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Numerical simulation of the 3D thermoelasticity problem*

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The paper deals with a numerical model based on the finite element discretization of the 3D thermoelasticity problem in compound parallelepipedal domain. The piece-wise trilinear functions are used. Iterative process is based on the Neumann–Dirichlet domain decomposition procedure, and numerical experiments demonstrate that the convergence rate does not depend on the grid parameters.

1. Statement of the problem and some notations

1.1. Geometry. Let Ω be a 3D domain, which is a union of the two non-overlapping parallelepipeds

$$\Omega^{(1)} = \{(x_1, x_2, x_3):\ 0 < x_1 < a,\ 0 < x_2 < b,\ 0 < x_3 < c\}$$
$$\Omega^{(2)} = \{(x_1, x_2, x_3):\ 0 < x_1 < a,\ d < x_2 < b,\ c < x_3 < f\}$$

Let us introduce the following notation for different parts of the boundaries of subdomains:

$$\Gamma^{(1)}_{1,0} = \{x_1 = 0,\ 0 \le x_2 \le b,\ 0 \le x_3 \le c\},$$
$$\Gamma^{(1)}_{1,a} = \{x_1 = a,\ 0 \le x_2 \le b,\ 0 \le x_3 \le c\},$$
$$\Gamma^{(1)}_{2,0} = \{0 \le x_1 \le a,\ x_2 = 0,\ 0 \le x_3 \le c\},$$
$$\Gamma^{(1)}_{2,b} = \{0 \le x_1 \le a,\ x_2 = b,\ 0 \le x_3 \le c\},$$
$$\Gamma^{(1)}_{3,0} = \{0 \le x_1 \le a,\ 0 \le x_2 \le b,\ x_3 = 0\},$$
$$\Gamma^{(1)}_{3,c} = \{0 \le x_1 \le a,\ 0 \le x_2 \le d,\ x_3 = c\},$$
$$\Gamma^{(2)}_{1,0} = \{x_1 = 0,\ d \le x_2 \le b,\ c \le x_3 \le f\},$$
$$\Gamma^{(2)}_{1,a} = \{x_1 = a,\ d \le x_2 \le b,\ c \le x_3 \le f\},$$
$$\Gamma^{(2)}_{2,d} = \{0 \le x_1 \le a,\ x_2 = d,\ c \le x_3 \le f\},$$
$$\Gamma^{(2)}_{2,b} = \{0 \le x_1 \le a,\ x_2 = b,\ c \le x_3 \le f\},$$
$$\Gamma^{(2)}_{3,f} = \{0 \le x_1 \le a,\ d \le x_2 \le b,\ x_3 = f\}.$$

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And, finally, there is an interface between $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$\Gamma = \{0 \le x_1 \le a,\ d \le x_2 \le b,\ x_3 = c\}.$$

1.2. Elasticity equations. The components of the stress tensor σ are defined for $i, j = 1, 2, 3$ as follows

$$\sigma_{ij} = \sigma_{ji} = 2\mu\,\varepsilon_{ij} + \lambda\Big(\sum_{k=1}^{3}\varepsilon_{kk}\Big)\delta_{ij} - (2\mu + 3\lambda)\alpha_T(T - T_1)\delta_{ij}, \qquad (1)$$

where $T = T(x_1, x_2, x_3)$ is the given temperature field, T_1 is the parameter,

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad \delta_{ii} = 1, \quad \delta_{ij} = 0 \ \ i \ne j,$$

u_1, u_2, u_3 are the components of the displacement vector $\boldsymbol{u}$. The Lamé coefficients λ and μ are connected with the Young module E and the Poison ratio ν according to the formulas

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

Then the equilibrium equation as the system for displacement $\boldsymbol{u}$ in the subdomains $\Omega^{(1)}$, $\Omega^{(2)}$ is the following:

$$\mathbf{div}\,\sigma(\boldsymbol{u}) = 0, \qquad (2)$$

where

$$\mathbf{div}\,\sigma = \begin{pmatrix} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} + \dfrac{\partial\sigma_{13}}{\partial x_3} \\ \dfrac{\partial\sigma_{12}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} \\ \dfrac{\partial\sigma_{13}}{\partial x_1} + \dfrac{\partial\sigma_{23}}{\partial x_2} + \dfrac{\partial\sigma_{33}}{\partial x_3} \end{pmatrix}.$$

The boundary and the interface conditions are the following:

- The Dirichlet conditions

$$u_1 = u_1^0\Big(\frac{1}{2} - \frac{x_1}{a}\Big), \quad u_2 = u_2^0\Big(\frac{1}{2} - \frac{x_2}{b}\Big), \quad u_3 = 0 \quad \text{on } \Gamma_{3,0}^{(1)}; \qquad (3)$$

- The Neumann conditions

$$\begin{aligned} &\sigma_{11} = \sigma_{12} = \sigma_{13} = 0 \quad \text{on } \Gamma_{1,0}^{(1)} \cup \Gamma_{1,a}^{(1)} \cup \Gamma_{1,0}^{(2)} \cup \Gamma_{1,a}^{(2)}, \\ &\sigma_{12} = \sigma_{22} = \sigma_{23} = 0 \quad \text{on } \Gamma_{2,0}^{(1)} \cup \Gamma_{2,b}^{(1)} \cup \Gamma_{2,d}^{(2)} \cup \Gamma_{2,b}^{(2)}, \\ &\sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \quad \text{on } \Gamma_{3,c}^{(1)} \cup \Gamma_{3,f}^{(2)}; \end{aligned} \qquad (4)$$

- The conditions on the interface

$$[u_1] = [u_2] = [u_3] = [\sigma_{13}] = [\sigma_{23}] = [\sigma_{33}] = 0 \quad \text{on } \Gamma. \qquad (5)$$

1.3. Variational form of elasticity problem. According to conditions (3)–(5) for any tensor τ we have the following formula of integration by parts:

$$\int_\Omega \varepsilon(\boldsymbol{u}) : \tau \, dx_1 \, dx_2 \, dx_3 = -\int_\Omega \boldsymbol{u} \cdot \operatorname{div} \tau \, dx_1 \, dx_2 \, dx_3,$$

where

$$\varepsilon : \tau = \sum_{i,j=1}^{3} \varepsilon_{ij} \tau_{ij}.$$

Then from (1), (2) it follows

$$2\int_\Omega \mu \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx_1 \, dx_2 \, dx_3 + \int_\Omega \lambda (\nabla \cdot \boldsymbol{u})(\nabla \cdot \boldsymbol{v}) \, dx_1 \, dx_2 \, dx_3 = \int_\Omega (2\mu + 3\lambda) \alpha_T (T - T_1)(\nabla \cdot \boldsymbol{v}) \, dx_1 \, dx_2 \, dx_3. \tag{6}$$

Here for the vector $\boldsymbol{u}$ we use the Dirichlet conditions (3) and $\boldsymbol{v} = 0$ on $\Gamma_{3,0}^{(1)}$. A more accurate formulation requires indication to the vector Sobolev spaces for these vectors. Let us note that the Lamé coefficients are discontinuous functions.

2. Finite element discretization of elasticity equations

2.1. Basis functions. We will use the finite element space of the rectangular partitioning of trilinear functions $(Q_1(e_0))^3$ [1] for each component of the displacement vector. Here $e_0 = [0,1]^3$ is the reference cube. The basis of the space $(Q_1(e_0))^3$ consists of 24 vector-functions

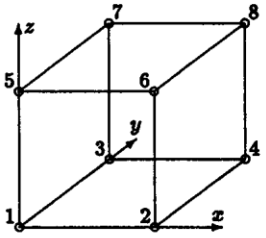

Figure 1

$$\phi_{i1}(\xi) = \begin{pmatrix} \phi_i(\xi) \\ 0 \\ 0 \end{pmatrix}, \quad \phi_{i2}(\xi) = \begin{pmatrix} 0 \\ \phi_i(\xi) \\ 0 \end{pmatrix}, \quad \phi_{i3}(\xi) = \begin{pmatrix} 0 \\ 0 \\ \phi_i(\xi) \end{pmatrix},$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ are reference variables, $i = 1, \ldots, 8$ are local numbers of the vertices of the cube e_0 (Figure 1). Then

$$\begin{aligned} \phi_1(\xi) &= (1-\xi_1)(1-\xi_2)(1-\xi_3), & \phi_5(\xi) &= (1-\xi_1)(1-\xi_2)\xi_3, \\ \phi_2(\xi) &= \xi_1(1-\xi_2)(1-\xi_3), & \phi_6(\xi) &= \xi_1(1-\xi_2)\xi_3, \\ \phi_3(\xi) &= (1-\xi_1)\xi_2(1-\xi_3), & \phi_7(\xi) &= (1-\xi_1)\xi_2\xi_3, \\ \phi_4(\xi) &= \xi_1\xi_2(1-\xi_3), & \phi_8(\xi) &= \xi_1\xi_2\xi_3. \end{aligned}$$

2.2. A local stiffness matrix. According to (6), the entries of 24×24 local stiffness matrix have the form

$$K^e_{(ik),(jn)} = V^e\left(2\int_{e_0} \mu\varepsilon_\xi(\phi_{ik}) : \varepsilon_\xi(\phi_{jn})\,d\xi + \int_{e_0} \lambda(\nabla_\xi \cdot \phi_{ik})(\nabla_\xi \cdot \phi_{jn})\,d\xi\right),$$
$$i,j = 1,\dots 8, \quad k,n = 1,2,3.$$

Here $V^e = l_1 l_2 l_3$ is a volume of the parallelepiped e, and l_k are the lengths of its edges. The differential operators ε_ξ and ∇_ξ correspond to the operators ε and ∇ for the mapping $e \to e_0$. Then, using the notation

$$K^0_{(ik),(jn)} = \int_{e_0} \frac{\partial\phi_i}{\partial\xi_k}\frac{\partial\phi_j}{\partial\xi_n}d\xi,$$

we have

$$K^e_{(ik),(jn)} = V^e\left(\frac{1}{l_k l_n}(\mu^e + \lambda^e)K^0_{(ik),(jn)} + \mu^e\sum_{m=1}^{3}\left(\frac{1}{l_m^2}K^0_{(im),(jm)}\right)\delta_{kn}\right).$$

The numbers $K^0_{(ik),(jn)}$ are independent of the elements e. The local matrix K^e can be presented in the block form

$$K^e = \begin{pmatrix} K^e_{11} & K^e_{12} & K^e_{13} \\ K^e_{21} & K^e_{22} & K^e_{23} \\ K^e_{31} & K^e_{32} & K^e_{33} \end{pmatrix},$$

where

$$K^e_{mn} = \begin{pmatrix} K^e_{(1m),(1n)} & K^e_{(1m),(2n)} & \cdots & K^e_{(1m),(8n)} \\ K^e_{(2m),(1n)} & K^e_{(2m),(2n)} & \cdots & K^e_{(2m),(8n)} \\ \vdots & \vdots & & \vdots \\ K^e_{(8m),(1n)} & K^e_{(8m),(2n)} & \cdots & K^e_{(8m),(8n)} \end{pmatrix}.$$

2.3. A local force vector. According to (6), the components of the local vector of the right-hand side have the form

$$F^e_{(ik)} = V^e\int_{e_0}(2\mu + 3\lambda)\alpha_T(T - T_1)(\nabla_\xi \cdot \phi_{ik})\,d\xi,$$

where, in the element e,

$$T = \sum_{j=1}^{8} T^e_j\phi_j(\xi).$$

Then

$$F^e_{(ik)} = V^e(2\mu^e+3\lambda^e)\frac{1}{l_k}\sum_{j=1}^{8}\alpha^e_T(T^e_j-T_1)F^0_{j,(ik)}, \quad i,j=1,\dots,8, \quad k,=1,2,3,$$

where

$$F^0_{j,(ik)} = \int_{e_0}\phi_j\frac{\partial\phi_i}{\partial\xi_k}\,d\xi$$

is independent of e.

3. Iterative method

After the finite element discretization, the grid elasticity problem for the known grid functions T and N can be written down as the following algebraic linear system

$$Au = f, \tag{7}$$

where the block presentation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}, \quad u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \quad f = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}.$$

corresponds to the vector displacement u.

We will use the preconditioned conjugate gradient (CG) method [2] to solve system (7). As preconditioner, the Neumann–Dirichlet decomposition [3] with the reflection operator for the extension of grid functions will be applied [4]. The solvers in subdomains will be realized with the use of the Fourier transform [2].

3.1. The generalized conjugate gradient method. Let us introduce 3×3 block symmetric positive definite matrix B (preconditioner). Then the generalized CG method can be written down as follows. Let u_0 be given. Then

$$g_k = B^{-1}r_0 \text{ at } k=1, \quad g_k = B^{-1}r_{k-1}+\alpha_k g_{k-1} \text{ at } k>1,$$

where $r_{k-1} = Au_{k-1}-f$. Then

$$u_k = u_{k-1}-\beta_k g_k, \quad k\geq 1,$$

$$\alpha_k = \frac{\|r_{k-1}\|^2_{B^{-1}}}{\|r_{k-2}\|^2_{B^{-1}}} \text{ at } k\geq 2, \qquad \beta_k = \frac{\|r_{k-1}\|^2_{B^{-1}}}{\|g_k\|^2_A} \text{ at } k\geq 1,$$

where

$$\|r_m\|^2_{B^{-1}} = (B^{-1}r_m, r_m), \qquad \|g_k\|^2_A = (Ag_k, g_k).$$

The purpose of preconditioning consists in designing a matrix B which, in a sense, is close to the original matrix A (the matrix AB^{-1} is close to the unit matrix), and the operation $B^{-1}r$ is efficiently realized (either by the direct method or by some inner iterative process using a special structure of B, whose convergence is sufficiently high in contrast to the matrix A). We will use the block-diagonal

$$B = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \tag{8}$$

where $(-\Lambda)$ is some "approximation" to the grid 7-point Laplacian Δ_h in Ω with the Dirichlet conditions on $\Gamma_{3,0}^{(1)}$ and the Neumann conditions on the other parts of the boundary. The matrix Λ is positive definite. We cannot efficiently inverse the operator Δ_h because Ω is not a parallelepiped. However, as Ω is a union of two parallelepipeds, we can use the domain decomposition technique in the form of the additive Swartz method with the Dirichlet–Neumann alternating conditions at the interface Γ.

3.2. Domain decomposition preconditioning. Let us represent the operator Δ_h in the following block form:

$$\Delta_h = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix},$$

where blocks of the matrix Δ_h correspond to the partitioning of a vector into two groups: the first group, consisting of variables in the grid vertices of the first subdomain $\overline{\Omega}^{(1)}$, except for $\Gamma_{3,0}^{(1)}$ and including the interface Γ between subdomains and variables in the grid vertices from $\overline{\Omega}^{(2)}$, except for the interface Γ, form the second group. Let us consider the matrix Δ_{1N}, which corresponds to the Neumann problem at the interface Γ in the first subdomain. Note that Δ_{22} corresponds to the Dirichlet problem at the interface Γ in the second subdomain. Let

$$\Lambda^{-1} = -R\Delta_{1N}^{-1}R^T - \begin{pmatrix} 0 & 0 \\ 0 & \Delta_{22}^{-1} \end{pmatrix}, \qquad R = \begin{pmatrix} I_{11} \\ R_{21} \end{pmatrix},$$

where I_{11} is the identity operator in the first subdomain and R_{21} is the operator of the extension of grid functions from $\Omega^{(1)}$ to $\Omega^{(2)}$, which can be readily designed as reflection with respect to the interface Γ. Note that the matrices Δ_{1N} and Δ_{22} are invertible because the Dirichlet conditions are valid at $\Gamma_{3,0}^{(1)}$ and Γ, respectively. And, finally, note that we have designed the matrix Λ^{-1} but not the matrix Λ. Let us recall that we need an efficient procedure to calculate $B^{-1}r$ (according to (8), $\Lambda^{-1}r^i$).

Now we will describe the realization of this preconditioner. Let h_1, h_2, h_3 be constant steps size of the spatial grid, such that

$$N_1 = \frac{a}{h_1}, \quad N_2^{(1)} = \frac{b}{h_2}, \quad N_2^{(2)} = \frac{b-d}{h_2}, \quad N_3^{(1)} = \frac{c}{h_3}, \quad N_3^{(2)} = \frac{f-c}{h_3}$$

are integer numbers, and let $N_2^{(1)} \geq N_2^{(2)}$ and $N_3^{(1)} \leq N_3^{(2)}$. Then let r_1 and r_2 be the partitioning of the vector r^i into two groups described above. Calculation of the vector $w = \Lambda^{-1} r^i = (w_1, w_2)$ can be represented as the following sequence of steps:

1. Calculation of $R^T r^i = r_1 + R_{21}^T r_2$:

$$v_1(i_1, i_2, i_3) = r_1(i_1, i_2, i_3) + r_2(i_1, i_2 - N_2^{(1)} + N_2^{(2)}, N_3^{(1)} + 1 - i_3),$$
$$_1 = 0, \ldots, N_1, \quad i_2 = N_2^{(1)} - N_2^{(2)}, \ldots, N_2^{(1)}, \quad i_3 = 1, \ldots, N_3^{(1)}.$$

2. Solution of the problem in $\Omega^{(1)}$: $-\Delta_{1N} w_1 = v_1$.
3. Extension from $\Omega^{(1)}$ to $\Omega^{(2)}$ (reflection):

$$v_2(i_1, i_2, i_3) = w_1(i_1, i_2 + N_2^{(1)} - N_2^{(2)}, N_3^{(1)} + 1 - i_3),$$
$$i_1 = 0, \ldots, N_1, \quad i_2 = 0, \ldots, N_2^{(2)}, \quad i_3 = 1, \ldots, N_3^{(1)}.$$

4. Solution of the problem in $\Omega^{(2)}$: $-\Delta_{22} y_2 = r_2$.
5. Calculation of $w_2 = v_2 + y_2$:

$$w_2(i_1, i_2, i_3) = v_2(i_1, i_2, i_3) + y_2(i_1, i_2, i_3),$$
$$i_1 = 0, \ldots, N_1, \quad i_2 = 0, \ldots, N_2^{(2)}, \quad i_3 = 1, \ldots, N_3^{(2)}.$$

In the subsection to follow, the exact inversion of the matrices Δ_{1N} and Δ_{22} will be described.

3.3. Solution in subdomains. Let us consider the solution to the linear algebraic system

$$Cu = g, \tag{9}$$

which is an ordinary 7-point approximation on rectangular grid of the Poisson equation in the parallelepiped $(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ with the Dirichlet conditions on a part of the boundary $\{(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2),\ x_3 = a_3\}$, and with the Neumann conditions on its other parts.

Let us note that both matrices $-\Delta_{1N}$ and $-\Delta_{22}$ have the form of the matrix C. In the first case

$$a_1 = 0, \quad b_1 = a, \quad a_2 = 0, \quad b_2 = b, \quad a_3 = 0, \quad b_3 = c,$$

and in the second case

$$a_1 = 0, \quad b_1 = a, \quad a_2 = d, \quad b_2 = b, \quad a_3 = c, \quad b_3 = f.$$

Let

$$l_1 = b_1 - a_1, \quad l_2 = b_2 - a_2, \quad l_3 = b_3 - a_3$$

and

$$N_1 = \frac{l_1}{h_1}, \quad N_2 = \frac{l_2}{h_2}, \quad N_3 = \frac{l_3}{h_3}.$$

This means that either $N_2 = N_2^{(1)}$, $N_3 = N_3^{(1)}$ or $N_2 = N_2^{(2)}$, $N_3 = N_3^{(2)}$. We will use the discrete Fourier transform for variables x_1 and x_3 and LU factorization for the variable x_2. The matrix C can be written down as follows:

$$C = \frac{h_2 h_3}{h_1} C_1 \otimes M_2 \otimes M_3 + \frac{h_1 h_3}{h_2} M_1 \otimes C_2 \otimes M_3 + \frac{h_1 h_2}{h_3} M_1 \otimes M_2 \otimes C_3,$$

where C_1 and M_1 are $(N_1 + 1) \times (N_1 + 1)$ matrices, C_2 and M_2 are $(N_2 + 1) \times (N_2 + 1)$ matrices, C_3 and M_3 are $N_3 \times N_3$ matrices:

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & & 0 & \\ & & \ddots & & \\ & 0 & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{1}{2} & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & & 0 & \\ & & \ddots & & \\ & 0 & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{1}{2} & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & 0 & \\ & & \ddots & & \\ & 0 & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix}.$$

Let us consider two generalized eigenvalue problems:

$$C_1 \psi_1 = \lambda_1 M_1 \psi_1, \qquad C_3 \psi_3 = \lambda_3 M_3 \psi_3.$$

Their solutions are the following:

$$\lambda_{1,k} = 4\sin^2\frac{k\pi}{2N_1}, \qquad k = 0,\dots,N_1,$$

$$\psi_{1,0}(i) = \sqrt{\frac{1}{N_1}}, \qquad \psi_{1,N_1}(i) = \sqrt{\frac{1}{N_1}}\cos i\pi, \qquad i = 0,\dots,N_1,$$

$$\psi_{1,k}(i) = \sqrt{\frac{2}{N_1}}\cos\frac{ki\pi}{N_1}, \qquad k = 1,\dots,N_1-1, \quad i = 0,\dots,N_1,$$

$$\lambda_{3,k} = 4\sin^2\frac{(2k-1)\pi}{4N_3}, \qquad k = 1,\dots,N_3,$$

$$\psi_{3,k}(i) = \sqrt{\frac{2}{N_3}}\cos\frac{(2k-1)i\pi}{2N_3}, \qquad k = 1,\dots,N_3, \quad i = 1,\dots,N_3.$$

The calculation of the solution to system (9) consists of the following three steps:

1. The direct Fourier transform:

$$g_{k_1}(i_2,i_3) = \sum_{i_1=0}^{N_1} g(i_1,i_2,i_3)\psi_{1,k_1}(i_1),$$

$$g_{k_1,k_3}(i_2) = \sum_{i_3=1}^{N_3} g_{k_1}(i_2,i_3)\psi_{3,k_3}(i_3).$$

2. *LU* factorization for the Fourier coefficients:

$$\alpha_{k_1,k_3}(1) = \frac{2}{2+(h_2/h_1)^2\lambda_{1,k_1}+(h_2/h_3)^2\lambda_{3,k_3}},$$

$$\beta_{k_1,k_3}(1) = \frac{h_2}{h_1h_3}g_{k_1,k_3}(0)\alpha_{k_1,k_3}(1);$$

$$\alpha_{k_1,k_3}(i_2+1) = \frac{1}{2+(h_2/h_1)^2\lambda_{1,k_1}+(h_2/h_3)^2\lambda_{3,k_3}-\alpha_{k_1,k_3}(i_2)},$$

$$\beta_{k_1,k_3}(i_2+1) = \left(\frac{h_2}{h_1h_3}g_{k_1,k_3}(i_2)+\beta_{k_1,k_3}(i_2)\right)\alpha_{k_1,k_3}(i_2+1),$$

$$i_2 = 1,\dots,N_2-1;$$

$$u_{k_1,k_3}(N_2) = \frac{2g_{k_1,k_3}(N_2)h_2/(h_1h_3)+2\beta_{k_1,k_3}(N_2)}{2+(h_2/h_1)^2\lambda_{1,k_1}+(h_2/h_3)^2\lambda_{3,k_3}-2\alpha_{k_1,k_3}(N_2)},$$

$$u_{k_1,k_3}(i_2-1) = \alpha_{k_1,k_3}(i_2)u_{k_1,k_3}(i_2)+\beta_{k_1,k_3}(i_2), \qquad i_2 = N_2,\dots,1.$$

3. The inverse Fourier transform:

$$u_{k_1}(i_2,i_3) = \sum_{k_3=1}^{N_3} u_{k_1,k_3}(i_2)\psi_{3,k_3}(i_3),$$

$$u(i_1,i_2,i_3) = \sum_{k_1=0}^{N_1} u_{k_1}(i_2,i_3)\psi_{1,k_1}(i_1).$$

The values $u(i_1, i_2, i_3)$ form the vector, which is the solution to problem (9).

4. Numerical results

As mentioned above, Ω consists of the two parallelepipeds: the lower one Ω_1 and the upper one Ω_2. The lower one $\Omega_1 = [0, 11.0] \times [0, 2.6] \times [0, 0.3]$ consists of staff with the following coefficients: Young's module $E = 5.0 \cdot 10^4$, Poisson's ratio $\nu = 0.3$ and the thermal expansion coefficient $\alpha_T = 10^{-5}$. The upper one $\Omega_1 = [1.0, 11.0] \times [0, 2.6] \times [0.3, 0.6]$ has the following coefficients: Young's module $E = 6.89 \cdot 10^4$, Poisson's ratio $\nu = 0.3$ and the thermal expansion coefficient $\alpha_T = 5 \cdot 10^{-5}$. The boundary conditions are the same as in Section 1.2, i.e., the same Dirichlet condition given on the lower plane, the free surface condition is given on the rest plane. The temperature T was constant and $T - T_1 = 180$. The experiment has been realized on three inserted grids. The character quantity and the resulting data are presented in the table, where h is a step size of the grid, N is the number of degrees of freedom, n is the number of iterations, which is necessary to provide the inequality $\|r^n\|/\|r^0\| \leq 10^{-6}$ for the residual r^n and, finally, τ is the time of task execution.

h	N	n	τ, s
0.1	51 526	52	8.8
0.05	402 588	51	76.1
0.025	3182 588	52	706.2

As we can see, the number of iterations practically does not depend on the conditioning number of the algebraic linear system. As most execution time of our task is spent on calculation process exactly, the ratio τ/n gives us an approximate duration of one iteration. Evidently, it is proportional to the problem dimension.

The scalar field-function $H = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$ is shown to confirm the convergence of solutions to one, as we can see in Figure 2.

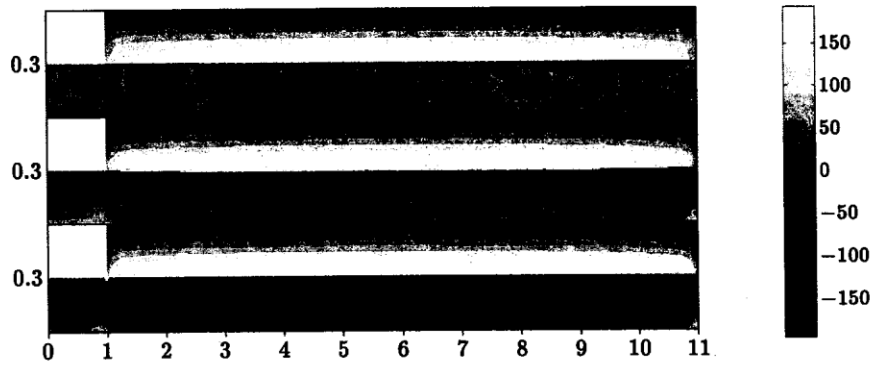

Figure 2. The scalar field section for $x_1 = 1.3$ for $h = 0.1$, 0.05, and 0.025

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