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Numerical methods in elasticity problems*

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The paper deals with some results, which are connected with the numerical solution of elasticity problems in the case of an arbitrary curvilinear coordinates system. The main ideas will be illustrated for the case of the Cartesian coordinates system.

Let x_1, x_2, x_3 be a Cartesian coordinates system, $\overrightarrow{u}(u_1, u_2, u_3)$ an elastic displacement vector (column) and ε_{ik} be components of symmetric elastic strain tensor for linear (geometric) medium. Then,

$$2\varepsilon_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}, \quad i,k = 1,2,3; \quad \overrightarrow{\varepsilon} = R\overrightarrow{u}. \tag{1}$$

Let the components of the symmetric elastic stress tensor for linear (physical) medium be denoted by σ_{ik}. Then, (Hooke's law):

$$\sigma_{ik} = \lambda\delta_{ik} div\,\overrightarrow{u} + 2\mu\varepsilon_{ik}, \quad i,k = 1,2,3; \quad \overrightarrow{\sigma} = K\overrightarrow{\varepsilon}. \tag{2}$$

The differential equations of equilibrium in the field of given mass forces $\overrightarrow{f}(f_1, f_2, f_3)$ may be formulated as follows:

$$-L\overrightarrow{\sigma} + \overrightarrow{f} = 0 \longleftrightarrow \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} + f_i = 0, \quad i = 1,2,3. \tag{3}$$

Assume that $D \subset R^3$ - single-connected region and $\gamma = \partial D$ is sufficiently smooth.

We must define fifteen functions $u_i(M)$, $\varepsilon_{ik}(M)$, $\sigma_{ik}(M)$, $M \in D$, which satisfy fifteen equations (1)–(3). Some functions: $u_i(M)$ or $\sigma_{ik}(M)$ must satisfy the given boundary condition. For example,

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$$\sum_{k=1}^{3} \sigma_{ik}(M')n_k = g_i(M'), \quad M' \in \gamma, \quad i = 1,2,3. \tag{4}$$

or

$$u_i(M') = g_i(M'), \quad M' \in \gamma, \quad i = 1,2,3. \tag{5}$$

As usually

$$\int\limits_D \overrightarrow{f}\, dM + \int\limits_\gamma \overrightarrow{g}\, dM' = 0, \quad \int\limits_D \overrightarrow{f} \times \overrightarrow{r}\, dM + \int\limits_\gamma \overrightarrow{g} \times \overrightarrow{r}\, dM' = 0. \tag{6}$$

There are two ways of solving the static problem. The first one is bound up with determination of the elastic displacement vector $\overrightarrow{u}$. After that we can deduce ε_{ik} and σ_{ik} from (1), (2). Equation of equilibrium (3) needs to be modified as follows:

$$A\overrightarrow{u} + \overrightarrow{f} = 0 \leftrightarrow \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Big[\lambda\delta_{ik} div\overrightarrow{u} + \mu\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\Big] + f_i = 0, \quad i = 1,2,3. \tag{7}$$

Instead of (4), we have

$$\sum_{k=1}^{3} \Big[\lambda\delta_{ik} div\overrightarrow{u} + \mu\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\Big] n_k = g(M'). \tag{8}$$

The unique solution can be obtained, for example, under such a condition

$$(\overrightarrow{u}, 1)_D \equiv \int\limits_D \overrightarrow{u}\, dM = 0, \quad (\overrightarrow{r} \times \overrightarrow{u}, 1)_D \equiv \int\limits_D \overrightarrow{r} \times \overrightarrow{u}\, dM = 0. \tag{9}$$

The present-day way of working up the numerical method of solving the static problem is as follows. Static problem (6)–(9) is substituted for the equivalent variational problem

$$\min\{J(\overrightarrow{u}) \equiv W(\overrightarrow{u}) + 2(\overrightarrow{f}, \overrightarrow{u})_D - 2(\overrightarrow{g}, \overrightarrow{u})_\gamma\}. \tag{10}$$

Here $W(\overrightarrow{u})$ is twice the elastic strain potential energy, expressed in terms of the elastic displacement vector. The methods of construction of the difference problem

$$\Lambda\overrightarrow{u}_h + \overrightarrow{f}_h = 0 \tag{11}$$

from (10) are well-known. These methods guarantee the self-adjoining of the operator Λ. Indeed, it is the result of the fact that the quadratic form $W(\overrightarrow{u})$ generates a symmetric bilinear form

$$(\varepsilon_{ik}(\overrightarrow{u}), \sigma_{ik}(\overrightarrow{u}))_D.$$

In the case of an arbitrary nonorthogonal curvilinear coordinates system, the main difficulties of solving problem (6)–(9) are both the construction of the operator Λ, and the proving of the positive definiteness of Λ.

Now let us consider the second way of solving the static problem. In this case the elastic stress tensor components σ_{ik} or the elastic strain tensor components ε_{ik} is first determined. With the help of (1) we then determine the elastic displacement vector components u_i.

It is not important which method we use when determining ε_{ik}, but u_i can be determined only from (1). Hence, the first we need is to know about the solvability of problem (1). The well-known conditions of solvability (Saint Venan's conditions) in the Cartesian coordinate system may be written as follows:

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 \varepsilon_{kk}}{\partial x_i \partial x_j} = \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_k} + \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_k}, \quad i,j,k = 1,2,3. \tag{12}$$

Only six equations from (12) are different. Let us denote them as

$$G_\alpha(\overrightarrow{\varepsilon}) = 0, \quad \alpha = 1, \ldots, 6. \tag{13}$$

Therefore, if we choose the second way of solving the static problem, the components ε_{ik} (or σ_{ik}) must satisfy conditions (13), regardless to a particular method of determination of ε_{ik} (or σ_{ik}).

In what follows, we will consider only homogeneous boundary conditions (4) or (5).

Let us write another form of conditions (13). To do this we will consider the column-vectors $\overrightarrow{\xi}(\xi_{11}, \xi_{22}, \xi_{33}, 2\xi_{12}, 2\xi_{13}, 2\xi_{23})$, $\overrightarrow{\eta}(\eta_{11}, \eta_{22}, \eta_{33}, \eta_{12}, \eta_{13}, \eta_{23})$, $\overrightarrow{v}(v_1, v_2, v_3)$, $\xi_{ik} = \xi_{ki}$, $\eta_{ik} = \eta_{ki}$. We define the connection between $\overrightarrow{\xi}$ and $\overrightarrow{\eta}$ from (2) as follows:

$$\overrightarrow{\eta} = K\overrightarrow{\xi}, \quad \overrightarrow{\xi} = K^{-1}\overrightarrow{\eta}. \tag{14}$$

Here K is a 6x6 matrix according to Hook's law (2).

Let us define $H(\overrightarrow{v})$ as the Hilbert vector space $L_2(D)$. The scalar product in $H(\overrightarrow{v})$ is defined as usual

$$\left[\overrightarrow{v}^{(1)}, \overrightarrow{v}^{(2)}\right]_H = \sum_{i=1}^{3} \left(v_i^{(1)}, v_i^{(2)}\right) = \sum_{i=1}^{3} \int_D v_i^{(1)} v_i^{(2)} dM. \tag{15}$$

Let us define $H_1(\overrightarrow{\eta})$ as the Hilbert vector space $L_2(D)$. The scalar product in $H_1(\overrightarrow{\eta})$ is defined as follows:

$$\left[\overrightarrow{\eta}^{(1)}, \overrightarrow{\eta}^{(2)}\right]_{H_1} = \left(\overrightarrow{\eta}^{(1)}, K^{-1}\overrightarrow{\eta}^{(2)}\right) = \left(\overrightarrow{\eta}^{(1)}, \overrightarrow{\xi}^{(2)}\right) =$$

$$\sum_{i,k=1}^{3} \left(\eta_{ik}^{(1)}, \xi_{ik}^{(2)}\right) = \sum_{i,k=1}^{3} \int_D \eta_{ik}^{(1)} \xi_{ik}^{(2)} dM, \quad k \geq i. \tag{16}$$

The operator $R : H(\overrightarrow{v}) \to H_1(\overrightarrow{\xi}) = H_1(K^{-1}\overrightarrow{\eta})$ will be defined with the help of the matrix-differential operator

$$R = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{pmatrix}^T . \tag{17}$$

Here the symbol T denotes a transposition, the space $H_1(\overrightarrow{\xi})$ is the image of $H_1(\overrightarrow{\eta})$ obtained by mapping K^{-1}. The operator $L : H_1(\overrightarrow{\eta}) \to H(\overrightarrow{v})$ will be defined with the help of the matrix-differential operator

$$L = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{pmatrix} . \tag{18}$$

If we consider problem (1)–(4), then the range of definition of the operator L consists of the vectors $\overrightarrow{\eta}$, whose components are sufficiently smooth in D and satisfy the boundary condition (4).

If we consider problem (1)–(3), (5), then the range of definition of the operator R consists of the vectors $\overrightarrow{v}$, whose components are sufficiently smooth in D and satisfy the boundary condition (5).

Theorem 1. *Operator $-L$ is conjugate to R : $-L = R^*$, or*

$$[R\overrightarrow{v}, \overrightarrow{\eta}]_{H_1} = -[\overrightarrow{v}, L\overrightarrow{\eta}]_H. \tag{19}$$

Therefore, instead of (7) we have

$$A\overrightarrow{u} + \overrightarrow{f} \equiv R^* K R \overrightarrow{u} + \overrightarrow{f} = 0. \tag{20}$$

Here and in (7) A is Lame's operator. Theorem 1 states that $A = R^*KR$.

Now it is easy to see that conditions of solvability of equation (1) $\overrightarrow{\varepsilon} = R\overrightarrow{u}$ are

$$[\vec{\varepsilon}, \vec{\psi}]_{H_1} = 0, \quad \forall \vec{\psi} : \ R^* \vec{\psi} = 0. \tag{21}$$

The conditions (21) and (13) define the same subspace $H_2(\vec{\varepsilon}) \subset H_1(\vec{\xi})$. The operator R^{-1} exists in this subspace.

Now we may give *a new statement* of the static problem of elasticity theory *"in terms of stresses"*: in subspace $H_2(\vec{\sigma}) = H_2(K\vec{\varepsilon}) \subset H_1(\vec{\eta})$ we define the vectors $\vec{\sigma}, \vec{\varepsilon}, \vec{u}$ from equations

$$RR^*\vec{\sigma} + R\vec{f} = 0 \quad \to \quad \vec{\varepsilon} = K^{-1}\vec{\sigma} \quad \to \quad \vec{u} = R^{-1}\vec{\varepsilon}. \tag{22}$$

Now we may give also a variational formulation of the problems (22) in terms of stresses

$$\min\{\Phi(\vec{\sigma}) \equiv (RR^*\vec{\sigma}, \vec{\sigma}) + 2(\vec{\sigma}, R\vec{f})\}, \quad \vec{\sigma} \in H_2(\vec{\sigma}). \tag{23}$$

But

$$\Phi(\vec{\sigma}) = (R^*\vec{\sigma} + \vec{f}, R^*\vec{\sigma} + \vec{f}) - (\vec{f}, \vec{f}) = F(\vec{\sigma}) - \| \vec{f} \|^2.$$

The functional $F(\vec{\sigma})$ is the functional of the least square method in the problem

$$R^*\vec{\sigma} + \vec{f} = 0.$$

In terms of the displacement vector $\vec{u}$, the functional $F(\vec{\sigma})$ is a functional of the least square method in the problem

$$R^*KR\vec{u} + \vec{f} = 0.$$

We can now determine the main requirement for any numerical method "in stresses": the method does not have to remove the vector $\vec{\sigma}$ from the subspace $H_2(\vec{\sigma})$.

Consider the next numerical method

$$\frac{\vec{\varepsilon}_h^{m+1} - \vec{\varepsilon}_h^{m}}{\tau_{m+1}} + R_h R_h^* \vec{\sigma}_h^m + R_h \vec{f}_h = 0. \tag{24}$$

Theorem 2. *If* $\vec{\sigma}_h^0 \in H_2(\vec{\sigma})$, *then*

$$\vec{\sigma}_h^m \in H_2(\vec{\sigma}), \quad m = 1, 2, \ldots. \tag{25}$$

The exploration carried out permits to notice general principles of construction of numerical methods for solving the static (dynamic) problems in the elasticity theory. The operators of the static problems $A = R^*KR$ and RR^* are factorized and self-adjont. By choosing R as a generating operator (in terms of A.A. Samarsky) and constructing its approximation, we get the approximation of the operator R^* from

$$[R_h \vec{u}_h \vec{\sigma}_h]_{H_1} = [\vec{u}_h, R_h^* \vec{\sigma}_h]_H. \tag{26}$$

This way always gives a self-adjoint approximation for the static problem both in case of the "displacement statement" ($\vec{u} \to \vec{\varepsilon} \to \vec{\sigma}$) and in case of the "stress statement" ($\vec{\sigma} \leftrightarrow \vec{\varepsilon} \to \vec{u}$). Besides, the approximations possess the positiveness too

$$\begin{aligned} [R_h^* K R_h \vec{u}_h, \vec{u}_h]_H &= [K^{\frac{1}{2}} R_h \vec{u}_h, K^{\frac{1}{2}} R_h \vec{u}_h]_{H_1} \geq 0, \\ [R_h R_h^* \vec{\sigma}_h, \vec{\sigma}_h]_{H_1} &= [R_h^* \vec{\sigma}_h, R_h^* \vec{\sigma}_h]_H \geq 0. \end{aligned}$$

Here we point out only changes connected with the curvilinear non-orthogonal coordinate system.

Let x_1, x_2, x_3 be a Cartesian rectangular coordinates system and y_1, y_2, y_3 a curvilinear non-ortogonal coordinates system. The mapping $x \to y$ is not singular. Hence, the covariant components g_{ik} and the contravariant components g^{ik} of the metric tensor are known.

The displacement vector $\vec{u}$ and the strain tensor $\vec{\varepsilon}$ will be given by the covariant components u_i and ε_{ik}. The vector of mass forces and the stress tensor will be given by the contravariant components f^i and σ^{ik}. In (17) and (18) we must substitute ∇_α instead of $\frac{\partial}{\partial x_\alpha}$, where ∇_α is a covariant derivative. Finally, instead of (2) we have

$$\vec{\sigma} = K_1 \vec{\varepsilon},$$

where $K_1 = C^* K C > 0$ and the matrix C gives the conformity between σ_{ik} in the system x_1, x_2, x_3 and σ^{ik} in the system y_1, y_2, y_3.

In conclusion, we have to mention the problem connected with the covariant derivative approximation. Let $D = \{a_\alpha \leq y_\alpha \leq b_\alpha\}$. Denoting the basis and the cobasis by $\vec{e}_p(M)$ and $\vec{e}^{\,p}(M)$, respectively, we have

$$(\vec{e}_p, \vec{e}^{\,q}) = \delta_{pq}.$$

Further $\vec{u} = u^p \vec{e}_p$ or $\vec{u} = u_p \vec{e}^{\,p}$, where

$$u^p \vec{e}_p = \sum_{p=1}^{3} u^p \vec{e}_p.$$

Let $\frac{\partial \vec{u}}{\partial y_\alpha}$ be a derivative of $\vec{u}$ with respect to y_α

$$\begin{aligned} \frac{\partial \vec{u}}{\partial y_\alpha} &= \frac{\partial}{\partial y_\alpha}(u_p \vec{e}^{\,p}) = \frac{\partial u_p}{\partial y_\alpha} \vec{e}^{\,p} + u_p \frac{\partial \vec{e}^{\,p}}{\partial y_\alpha} \\ &= \left(\frac{\partial u_p}{\partial y_\alpha} \vec{e}^{\,p} - \Gamma_{p\alpha}^s u_s\right) \vec{e}^{\,p} = (\nabla_\alpha u_p) \vec{e}^{\,p}. \end{aligned} \tag{27}$$

Thus, formula (27) defines the covariant derivative of the covariant vector components

$$\nabla_\alpha u_p = \frac{\partial u_p}{\partial y_\alpha} - \Gamma^s_{.p\alpha} u_s. \tag{28}$$

We would like to notice two main properties of the covariant derivative:

- the covariant derivative of a metric tensor is equal to zero

$$\nabla_\alpha\, g_{sp} = 0, \tag{29}$$

- the property of the lowering of indices

$$\nabla_\alpha\, u_p = \nabla_\alpha u^s g_{sp}. \tag{30}$$

Assume that $N_p h_p = b_p - a_p$. Denote by $(\overrightarrow{u})_{y_\alpha}$ a difference derivative "forfward". We will give below a new definition of the difference covariant derivative connected with definition (27) in the continuons case. From the definition of the difference derivative with respect to y_α ("forward") we have

$$(\overrightarrow{u})_{y_\alpha} = \frac{T_{+\alpha}\overrightarrow{u} - \overrightarrow{u}}{h_\alpha} = (u_p \overrightarrow{e}^p)_{y_\alpha},$$

where $T_{\pm\alpha}\varphi(y_\alpha) = \varphi(y_\alpha \pm h_\alpha)$. Using the formula of the differencing of the product, we have

$$(\overrightarrow{u})_{y_\alpha} = (u_p)_{y_\alpha}\overrightarrow{e}^p + (T_{+\alpha}u_p)(\overrightarrow{e}^p)_{y_\alpha}. \tag{31}$$

If we want to obtain covariant components of the vector $(u_p)_{y_\alpha}$, then we must multiply (31) by the vector $\overrightarrow{e}^p$

$$(\overrightarrow{u}_{y_\alpha}, \overrightarrow{e}_p)\overrightarrow{e}^p = \Big[(u_p)_{y_\alpha} + (T_{+\alpha}u_s)(\overrightarrow{e}^s_{y_\alpha}, \overrightarrow{e}_p)\Big]\overrightarrow{e}^p. \tag{32}$$

The expression in the brackets is called the difference covariant derivative ("forward")

$$\nabla^{+h_\alpha}_\alpha\, u_p = (\overrightarrow{u}_{y_\alpha}, \overrightarrow{e}_p) = (u_p)_{y_\alpha} + (T_{+\alpha}u_s)(\overrightarrow{e}^s_{y_\alpha}, \overrightarrow{e}_p). \tag{33}$$

This "very good" definition keeps two main properities of the continuous covariant derivatives (29), (30).

Here as an example we give the difference covariant derivative "forward" of covariant components u_α for curvilinear cylindrical coordinates system r, φ, z. Here $u_1 = u$, $u_2 = v$, $u_3 = w$ and

$$\varepsilon_{22} = \nabla_2 u_2 = \varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r},$$

$$2\varepsilon_{12} = \nabla_1 u_2 + \nabla_2 u_1 = 2\varepsilon_{r\varphi} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \varphi} - \frac{v}{r}.$$

In the conformity of (33) we will have

$$\nabla_2^{+h_2} u_2 = \frac{1}{r_{i+\frac{1}{2}}}\Big[cos\frac{h_2}{2}\frac{v_{j+1} - v_j}{h_2} + \frac{2}{h_2} sin\frac{h_2}{2}\frac{u_{j+1} + u_j}{2}\Big],$$

$$\nabla_1^{+h_1} u_2 + \nabla_2^{+h_2} u_1 =$$

$$\frac{v_{i+1} - v_i}{h_1} + \frac{1}{r_{i+\frac{1}{2}}}\Big[cos\frac{h_2}{2}\frac{u_{j+1} - u_j}{h_2} - \frac{2}{h_2} sin\frac{h_2}{2}\frac{v_{j+1} + v_j}{2}\Big].$$

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