porous medium for the case of energy loss due to intercomponent friction.

2. Problem statement

Let us consider an unbounded porous isotropic medium with elastic constants λ, μ, α, friction coefficient χ, partial density ρ_s of elastic porous body, and partial density ρ_l of liquid. In this medium, there is a crack (defect) which is considered to be the half-plane

*Supported by the Russian Foundation for Basic Research under Grants 96-15-98542, 99-05-64538, and the Youth Grant of the Siberian Branch of the Russian Academy of Sciences.

$$z = 0, \quad 0 \le x < \infty, \quad -\infty < y < \infty. \tag{1}$$

A SH-wave of the form

$$u^0(t,x,z) = \tilde{\varphi}\Big(t - \frac{x\sin\alpha - z\cos\alpha}{c_t}\Big), \quad c_t^2 = \mu\rho_s^{-1}, \tag{2}$$

is incident on this crack. Here $\tilde{\varphi}(\xi)|_{\xi<0} \equiv 0$ (the incident wave $\tilde{\varphi}$ has a leading edge), α is the angle of incidence of the wave. In the absence of a crack, this wave causes the following tangential stresses in the porous isotropic medium:

$$\tau_{yz}^0(t,x,z) = \mu\frac{\partial u^0}{\partial z} = \mu\cos\alpha\frac{d\tilde{\varphi}(\xi)}{d\xi}\bigg|_{\xi=t-(x\sin\alpha - z\cos\alpha)/c_t} \tag{3}$$

Problem. *Find the distributions of stresses and displacements of the elastic porous body, and the liquid $\tau_{yz}(t,x,z)$, $u(t,x,z)$, $v(t,x,z)$ after the incidence of the wave* (2) *on the crack* (1).

Our problem is reduced to solving the system of equations [5]:

$$\frac{1}{c_t^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2} + \frac{\chi}{c_t^2}\frac{\rho_l^2}{\rho_s}\Big(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\Big) = 0, \quad |x| < \infty, \ z \ne 0, \ t > 0, \tag{4}$$

$$\frac{\partial^2 v}{\partial t^2} - \chi\rho_l\Big(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\Big) = 0, \quad |x| < \infty, \ z \ne 0, \ t > 0, \tag{5}$$

at zero initial conditions, and under the following conditions on the crack (1):

$$\langle u(t,x,0)\rangle = u(t,x,-0) - u(t,x,+0) = \varphi(t,x) \ne 0, \quad x > 0, \tag{6}$$

$$\langle v(t,x,0)\rangle = v(t,x,-0) - v(t,x,+0) = \psi(t,x) \ne 0, \quad x > 0, \tag{7}$$

$$\tau_{yz}(t,x,\pm 0) = 0, \quad x \ge 0. \tag{8}$$

3. Algorithm of constructing the solution

We construct a solution to problem (4)–(8) in the following form:

$$\begin{aligned} u(t,x,z) &= u^0(t,x,z) + u^1(t,x,z), \\ v(t,x,z) &= v^0(t,x,z) + v^1(t,x,z), \\ \tau_{yz}(t,x,z) &= \tau_{yz}^0(t,x,z) + \tau_{yz}^1(t,x,z). \end{aligned}$$

Here u^0, τ_{yz}^0 are determined with the help of formulas (2) and (3), u^1, v^1 are discontinuous solutions to the system of equations (4), (5). Let us show that the functions ψ and v^0 are expressed in terms of the functions φ and u^0. Applying the integral Laplace transform with respect to the variable t to both sides of equations (5) and (6), after simple transformations we obtain:

$$\frac{p^2}{c_t^2}\Big(1+\frac{\rho_l^2}{\rho_s}\frac{\chi}{\chi\rho_l-p}\Big)u_p(x,z)-\frac{\partial^2 u_p(x,z)}{\partial x^2}-\frac{\partial^2 u_p(x,z)}{\partial z^2}=0, \tag{9}$$

$$v_p(x,z)=\frac{\chi\rho_l}{\chi\rho_l-p}u_p(x,z), \qquad |x|<\infty,\ \ z\neq 0, \tag{10}$$

$$u_p(x,z)=\int_0^\infty \mathrm{e}^{-pt}u(t,x,z)\,dt.$$

Hence, the functions ψ, v^0 and φ, u^0 are related as follows:

$$v_p^0(x,z)=\frac{\chi\rho_l}{\chi\rho_l-p}u_p^0(x,z), \qquad \psi_p(x)=\frac{\chi\rho_l}{\chi\rho_l-p}\varphi_p(x), \quad x>0. \tag{11}$$

Then, using a method proposed in [4], we construct a discontinuous solution to equations (9) for the crack (1). We use the integral Fourier transform first with respect to the variable x

$$u_{p\beta}^1(z)=\int_{-\infty}^{\infty}u_p^1(x,z)\mathrm{e}^{i\beta x}\,dx,$$

and then with respect to the variable z

$$u_{p\beta\gamma}^1=\Big(\int_{-\infty}^{-0}+\int_{+0}^{\infty}\Big)u_{p\beta}^1(z)\mathrm{e}^{i\gamma z}\,dz,$$

in accordance with a generalized scheme proposed in [6]. Taking the Fourier inverse transform of the sought-for discontinuous solution, we obtain

$$u_p^1(x,z)=\frac{1}{\pi}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\varphi_p(\xi)K_0(PY)\,d\xi, \tag{12}$$

$$-\infty<x,z<\infty, \quad P=\frac{p}{c_t}\sqrt{1+\frac{\rho_l^2}{\rho_s}\frac{\chi}{\chi\rho_l-p}}, \quad Y=\sqrt{(x-\xi)^2+z^2}.$$

Here $K_0(z)$ is the MacDonald function.

Substituting this formula into condition (8) written in terms of the Laplace transforms, in the same way as in [4], we obtain the following integral differential equation for the function $\varphi_p(x)$ for $x\geq 0$:

$$\Big(P^2-\frac{\partial^2}{\partial x^2}\Big)\frac{1}{\pi}\int_0^\infty\varphi_p(\xi)K_0(P|x-\xi|)\,d\xi=F_p(x)\equiv-\frac{p}{c_t}\cos\alpha\tilde{\varphi}_p\mathrm{e}^{\frac{p}{c_t}x\sin\alpha},$$

We change variables in this equation as follows:

$$x=\frac{s}{P}, \quad \xi=\frac{\sigma}{P}, \quad P\varphi_p\Big(\frac{\sigma}{P}\Big)=\tilde{\psi}(\sigma,p), \quad F_p\Big(\frac{s}{P}\Big)=g(s,p). \tag{13}$$

Then it takes the form

$$\Big(1-\frac{d^2}{ds^2}\Big)\frac{1}{\pi}\int_0^\infty\tilde{\psi}(\sigma,p)K_0(|s-\sigma|)\,d\sigma=g(s,p). \tag{14}$$

We seek a solution to the integral differential equation (14) in the following form:

$$\tilde{\psi}(\sigma,p) = \sum_{n=0}^{\infty} \sqrt{\sigma}\mathrm{e}^{-\sigma} L_n^{1/2}(2\sigma)\tilde{\psi}_n(p). \tag{15}$$

Here $L_n^{\beta}(z)$ is a Chebyshev–Laguerre polynomial.

We substitute (15) into (14). Then, using the spectral relation [4]

$$\left(1 - \frac{d^2}{ds^2}\right)\frac{1}{\pi}\int_0^{\infty} \sqrt{\sigma}\mathrm{e}^{-\sigma} L_n^{1/2}(2\sigma)K_0(|s-\sigma|)\,d\sigma = \frac{\sqrt{2}\Gamma(n+3/2)L_n^{1/2}(2s)}{n!\,\mathrm{e}^s}$$

and the orthogonality of the Chebyshev–Laguerre polynomials, we obtain an expression for the coefficients of expansion in (15),

$$\tilde{\psi}_n(p) = \frac{2(n!)^2}{\Gamma(n+3/2)}\int_0^{\infty} \sqrt{\sigma}\mathrm{e}^{-\sigma} L_n^{1/2}(2\sigma)g(\sigma,p)\,d\sigma.$$

Substituting these expressions into (15) and returning to the original variables in accordance with (13), we find the jumps of displacements in terms of the Laplace transforms. Hence, the sought-for discontinuous solution is calculated by (12). Taking the Laplace transform of the discontinuous solution, we obtain a discontinuous solution to the problem formulated in the coordinates (t,x,z).

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