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Forward seismic modeling based on combination of finite Fourier transforms with matrix decomposition method

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This paper is the latest version of the numerical-analytical algorithm for solving the forward seismic problem and migration. The main concept of the algorithm is in splitting of 2D and 3D seismic problems to a series of 1D problems with the help of the finite integral Fourier, Fourier–Bessel or Legendre transforms and with the use of a 1D finite difference technique (see [1–3]). In this paper, instead of the finite difference method with respect to one spatial coordinate and time, we propose to use the finite difference approximation with respect to only the spatial coordinate with a subsequent analytical solution to the obtained system of ordinary differential equations with respect to time.

1. Statement of the problem

Let us illustrate the main stages of the method on the wave equation in the Cartesian coordinate system

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} = \frac{1}{v_p^2(x,z)}\frac{\partial^2 U}{\partial t^2} + \delta(x - x_0)\delta(z - z_0)f(t). \tag{1}$$

We consider $v_p(x, z)$ to be a piecewise-continuous function of two coordinates x, z. Here x_0, z_0 are the coordinates of the source simulated by the right-hand side of equation (1), $f(t)$ is a band-limited source function.

The problem is solved with zero initial data

$$U\Big|_{t=0} = \frac{\partial U}{\partial t}\Big|_{t=0} = 0 \tag{2}$$

and the boundary conditions on the free surface in the form

$$\frac{\partial U}{\partial z}\Big|_{z=0} = 0. \tag{3}$$

Assume that the function $U(z, x, t)$ possesses sufficient smoothness for using the subsequent transformations.

2. Numerical-analytical method for solving the problem

For solving problem (1)–(3) let us make use of the finite integral cosine-Fourier transform

$$R(z,n,t) = \int_0^a U(z,x,t)\cos\left(\frac{n\pi x}{a}\right)dx, \qquad (4)$$

with the inversion formula

$$U(z,x,t) = \frac{1}{a}R(z,0,t) + \frac{2}{a}\sum_{n=1}^{\infty} R(z,n,t)\cos\left(\frac{n\pi x}{a}\right). \qquad (5)$$

The equation obtained after the transformation contains the term $\partial U/\partial x$ at $x=0$ and $x=a$.

Let us introduce new additional boundary conditions

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial U}{\partial x}\right|_{x=a} = U\Big|_{z=b} = 0 \qquad (6)$$

and consider the wave field up to the time $t < T$, where T is the minimal time of propagation of the leading wave front from the reflecting surfaces $x=a$, $z=b$. We are able to do it due to hyberbolicity of the problem. Another possibility to eliminate artificial reflections from the boundaries is to introduce instead of boundary conditions (6) the absorbing boundary conditions (see, for example [4]). Let us consider numerical-analytical algorithm for two cases.

2.1. Velocity is a piecewise-continuous function only of the coordinate z

In this case after applying a finite integral cosine transform (4), (5) the new boundary problem for $R(z,n,t)$ is of the form

$$\frac{\partial^2 R}{\partial z^2} - k_n^2 R = \frac{1}{v_p^2(z)}\frac{\partial^2 R}{\partial t^2} + \cos(k_n x_0)\delta(z-z_0)f(t), \qquad (7)$$

$$\left.\frac{\partial R}{\partial z}\right|_{z=0} = R\Big|_{z=b} = 0, \qquad (8)$$

$$R\Big|_{t=0} = \left.\frac{\partial R}{\partial t}\right|_{t=0} = 0, \qquad (9)$$

where

$$k_n = \frac{n\pi}{a}, \qquad n = 0,1,2,\dots .$$

For solving the system of problems (7)–(9) let us make use of the finite difference approximation with respect only to the spatial coordinate z with a subsequent analytical solution to the obtained system of ordinary differential equations with respect to time.

Let us introduce in the variable z the uniform grid

$$\omega = \{z_i = (i-1)h; \quad i = 1, \dots, N+1, \; b = Nh\}.$$

If $v_p(z)$ is continuous at the node $z_i \in \omega$, we assume $v_i = v_p(z_i)$, otherwise $v_i = (v_{i+1} + v_{i-1})/2$. The coordinate z_0, which determines the location of a source, is calculated by the formula

$$z_0 = (l-1)h.$$

Determination of the functions $R_i(n,t)$ on the lines $z = z_i$ reduces to solving the Cauchy problem for the system of N linear differential second order equations (see [5]). Write down the system in the vector form

$$\frac{d^2\overline{Z}}{dt^2} + A_n\overline{Z} = f(t)\overline{F}, \tag{10}$$

$$\overline{Z}\Big|_{t=0} = \frac{d\overline{Z}}{dt}\Big|_{t=0} = 0. \tag{11}$$

Note, that the original system has been reduced to the form of (10), (11) by means of the preliminary replacement of the variables

$$\overline{Z}(n,t) = D\overline{R}(n,t), \quad D = \operatorname{diag}\left\{\frac{1}{v_1}, \frac{\sqrt{2}}{v_2}, \dots, \frac{\sqrt{2}}{v_N}\right\},$$
$$\overline{R}(n,t) = (R_1(n,t), \dots, R_N(n,t))^T,$$

providing the symmetry of the matrix A_n. In equation (10) the vector $\overline{F}$ is determined by the components

$$F_i = 0, \qquad i = 1, \dots, N, \quad i \neq l;$$
$$F_l = -\cos(k_n x_0)\frac{\sqrt{2}}{h}v_l, \qquad l \neq 1;$$
$$F_l = -\cos(k_n x_0)\frac{v_l}{h}, \qquad l = 1.$$

Let us distinguish the dependence on the parameter n in the square matrix A_n by representing it as a sum of two constants for these medium matrices

$$A_n \doteq A + k_n^2 B, \tag{12}$$

where B is a diagonal matrix

$$B = \operatorname{diag}\left\{v_1^2, v_2^2, \ldots, v_N^2\right\}$$

and A is a three-diagonal symmetric positive matrix

$$A = \frac{1}{h^2}\begin{pmatrix} 2v_1^2 & -\sqrt{2}v_1v_2 & \cdots & \cdots & 0 \\ -\sqrt{2}v_1v_2 & 2v_2^2 & -v_2v_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -v_{N-2}v_{N-1} & 2v_{N-1}^2 & -v_{N-1}v_N \\ 0 & \cdots & \cdots & -v_{N-1}v_N & 2v_N^2 \end{pmatrix}.$$

Using the orthonormal decomposition [6]:

$$A_n = Q\operatorname{diag}\{\lambda_1, \ldots, \lambda_N\}Q^{-1} \tag{13}$$

and replacing the variables:

$$\overline{Y}(n,t) = Q^{-1}\overline{Z}(n,t)$$

problems (10), (11) falls into N independent Cauchy problems for each component Y_i of the vector $\overline{Y}(n,t)$:

$$\frac{d^2Y_i}{dt^2} + \lambda_i Y_i = \varphi_i, \quad \varphi_i = F_l Q_{l,i} f(t), \tag{14}$$

$$Y_i\Big|_{t=0} = \frac{dY_i}{dt}\Big|_{t=0} = 0. \tag{15}$$

The solution to (14), (15) is written down in the form (see [5]):

$$Y_i(t) = \int_0^t \varphi_i(\tau)\frac{1}{\sqrt{\lambda_i}}\sin\left(\sqrt{\lambda_i}(t-\tau)\right)d\tau. \tag{16}$$

Depending on the type of the signal $f(t)$ the values $Y_i(t)$ can be obtained either analytically or numerically by calculating the integral in (16). Let, for example,

$$f(t) = \exp\left(-\frac{(2\pi f_0(t-t_0))^2}{\gamma^2}\right)\sin\left(2\pi f_0(t-t_0)\right),$$

where t_0, f_0, γ are certain constants. Then for all $2t_0 \le t < T$ formula (16) takes the form

$$Y_i(t) \approx \frac{F_l Q_{l,i}\gamma}{4\sqrt{\pi\lambda_i}f_0}\cos\left(\sqrt{\lambda_i}(t-t_0)\right)\left\{\exp\left(-\left[\frac{(2\pi f_0+\sqrt{\lambda_i})\gamma}{4\pi f_0}\right]^2\right)\right.$$
$$\left. -\exp\left(-\left[\frac{(2\pi f_0-\sqrt{\lambda_i})\gamma}{4\pi f_0}\right]^2\right)\right\}.$$

After we have found $\overline{Y}(n,t)$ it is sufficient to return to the original variable

$$\overline{R}(n,t) = D^{-1}Q\overline{Y}(n,t)$$

and then to find the solution $U(x, z_i, t)$ to the original problem by (5).

2.2. Velocity function is a piecewise-continuous function of coordinates x, z

Now we consider the case when the velocity $v_p(x,z)$ is an arbitrary function of two coordinates. After applying a finite integral cosine transform (4), (5) to problem (1)–(3) the following equations results

$$\sum_{l=0}^{M} c(l,n,z)\left[\frac{\partial^2 R(z,l,t)}{\partial z^2} - k_n^2 R(z,l,t)\right]$$
$$= \frac{\partial^2 R(z,n,t)}{\partial t^2} + v_p^2(x_0,z)\cos(k_n x_0)\delta(z-z_0)f(t), \tag{17}$$

$$\left.\frac{\partial R}{\partial z}\right|_{z=0} = \left.R\right|_{z=b} = 0, \tag{18}$$

$$\left.R\right|_{t=0} = \left.\frac{\partial R}{\partial t}\right|_{t=0} = 0, \tag{19}$$

where

$$c(l,n,z) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^a v_p^2(x,z)\cos(k_n x)\,dx, & l=0, \\ \dfrac{2}{\pi}\displaystyle\int_0^a v_p^2(x,z)\cos(k_n x)\cos(k_l x)\,dx, & l=1,2,\dots,M. \end{cases} \tag{20}$$

The dimension of system (17) that is the number of terms (M) needed to approximate the infinite sum is dependent on the Fourier spectrum width of the signal's time dependence $f(t)$. Also, the additional convergence is dependent on the smoothness of the function $\beta_k(x) = v_p^2(x, z_k)$ used for calculation of the coefficients $c(l,n,z_k)$. We may improve decreasing these coefficients by approximating $\beta_k(x)$ by fifth-order spline function. For this purpose the interval of integration from zero to a in integral (20) should be partitioned into L non-uniform parts. If for the approximation of the function $\beta_k(x)$ one uses fifth-order splines $S(x)$, after integration by parts in each interval, the expression for the coefficient $c(p, z_k)$ has the form

$$
\begin{aligned}
c(p, z_k) &= \frac{a^4}{p^5\pi^5}\left[M_L \sin\frac{p\pi x_L}{a} - M_1 \sin\frac{p\pi x_1}{a}\right] - \\
&\frac{a^5}{p^6\pi^6}\sum_{i=2}^{L}\left(\frac{M_i - M_{i-1}}{x_i - x_{i-1}}\right)\left(\cos\frac{p\pi x_{i-1}}{a} - \cos\frac{p\pi x_i}{a}\right), \qquad (21)\\
c(0, z_k) &= \frac{1}{a}\left[\frac{1}{120}(M_L x_L^5 - M_1 x_1^5) - \frac{1}{720}\sum_{i=2}^{L}(M_i - M_{i-1})\frac{x_i^6 - x_{i-1}^6}{x_i - x_{i-1}}\right], \qquad (22)
\end{aligned}
$$

where

$$M_i = \left.\frac{d^4 S(x)}{dx^4}\right|_{x=x_i} \qquad p = l \pm n.$$

We see that the coefficients $c(p, z_k)$ asymptotically decrease as p^{-6}. The first term of equation (21) is identically zero because $x_L = a$ and $x_1 = 0$. The complexity of subsurface geometries in the direction of the coordinate x does not cause additional computational difficulties as long as the interval of integration between 0 and a is partitioned into a sufficiently large number of non-uniform intervals L in which the function $\beta_k(x)$ is well approximated by the spline function.

Problem (17)–(19) can be presented in the vector form

$$\frac{\partial^2 \overline{R}}{\partial t^2} = C\left[\frac{\partial}{\partial z^2} - K\right]\overline{R} - \delta(z - z_0)\overline{F}, \qquad (23)$$

$$\left.\frac{\partial \overline{R}}{\partial z}\right|_{z=0} = \left.\overline{R}\right|_{z=b} = 0, \qquad (24)$$

$$\left.\overline{R}\right|_{t=0} = \left.\frac{\partial \overline{R}}{\partial t}\right|_{t=0} = 0. \qquad (25)$$

Here

$$
\begin{aligned}
&\overline{R}(z,t) = (R(k_0, z, t), R(k_1, z, t), \ldots, R(k_M, z, t))^T, \\
&\overline{F} = f(t)v_p^2(x_0, z)\,(\cos(k_0 x_0), \cos(k_1 x_0), \ldots, \cos(k_M x_0))^T, \\
&K = \operatorname{diag}\left\{k_0^2, k_1^2, \ldots, k_M^2\right\},
\end{aligned}
$$

$$C = \begin{pmatrix} c(0,0,z) & c(1,0,z) & \ldots & c(M,0,z) \\ c(0,1,z) & c(1,1,z) & \ldots & c(M,1,z) \\ \vdots & \vdots & \ddots & \vdots \\ c(0,M,z) & c(1,M,z) & \ldots & c(M,M,z) \end{pmatrix}. \qquad (26)$$

As indicated above, for solving the system of equations (23)–(25) we may use the finite difference approximation with respect to only the spatial coordinate z. In this case the problem reduces to the form (10), (11) whose algorithm of the solution is described above.

3. Application of the method for forward seismic problem in the polar coordinate system

Let us illustrate the main stages of the method on the wave equation in the polar coordinate system (r, φ):

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} = \frac{1}{v_p^2(r,\varphi)}\frac{\partial^2 U}{\partial t^2} + \frac{4\pi}{r}\delta(r - r_0)\delta(\varphi - \varphi_0)f(t) \quad (27)$$

on the half-space $0 \le \varphi \le \pi$, $0 \le r < \infty$.

The problem is considered with zero initial data

$$U\Big|_{t=0} = \frac{\partial U}{\partial t}\Big|_{t=0} = 0 \quad (28)$$

and with the boundary conditions on the free surface in the form

$$\frac{\partial U}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial U}{\partial \varphi}\Big|_{\varphi=\pi} = 0. \quad (29)$$

We consider $v_p(r, \varphi)$ to be a piecewise-continuous function of the two coordinates r, φ; (r_0, φ_0) are the coordinates of the source.

For solving problem (27)–(29) let us make use of the finite integral cosine-Fourier transform with respect to φ:

$$R_n(r,t) = \int_0^\pi U(\varphi, r, t)\cos(n\varphi)d\varphi \quad (30)$$

with the inversion formula

$$U(\varphi, r, t) = \frac{1}{\pi}R_0(r,t) + \frac{2}{\pi}\sum_{n=1}^{\infty} R_n(r,t)\cos(n\varphi). \quad (31)$$

After applying transform (30), (31) a new boundary value problem is of the form

$$\sum_{m=0}^{M}\left[\frac{\partial^2 R_m(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial R_m(r,t)}{\partial r} - \frac{m^2}{r^2}R_m(r,t)\right]c(n,m,r) = \frac{\partial^2 R_n(r,t)}{\partial t^2} + f_n(r,t) \quad (32)$$

with initial data

$$R_n(r,t)\Big|_{t=0} = 0, \qquad \frac{\partial R_n(r,t)}{\partial t}\Big|_{t=0} = 0, \tag{33}$$

where

$$f_n(r,t) = \frac{4\pi}{r} v_p^2(r,\varphi_0)\delta(r-r_0)\cos(n\varphi_0)f(t),$$

$$c(n,m,r) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\pi v_p^2(r,\varphi)\cos(n\varphi)\,d\varphi, & m=0, \\ \dfrac{2}{\pi}\displaystyle\int_0^\pi v_p^2(r,\varphi)\cos(n\varphi)\cos(m\varphi)\,d\varphi, & m=1,2,\ldots,M. \end{cases}$$

System (32), (33) is solved for $n, m = 0, 1, \ldots, M$, where M is the number of terms of the row, needed to approximate the infinite sum (31) with required accuracy.

We rewrite the problem in the vector form

$$\frac{\partial^2 \vec{R}}{\partial t^2} = C\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)E - \frac{1}{r^2}P\right]\vec{R} - \frac{\delta(z-z_0)}{r}\vec{F}. \tag{34}$$

Here C is a matrix of form (26), E is a unit matrix,

$$\vec{R}(r,t) = (R_0(r,t), R_1(r,t), \ldots, R_M(r,t))^T,$$
$$\vec{F} = 4\pi v_p^2(r_0,\varphi_0)f(t)\,(1, \cos\varphi_0, \cos 2\varphi_0, \ldots, \cos M\varphi_0)^T,$$
$$P = \operatorname{diag}\left\{0, 1, 2^2, \ldots, M^2\right\}.$$

For solving problem (34) we make use of the finite difference approximation with respect to only the spatial coordinate r. After that we reduce the problem to form (10), (11) whose algorithm of solution is described above.

4. Construction of 2D depth migration algorithms

In order to construct a migration algorithm we make use of the numerical-analytical operator as we have done for the forward modeling. In this connection we use the full wave equation, where velocity distributions are the smoothed versions of the actual velocities to suppress secondary scattering [7]. In the case of migration, however, we must take into consideration data acquisition parameters such as trace interval, frequency content of the data to be migrated, etc.

One of the parts of the depth migration is the reverse-time propagation of the wave field recorded at the free surface backward in time. This procedure is simulated by equation (1) with the right-hand side in the form

$$\delta(z - z_0) \sum_{j=0}^{p} \delta(x - x_j) f_j(T_0 - t), \tag{35}$$

where $z_0 = 0$, $f_j(T_0 - t)$ are reverse seismic traces, T_0 is the time trace length, p is the number of traces.

The numerical-analytical algorithm which we used for the forward seismic modeling without essential changes is applicable to the migration problem. After application to it the finite cosine transformation (4), (5) and presenting it in the vector form, the migration problem differs from problem (23)–(25) only in the vector

$$\overline{F} = \left(\sum_{j=1}^{p} \cos(k_0 x_j) f_j(T_0 - t), \sum_{j=1}^{p} \cos(k_1 x_j) f_j(T_0 - t), \ldots, \right.$$
$$\left. \sum_{j=1}^{p} \cos(k_M x_j) f_j(T_0 - t) \right)^T. \tag{36}$$

Further, the migration problem can be written down in the closed form. For example, for the case when the velocity v_p depends only on the coordinate z the solution is obtained in the following form:

$$Y_i(n, T_0) = \int_0^{T_0} \varphi_i(n, \tau) \frac{1}{\sqrt{\lambda_i}} \sin\left(\sqrt{\lambda_i}(T_0 - \tau)\right) d\tau, \tag{37}$$

where

$$\varphi(n, \tau) = -\frac{v_1}{h} Q_{1,i} \sum_{j=1}^{p} \cos(k_n x_j) f_j(T_0 - \tau).$$

Integral (37) is numerically calculated.

In order to calculate the displacement $U(x, z, t = T_0)$ and to construct a migration seismic section we make use of formula (5). In the case when the velocity is an arbitrary function of two coordinates we also obtain a solution similar to (37). The additional convergence of the migration algorithm is dependent on smoothness of the function $\beta_k(x) = v_p^2(x, z_k)$ in integral (20). In this case the function $\beta_k(x)$ is expanded in a Fourier series. We may take a small number of terms in the Fourier series if the function $\beta_k(x)$ is slowly varying along the coordinate x.

5. Conclusion

Let us dwell on the advantages of the given algorithm. The algorithm for seismic forward modeling allows the calculation of a wave field at any moment of time without recurrent recalculation procedure from one time level to another as it takes place when we use the finite difference approximation with respect to time. In addition, this approach makes it possible to obtain solutions to many sources without essential computer costs, because we need to transform the matrix A_n to the diagonal form only one time for all the sources.

The above-presented migration algorithm is based on the full wave equation, where velocity distributions are the smoothed versions of the actual velocities to suppress secondary scattering. In this case the algorithm possesses super-convergence.

At present time, this algorithm without essential changes has been developed for elastic inhomogeneous media.

References

[1] Alekseev A.S., Mikhailenko B.G. The solution of dynamic problems of elastic wave propagation in inhomogeneous media by a combination of partial separation of variables and finite-difference methods // J. Geoph. – 1980. – Vol. 40. – P. 161–172.

[2] Alekseev A.S., Mikhailenko B.G. Numerical modeling in seismic prospecting // Expanded Abstr. 55th Annual Intern. SEG Meeting. – Washington, 1985. – P. 535–539.

[3] Mikhailenko B.G. Numerical experiment in seismic investigations // J. Geoph. – 1985. – Vol. 58. – P. 101–124.

[4] Clayton R., Engquist B. Absorbing boundary conditions for acoustic and elastic wave equations // Bull. Seism. Soc. Am. – 1977. – Vol. 67. – P. 1529–1540.

[5] Berezin I.S., Zhidkov N.P. Computational Methods. – Moscow: State Publishing House of Physics–Mathematics Literature, 1960. – Vol. II (in Russian).

[6] Grantmacher F.R. The Theory of Matrices. – New York: Chelsea Publ. Co., 1959. – Vol. II.

[7] Lowenthal D., Stoffa P.L., Faria E.L. Supressing the unwanted reflections of the full wave equation // Geophysics. – 1987. – Vol. 52. – P. 1007–1012.