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Integral Laguerre transform as applied to forward seismic modeling

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The paper presents some efficient algorithms based on the application of the integral Laguerre transforms for approximation of temporal derivatives. Some specific features of employing this algorithm for the first and the second order equations with respect to time are considered. A few examples of calculation of seismic fields for the layered medium model with drastically contrast elastic parameters are presented.

Introduction

The finite difference frequency-domain modeling for the generation of synthetic seismograms has achieved considerable success and is currently an active field of research [1–3]. It is well-known that simulation of seismic fields in the time domain is widely used as it is sufficiently accurate and easy to realize. As for the frequency domain modeling, it has some advantages over the time domain simulation when, for example, the complete prestack seismic response is required for a multiple-source experiment. The space frequency domain modeling does not have any stability problem, while the accuracy of the simulation based on the space-time domains is determined by the stability limit dependent on the greatest velocity in the model. Also, it is possible to employ a function of frequency as the coefficient of damping, therefore simulation of the damping effects appears more flexible. Unfortunately, after discretization of the frequency-domain equations we arrive at the large matrix equations, and their solution for many temporal frequencies is a time-consuming task.

Recently, we have proposed the new approach based on applying the integral Laguerre transform along the time coordinate instead of the integral Fourier transform [4, 5]. Applying this approach, we obtain an analogue of the frequency-domain forward modeling, where instead of the frequency ω we have the number m – the degree of the Laguerre polynomials. After using high-order finite differences or the spectral technique, the resulting linear system has a sparse matrix independent of m, only the right-hand side of the system has the recurrent dependence on the parameter m. In this case, we can use fast methods such as the Cholesky method for solving the obtained system with a great number of the right-hand sides. As this takes place, the matrix is only once transformed as compared to the frequency-domain forward modeling. In this paper, our approach is applied to the elastic wave equations of the first order system for the velocity and stresses as well as to the acoustic wave equation of the second order with a variable velocity. Some examples of the calculation of the wave fields for drastically contrast, in terms of elastic parameters, media are presented.

1. The integral Laguerre transform

Let us introduce the integral Laguerre transform

$$F_m = \int_0^\infty F(t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht) \tag{1}$$

with the inversion formula

$$F(t) = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} F_m l_m^\alpha(ht), \tag{2}$$

where $l_m^\alpha(ht)$ are the orthonormal Laguerre functions:

$$\int_0^\infty l_m^\alpha(ht) l_n^\alpha(ht)\, dt = \delta_{mn} \frac{(m+\alpha)!}{m!}. \tag{3}$$

The Laguerre functions $l_m^\alpha(ht)$ are expressed by the classical Laguerre polynomials $L_m^\alpha(ht)$ [6]. We select the parameter α to be integer and positive, then

$$l_m^\alpha(ht) = (ht)^{\alpha/2} e^{-ht/2} L_m^\alpha(ht). \tag{4}$$

In formulas (1)–(4), $m = 0, 1, 2, \ldots$. In addition, the new shift parameter $h > 0$ is introduced, whose features and application are discussed below. Further, for the substantiation of validity of application of the integral transform (1), (2) to seismic problems, we assume all the functions to be, at least, piecewise-continuous and to have some limitations on their behaviour in 0 and ∞, see [6].

Hereafter, we will need formulas of the first and the second derivatives of the Laguerre polynomials in the variable t. Making use of the definition of the Laguerre polynomials it is easy to obtain the following formulas:

$$\frac{\partial}{\partial t} L_m^\alpha(ht) = -h \sum_{k=0}^{m-1} L_k^\alpha(ht), \tag{5}$$

$$\frac{\partial^2}{\partial t^2} L_m^\alpha(ht) = h^2 \sum_{k=0}^{m-2} (m-k-1) L_k^\alpha(ht). \tag{6}$$

2. Application of the integral Laguerre transform for the 2D vertically heterogeneous elastic problem

Let us consider the application of the integral Laguerre transform for the approximation of temporal derivatives. As for the approximation of spatial derivatives, one can use the finite difference method of the high order accuracy, the spectral technique or their combination [7–9]. The latter approach will be used in the present paper.

Seismic waves propagation in an elastic medium in the Cartesian system of the coordinates (x, z) for velocities and stresses may be written down as the following system of equations:

$$\begin{aligned}
\frac{\partial u_x}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}\right) + F_x(x,z)f(t),\\
\frac{\partial u_z}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}\right) + F_z(x,z)f(t),\\
\frac{\partial \tau_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z},\\
\frac{\partial \tau_{zz}}{\partial t} &= (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{\partial u_x}{\partial x},\\
\frac{\partial \tau_{xz}}{\partial t} &= \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right).
\end{aligned} \tag{7}$$

Here, (u_x, u_z) are the velocity vector components, $(\tau_{xx},\ \tau_{zz},\ \tau_{xz})$ are the stress tensor components, ρ is the medium density, λ and μ are the Lame parameters. In these equations, F_x, F_z are the components of the body forces $\vec{F}(x,z) = F_x\vec{e_x} + F_z\vec{e_z}$. Depending on the type of a source, these components will have the following values:

1) $F_x = 0$, $F_z = \delta(x - x_0)\delta(z - z_0)$ for the vertical type source;

2) $F_x = \dfrac{\partial\delta(x - x_0)}{\partial x}\delta(z - z_0)$, $F_z = \delta(x - x_0)\dfrac{\partial\delta(z - z_0)}{\partial z}$ for the explosive type source.

In the above formulas, (x_0, z_0) are the coordinates of the source and $f(t)$ represents the time variation of the source.

The problem is solved with zero initial data:

$$u_x\big|_{t=0} = u_z\big|_{t=0} = \tau_{xx}\big|_{t=0} = \tau_{zz}\big|_{t=0} = \tau_{xz}\big|_{t=0} = 0. \tag{8}$$

Consider this solution on the half-space $z \geq 0$ with the boundary conditions on the free surface defined in the form:

$$\tau_{xz}(x,z,t)|_{z=0} = \tau_{zz}(x,z,t)|_{z=0} = 0. \tag{9}$$

Let us assume the medium to be vertically heterogeneous, i.e., $\rho(z)$, $\lambda(z)$, and $\mu(z)$ are piecewise-continuous functions of the coordinate z. Assume, also, that the functions $u_x(x,z,t)$, $u_z(x,z,t)$, $\tau_{xx}(x,z,t)$, $\tau_{zz}(x,z,t)$, and $\tau_{xz}(x,z,t)$ are sufficiently smooth to be applied to subsequent transformations.

Let us make use of the finite integral cosine-sine Fourier transforms for each component of system (7):

$$\begin{Bmatrix} \bar{u}_x(n,z,t) \\ \bar{u}_z(n,z,t) \\ \bar{\tau}_{xx}(n,z,t) \\ \bar{\tau}_{zz}(n,z,t) \\ \bar{\tau}_{xz}(n,z,t) \end{Bmatrix} = \int_0^a \begin{Bmatrix} u_x(x,z,t)\sin(k_n x) \\ u_z(x,z,t)\cos(k_n x) \\ \tau_{xx}(x,z,t)\cos(k_n x) \\ \tau_{zz}(x,z,t)\cos(k_n x) \\ \tau_{xz}(x,z,t)\sin(k_n x) \end{Bmatrix} dx, \tag{10}$$

where $k_n = n\pi/a$. The respective inverse formulas are of the form:

$$\begin{Bmatrix} u_x(x,z,t) \\ u_z(x,z,t) \\ \tau_{xx}(x,z,t) \\ \tau_{zz}(x,z,t) \\ \tau_{xz}(x,z,t) \end{Bmatrix} = \sum_{n=0}^{\infty} d_n \begin{Bmatrix} \bar{u}_x(n,z,t)\sin(k_n x) \\ \bar{u}_z(n,z,t)\cos(k_n x) \\ \bar{\tau}_{xx}(n,z,t)\cos(k_n x) \\ \bar{\tau}_{zz}(n,z,t)\cos(k_n x) \\ \bar{\tau}_{xz}(n,z,t)\sin(k_n x) \end{Bmatrix}, \tag{11}$$

where

$$d_n = \begin{cases} 1/\pi, & \text{for } n = 0, \\ 2/\pi, & \text{for } n \geq 1. \end{cases}$$

After applying the transformations to (7)-(9), the obtained system will contain the terms

$$u_x\Big|_{\substack{x=0 \\ x=a}}, \quad \frac{\partial u_z}{\partial x}\Big|_{\substack{x=0 \\ x=a}}, \quad \frac{\partial \tau_{xx}}{\partial x}\Big|_{\substack{x=0 \\ x=a}}, \quad \frac{\partial \tau_{zz}}{\partial x}\Big|_{\substack{x=0 \\ x=a}}, \quad \tau_{xz}\Big|_{\substack{x=0 \\ x=a}}.$$

Introduce the supplementary boundary conditions assuming these terms on the boundaries $x = 0$ and $x = a$ be equal to zero. We will consider the wave field up to the moment $t < T$, where T is the minimal time of propagation of the leading wave front up to the reflecting surfaces $x = 0$, $x = a$. It can be done due to hyperbolicity of our problem.

As a result of using (10), (11), problem (7)-(9) takes the form:

$$
\begin{aligned}
\frac{\partial \bar{u}_x}{\partial t} &= \frac{1}{\rho}\Big(\frac{\partial \bar{\tau}_{xz}}{\partial z} - k_n \bar{\tau}_{xx}\Big) + F_x(n,z) f(t),\\
\frac{\partial \bar{u}_z}{\partial t} &= \frac{1}{\rho}\Big(\frac{\partial \bar{\tau}_{zz}}{\partial z} + k_n \bar{\tau}_{xz}\Big) + F_z(n,z) f(t),\\
\frac{\partial \bar{\tau}_{xx}}{\partial t} &= \lambda \frac{\partial \bar{u}_z}{\partial z} + k_n(\lambda + 2\mu)\bar{u}_x,\\
\frac{\partial \bar{\tau}_{zz}}{\partial t} &= (\lambda + 2\mu)\frac{\partial \bar{u}_z}{\partial z} + k_n \lambda \bar{u}_x,\\
\frac{\partial \bar{\tau}_{xz}}{\partial t} &= \mu\Big(\frac{\partial \bar{u}_x}{\partial z} - k_n \bar{u}_z\Big).
\end{aligned}
\tag{12}
$$

Here

$$
\begin{aligned}
F_x(n,z) &= \int_0^a F_x(x,z)\sin(k_n x)\,dx,\\
F_z(n,z) &= \int_0^a F_z(x,z)\cos(k_n x)\,dx.
\end{aligned}
$$

The system of equations (12) is solved at zero initial data and the following boundary conditions:

$$
\bar{\tau}_{xz}(n,z,t)\Big|_{z=0} = \bar{\tau}_{zz}(n,z,t)\Big|_{z=0} = 0. \tag{13}
$$

Now, to problem (12), (13), we apply the integral Laguerre transforms along the temporal variable t of the form:

$$
\left\{\begin{matrix} \bar{u}_x^m(n,z)\\ \bar{u}_z^m(n,z)\\ \bar{\tau}_{xx}^m(n,z)\\ \bar{\tau}_{zz}^m(n,z)\\ \bar{\tau}_{xz}^m(n,z)\end{matrix}\right\} = \int_0^\infty \left\{\begin{matrix} \bar{u}_x(n,z,t)\\ \bar{u}_z(n,z,t)\\ \bar{\tau}_{xx}(n,z,t)\\ \bar{\tau}_{zz}(n,z,t)\\ \bar{\tau}_{xz}(n,z,t)\end{matrix}\right\} (ht)^{-\alpha/2} l_m^\alpha(ht)\,d(ht), \tag{14}
$$

with the respective inversion formulas:

$$
\left\{\begin{matrix} \bar{u}_x(n,z,t)\\ \bar{u}_z(n,z,t)\\ \bar{\tau}_{xx}(n,z,t)\\ \bar{\tau}_{zz}(n,z,t)\\ \bar{\tau}_{xz}(n,z,t)\end{matrix}\right\} = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} \left\{\begin{matrix} \bar{u}_x^m(n,z)\\ \bar{u}_z^m(n,z)\\ \bar{\tau}_{xx}^m(n,z)\\ \bar{\tau}_{zz}^m(n,z)\\ \bar{\tau}_{xz}^m(n,z)\end{matrix}\right\} l_m^\alpha(ht). \tag{15}
$$

Based on inversion formulas (15), it is obvious that the value of the parameter α (the order of the Laguerre functions) should be either one or greater than one to satisfy the initial conditions. The value of the parameter α

affects the accuracy of the numerical implementation of the algorithm. We will talk about it in more detail in Section 5.

The application of (14), (15) results in the system of equations (12) of the following form:

$$\begin{aligned}
&\frac{h}{2}\bar{\tau}_{zz}^m - (\lambda+2\mu)\frac{\partial \bar{u}_z^m}{\partial z} - k_n\lambda\bar{u}_x^m = f_1^{m-1}(n,z),\\
&\frac{h}{2}\bar{\tau}_{xx}^m - \lambda\frac{\partial \bar{u}_z^m}{\partial z} - k_n(\lambda+2\mu)\bar{u}_x^m = f_2^{m-1}(n,z),\\
&\frac{h}{2}\bar{u}_x^m - \frac{1}{\rho}\Big(\frac{\partial \bar{\tau}_{xz}^m}{\partial z} - k_n\bar{\tau}_{xx}^m\Big) = f_3^{m-1}(n,z),\\
&\frac{h}{2}\bar{u}_z^m - \frac{1}{\rho}\Big(\frac{\partial \bar{\tau}_{zz}^m}{\partial z} + k_n\bar{\tau}_{xz}^m\Big) = f_4^{m-1}(n,z),\\
&\frac{h}{2}\bar{\tau}_{xz}^m - \mu\Big(\frac{\partial \bar{u}_x^m}{\partial z} - k_n\bar{u}_z^m\Big) = f_5^{m-1}(n,z),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
f_1^{m-1}(n,z) &= -h\sum_{j=0}^{m-1}\bar{\tau}_{zz}^j,\\
f_2^{m-1}(n,z) &= -h\sum_{j=0}^{m-1}\bar{\tau}_{xx}^j,\\
f_3^{m-1}(n,z) &= -h\sum_{j=0}^{m-1}\bar{u}_x^j + F_x(n,z)f^m,\\
f_4^{m-1}(n,z) &= -h\sum_{j=0}^{m-1}\bar{u}_z^j + F_z(n,z)f^m,\\
f_5^{m-1}(n,z) &= -h\sum_{j=0}^{m-1}\bar{\tau}_{xz}^j,
\end{aligned} \tag{17}$$

$$f^m = \int_0^\infty f(t)(ht)^{-\alpha/2}l_m^\alpha(ht)\,d(ht),$$

and the boundary conditions are the following:

$$\bar{\tau}_{xz}^m(n,z)\Big|_{z=0} = \bar{\tau}_{zz}^m(n,z)\Big|_{z=0} = 0. \tag{18}$$

System (16) can be written down in the vector form:

$$\frac{h}{2}\vec{V}(z,n,m) + A(z,n)\vec{V}(z,n,m) = \vec{F}(z,n,m), \tag{19}$$

where

$$\vec{V}(z,n,m) = \begin{pmatrix} \bar{\tau}^m_{zz}(z,n) \\ \bar{\tau}^m_{xx}(z,n) \\ \bar{u}^m_x(z,n) \\ \bar{u}^m_z(z,n) \\ \bar{\tau}^m_{xz}(z,n) \end{pmatrix}, \qquad \vec{F}(z,n,m-1) = \begin{pmatrix} f_1^{m-1}(z,n) \\ f_2^{m-1}(z,n) \\ f_3^{m-1}(z,n) \\ f_4^{m-1}(z,n) \\ f_5^{m-1}(z,n) \end{pmatrix},$$

$$A(z,n) = \begin{pmatrix} 0 & 0 & -k_n\lambda & -(\lambda+2\mu)\dfrac{\partial}{\partial z} & 0 \\ 0 & 0 & -k_n(\lambda+2\mu) & -\lambda\dfrac{\partial}{\partial z} & 0 \\ 0 & \dfrac{k_n}{\rho} & 0 & 0 & -\dfrac{1}{\rho}\dfrac{\partial}{\partial z} \\ -\dfrac{1}{\rho}\dfrac{\partial}{\partial z} & 0 & 0 & 0 & -\dfrac{k_n}{\rho} \\ 0 & 0 & -\mu\dfrac{\partial}{\partial z} & \mu k_n & 0 \end{pmatrix}. \tag{20}$$

As is seen, after application of the finite integral cosine-sine Fourier and Laguerre transforms the solution of the original problem (7)–(9) is reduced to the solution of the 1D problems (19) with respect to the coordinate z. In order to solve these problems, let us use the finite difference approximation of the fourth order accuracy scheme [10] on the staggered-grid space:

$$\begin{aligned} \omega &= \{z_i = i\Delta z; \quad i = 0,\dots,K\}, \\ \omega_{1/2} &= \{z_{i+1/2} = (i+0.5)\Delta z; \quad i = 0,\dots,K-1\}. \end{aligned}$$

The values of the components $\bar{\tau}^m_{zz}(n,z)$, $\bar{\tau}^m_{xx}(n,z)$, and $\bar{u}^m_x(n,z)$ are defined on the grid ω, and the values of the components $\bar{\tau}^m_{xz}(n,z)$ and $\bar{u}^m_z(n,z)$ are defined on the grid $\omega_{1/2}$. Let us introduce the vector

$$\vec{W}(n,m) = \left(\vec{V}_0(n,m), \vec{V}_1(n,m), \dots, \vec{V}_K(n,m)\right)^T,$$

where

$$\vec{V}_0 = \begin{pmatrix} \bar{\tau}_{zz}(z_0) \\ \bar{\tau}_{xx}(z_0) \\ \bar{u}_x(z_0) \end{pmatrix}, \qquad \vec{V}_i = \begin{pmatrix} \bar{u}_z(z_{i-1/2}) \\ \bar{\tau}_{xz}(z_{i-1/2}) \\ \bar{\tau}_{zz}(z_i) \\ \bar{\tau}_{xx}(z_i) \\ \bar{u}_x(z_i) \end{pmatrix}, \quad i = 1,\dots,K.$$

As a result, our problem reduces to the system of linear algebraic equations formally written down in the vector form

$$\left(A_\Delta(n) + \frac{h}{2}E\right)\vec{W}(n,m) = \vec{F}_\Delta(n,m-1), \tag{21}$$

where

$$\vec{F}_\Delta(n,m-1) = \left(\vec{F}_0, \vec{F}_1, \dots, \vec{F}_K\right)^T,$$

$$\vec{F}_0 = \begin{pmatrix} f_1^{m-1}(z_0,n) \\ f_2^{m-1}(z_0,n) \\ f_3^{m-1}(z_0,n) \end{pmatrix}, \qquad \vec{F}_i = \begin{pmatrix} f_1^{m-1}(z_{i-1/2},n) \\ f_2^{m-1}(z_{i-1/2},n) \\ f_3^{m-1}(z_i,n) \\ f_4^{m-1}(z_i,n) \\ f_5^{m-1}(z_i,n) \end{pmatrix}, \qquad i = \overline{1,K}.$$

Here $\vec{W}(m,n)$ is the solution vector, $A_\Delta(n)$ is a band matrix of the system independent of the parameter m, $A_\Delta(n)$ is the finite difference approximation of the operator $A(z,n)$ (20), obtained after approximation of the spatial derivatives with respect to z with the fourth order of accuracy on the difference grids ω, $\omega_{1/2}$. The right-hand side of the system $\vec{F}_\Delta(n,m-1)$ has the recurrent dependence on the parameter m. It gives us the possibility to use fast methods, for example, the Cholesky method, for solving the linear system with a great number of the right-hand sides. Thus, the solution of the original problem (7)–(9) can be calculated from the solution of the linear algebraic system of equations (21) with a subsequent application of the inversion formulas (15) and (11).

Note, that selection of a sequence of components of the vector $\vec{V}(z,n,m)$ is defined by the condition of minimization of the number of diagonals of the matrix $A_\Delta(n)$. In addition, the components of $\vec{V}(z,n,m)$ are selected in such a way that the terms of system (16), having the parameter $h/2$ as a co-factor, should be arranged on the main diagonal of the matrix $A_\Delta(n)$. By varying h, this choice makes it possible to essentially affect the condition number of the matrix $A_\Delta(n)$.

3. Application of the integral Laguerre transform for the second order wave equation

Let us consider the heterogeneous wave equation:

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} - \frac{1}{v_p^2(x,z)}\frac{\partial^2 u}{\partial t^2} = \delta(x-x_0)\delta(z-z_0)f(t). \tag{22}$$

We search for its solution satisfying zero initial data

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{23}$$

and the boundary conditions on the free surface:

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0. \tag{24}$$

We assume $v_p(x, z)$ to be a piecewise-continuous function of two variables. The source with the coordinates x_0, z_0 is simulated by the right-hand side of equation (22), where $f(t)$ represents the time variation of the source.

For solving problem (22)–(24), let us make use of the finite integral cosine Fourier transform

$$R(z, n, t) = \int_0^a U(z, x, t) \cos\frac{n\pi x}{a}\, dx, \tag{25}$$

with the inversion formula

$$U(z, x, t) = \frac{1}{a} R(z, 0, t) + \frac{2}{a} \sum_{n=1}^{\infty} R(z, n, t) \cos\frac{n\pi x}{a}. \tag{26}$$

The equation obtained after the transformation contains the terms $\left.\frac{\partial U}{\partial x}\right|_{x=0}$ and $\left.\frac{\partial U}{\partial x}\right|_{x=a}$. Let us introduce the new additional boundary conditions

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial U}{\partial x}\right|_{x=a} = U|_{z=b} = 0$$

and consider the wave field up to the time $t < T$, where T is the minimal time of propagation of the leading wave front up to the reflecting surfaces $x = a$, $z = b$.

After applying the finite integral cosine transform (25), (26) to problem (22)–(24), we arrive at the following equations:

$$\sum_{l=0}^{M} c(l, n, z) \left[\frac{\partial^2 R(z, l, t)}{\partial z^2} - k_n^2 R(z, l, t)\right] = \frac{\partial^2 R(z, n, t)}{\partial t^2} - \Phi(z, n) f(t), \tag{27}$$

$$\left.\frac{\partial R}{\partial z}\right|_{z=0} = R|_{z=b} = 0, \qquad R|_{t=0} = \left.\frac{\partial R}{\partial t}\right|_{t=0} = 0, \tag{28}$$

where

$$k_n = \frac{n\pi}{a}, \qquad \Phi(z, n) = -v_p^2(x_0, z) \cos(k_n x_0) \delta(z - z_0),$$

$$c(l,n,z) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^a v_p^2(x,z)\cos(k_n x)\,dx, & l=0, \\ \dfrac{2}{\pi}\displaystyle\int_0^a v_p^2(x,z)\cos(k_n x)\cos(k_l x)\,dx, & l=1,2,\ldots,M. \end{cases}$$

The dimension of system (27), i.e., the number of the terms (M) needed to approximate the infinite sum, is dependent on the Fourier spectrum width of the wavelet $f(t)$. Problem (27), (28) can be presented in the vector form

$$\frac{\partial^2 \vec{R}(z,t)}{\partial t^2} + A(z)\vec{R}(z,t) = \vec{\Phi}(z)f(t), \tag{29}$$

$$\left.\frac{\partial \vec{R}}{\partial z}\right|_{z=0} = \vec{R}|_{z=b} = 0, \quad \vec{R}|_{t=0} = \left.\frac{\partial \vec{R}}{\partial t}\right|_{t=0} = 0. \tag{30}$$

Here

$$\begin{aligned} \vec{R}(z,t) &= (R(z,0,t), R(z,1,t), \ldots, R(z,M,t))^T, \\ \vec{\Phi}(z) &= (\Phi(z,0), \Phi(z,1), \ldots, \Phi(z,M))^T. \end{aligned}$$

The matrix-operator $A(z)$ is represented as

$$A(z) = C(z)K(z), \tag{31}$$

where the matrix-operators $C(z)$ and $K(z)$ are of the form:

$$\begin{aligned} K(z) &= \operatorname{diag}\left(k_0^2 - \frac{\partial^2}{\partial z^2}, k_1^2 - \frac{\partial^2}{\partial z^2}, \ldots, k_M^2 - \frac{\partial^2}{\partial z^2}\right), \\ C(z) &= \begin{pmatrix} c(0,0,z) & c(1,0,z) & \ldots & c(M,0,z) \\ c(0,1,z) & c(1,1,z) & \ldots & c(M,1,z) \\ \cdots & \cdots & \cdots & \cdots \\ c(0,M,z) & c(1,M,z) & \ldots & c(M,M,z) \end{pmatrix}. \end{aligned}$$

In order for the spatial derivative be approximated with respect to the coordinate z, we can make use of the finite differences of a high order accuracy. Here, for the sake of simplicity, we are going to employ the finite difference approximation of the second order of accuracy.

Let us introduce in the variable z the uniform difference grid

$$\omega = \{z_i = (i-1)\Delta z; \quad i = 1, \ldots, N+1; \quad b = N\Delta z\}.$$

After discretization, problem (29), (30) reduces to the Cauchy problem for the system of the linear differential second order equations. In the vector form, it is as follows:

$$\frac{\partial^2 \vec{Z}(t)}{\partial t^2} + A_\Delta \vec{Z}(t) = \vec{\Phi}_\Delta f(t), \tag{32}$$

$$\vec{Z}|_{t=0} = \left.\frac{d\vec{Z}}{dt}\right|_{t=0} = 0, \tag{33}$$

where A_Δ is a positive block three-diagonal symmetric matrix, A_Δ is a finite difference approximation of the operator A from (31). Here

$$\vec{Z}(t) = (\vec{R}_1(t), \ldots, \vec{R}_N(t))^T, \qquad \vec{\Phi}_\Delta = (\vec{\Phi}(z_0), \ldots, \vec{\Phi}(z_N))^T.$$

Let us apply to problem (32), (33) the integral Laguerre transform along the variable t:

$$\vec{Q}(m) = \int_0^\infty (ht)^{-\alpha/2} l_m^\alpha(ht) \vec{Z}(t)\, d(ht), \tag{34}$$

with the inversion formula

$$\vec{Z}(t) = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} \vec{Q}(m) l_m^\alpha(ht). \tag{35}$$

As is seen from formula (35), the value of the parameter α should be either two or greater to satisfy the initial conditions (33). We obtain the problem for the coefficients $\vec{Q}(m)$:

$$\frac{h^2}{4}\vec{Q}(m) + A_\Delta \vec{Q}(m) = \vec{\Psi}(m-1), \tag{36}$$

$$\left.\frac{d\vec{Q}(m)}{dz}\right|_{z=0} = \vec{Q}(m)|_{z=b} = 0, \tag{37}$$

where

$$\vec{\Psi}(m-1) = f_m \vec{\Phi}_\Delta + h^2 \sum_{j=0}^{m-1} (m-j)\vec{Q}(j),$$

$$f_m = \int_0^\infty (ht)^{-\alpha/2} l_m^\alpha(ht) f(t)\, d(ht).$$

We come to a system of the linear algebraic equations which is written down in the compact form:

$$\left(A_\Delta + \frac{h^2}{4}E\right)\vec{Q}(m) = \vec{\Psi}(m-1). \tag{38}$$

The resulting linear algebraic system of equations has a symmetric, positive block three-diagonal matrix. Like in Section 2, this matrix is independent of number m and the right-hand side of the system has the recurrent dependence on this parameter. The solution of the original problem (22)–(24) can be calculated from the solution of the linear algebraic system of equations (38) with a subsequent application of the inversion formulas (26) and (35).

4. Application of the integral Laguerre transform for the 3D axially-symmetric vertically-heterogeneous elastic problem

Let us consider the application of the integral Laguerre transform on an example of solution of the first order elastic equations in the cylindrical coordinates (r, θ, z) for the 3D axially-symmetric, vertical heterogeneous half-space $z \geq 0$. The selected physical model can be described by the following system of equations:

$$
\begin{aligned}
\rho \frac{\partial u_r}{\partial t} &= \frac{\partial \tau_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_r - \tau_\theta}{r}, \\
\rho \frac{\partial u_z}{\partial t} &= \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_z}{\partial z} + \frac{\tau_{rz}}{r}, \\
\frac{\partial \tau_r}{\partial t} &= (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + \lambda \Big(\frac{\partial u_z}{\partial z} + \frac{u_r}{r} \Big), \\
\frac{\partial \tau_z}{\partial t} &= (\lambda + 2\mu) \frac{\partial u_z}{\partial z} + \lambda \Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} \Big), \\
\frac{\partial \tau_\theta}{\partial t} &= (\lambda + 2\mu) \frac{u_r}{r} + \lambda \Big(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} \Big), \\
\frac{\partial \tau_{rz}}{\partial t} &= \mu \Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \Big)
\end{aligned}
\tag{39}
$$

with zero initial data

$$
u_r|_{t=0} = u_z|_{t=0} = \tau_r|_{t=0} = \tau_z|_{t=0} = \tau_\theta|_{t=0} = \tau_{rz}|_{t=0} = 0 \tag{40}
$$

and the boundary conditions

$$
\tau_{rz}|_{z=0} = 0, \qquad \tau_z|_{z=0} = F(r) f(t). \tag{41}
$$

In equations (39)–(41), the elastic constants $\lambda(z)$, $\mu(z)$ and the medium density $\rho(z)$ are arbitrary piecewise-continuous functions of the variable z; σ_{ij} denotes a component of the symmetric stress tensor, u_i denotes a velocity component, $f(t)$ represents the time variation of the source, $F(r)$ is a function of distribution of the source on the plane $z = 0$. We can choose $F(r)$ in the form

$$
F(r) = \frac{n_0^2}{2\pi(1 + n_0^2 r^2)^{3/2}}
$$

to be suitable to simulate a point source when $n_0 \to \infty$.

Let us make use of the representation of the solution to (39)–(41) as a combination of the Fourier–Bessel series [8]:

$$\begin{Bmatrix} u_r \\ \sigma_{rz} \end{Bmatrix} = \frac{2}{a^2} \sum_{n=1}^{\infty} \begin{Bmatrix} W_5(k_n, z, t) \\ W_2(k_n, z, t) \end{Bmatrix} \frac{J_1(k_n r)}{[J_0(k_n a)]^2}, \tag{42}$$

$$\begin{Bmatrix} u_z \\ \sigma_z \end{Bmatrix} = \frac{2}{a^2} \sum_{n=1}^{\infty} \begin{Bmatrix} W_6(k_n, z, t) \\ W_1(k_n, z, t) \end{Bmatrix} \frac{J_0(k_n r)}{[J_0(k_n a)]^2}, \tag{43}$$

$$\begin{aligned} \sigma_r = {} & \frac{2}{a^2} \sum_{n=1}^{\infty} [k_n W_3(k_n, z, t) + W_4(k_n, z, t)] \frac{J_0(k_n r)}{[J_0(k_n a)]^2} - \\ & \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{1}{r} W_3(k_n, z, t) \frac{J_1(k_n r)}{[J_0(k_n a)]^2}, \end{aligned} \tag{44}$$

$$\sigma_\theta = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{1}{r} W_3(k_n, z, t) \frac{J_1(k_n r)}{[J_0(k_n a)]^2} + \frac{2}{a^2} \sum_{n=1}^{\infty} W_4(k_n, z, t) \frac{J_0(k_n r)}{[J_0(k_n a)]^2}, \tag{45}$$

where J_0 and J_1 are the Bessel functions of the first kind and k_n are the roots of the transcendental equation $J_1(k_n a) = 0$. We choose a parameter a to be sufficiently large to consider the wave field up to the time $t < T$, where T is the minimal propagation time of the leading wave from the reflecting surface $r = a$.

After applying (42)–(45) to problem (39)–(41), we arrive at the following system of equations:

$$\begin{aligned} \rho \frac{\partial W_5}{\partial t} &= \frac{\partial W_2}{\partial z} - k_n^2 W_3 - k_n W_4, \\ \rho \frac{\partial W_6}{\partial t} &= \frac{\partial W_1}{\partial z} + k_n W_2, \\ \frac{\partial W_1}{\partial t} &= (\lambda + 2\mu) \frac{\partial W_6}{\partial z} + \lambda k_n W_5, \\ \frac{\partial W_2}{\partial t} &= \mu \frac{\partial W_5}{\partial z} - \mu k_n W_6, \\ \frac{\partial W_3}{\partial t} &= 2\mu W_5, \\ \frac{\partial W_4}{\partial t} &= \lambda \frac{\partial W_6}{\partial z} + \lambda k_n W_5, \end{aligned} \tag{46}$$

with the initial data

$$W_s|_{t=0} = 0, \quad s = 1, \dots, 6, \tag{47}$$

and the boundary conditions

$$W_2|_{z=0} = 0, \quad W_1|_{z=0} = \frac{1}{2\pi} \exp\left(\frac{k_n}{n_0}\right) f(t). \tag{48}$$

Now we apply to problem (46)–(48) the integral Laguerre transform with respect to the time coordinate [4]:

$$\bar{W}_s^m(k_n, z) = \int_0^\infty W_s(k_n, z, t)(ht)^{-\alpha/2} l_m^\alpha(ht) d(ht), \quad s = 1, \dots, 6, \tag{49}$$

with the inverse formulas

$$W_s(k_n, z, t) = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} \bar{W}_s^m(k_n, z) l_m^\alpha(ht), \quad s = 1, \dots, 6. \tag{50}$$

We arrive at the following system of equations:

$$\begin{aligned}
\frac{h}{2}\bar{W}_5^m + h\sum_{j=0}^{m-1}\bar{W}_5^j &= \frac{1}{\rho}\frac{\partial \bar{W}_2^m}{\partial z} - \frac{k_n^2}{\rho}\bar{W}_3^m - \frac{k_n}{\rho}\bar{W}_4^m, \\
\frac{h}{2}\bar{W}_6^m + h\sum_{j=0}^{m-1}\bar{W}_6^j &= \frac{1}{\rho}\frac{\partial \bar{W}_1^m}{\partial z} + \frac{k_n}{\rho}\bar{W}_2^m, \\
\frac{h}{2}\bar{W}_1^m + h\sum_{j=0}^{m-1}\bar{W}_1^j &= (\lambda + 2\mu)\frac{\partial \bar{W}_6^m}{\partial z} + \lambda k_n \bar{W}_5^m, \\
\frac{h}{2}\bar{W}_2^m + h\sum_{j=0}^{m-1}\bar{W}_2^j &= \mu\frac{\partial \bar{W}_5^m}{\partial z} - \mu k_n \bar{W}_6^m, \\
\frac{h}{2}\bar{W}_3^m + h\sum_{j=0}^{m-1}\bar{W}_3^j &= 2\mu \bar{W}_5^m, \\
\frac{h}{2}\bar{W}_4^m + h\sum_{j=0}^{m-1}\bar{W}_4^j &= \lambda\frac{\partial \bar{W}_6^m}{\partial z} + \lambda k_n \bar{W}_5^m,
\end{aligned} \tag{51}$$

with the boundary conditions

$$\bar{W}_2^m|_{z=0} = 0, \quad \bar{W}_1^m|_{z=0} = \frac{1}{2\pi}\exp\left(\frac{k_n}{n_0}\right) f_m, \tag{52}$$

where

$$f_m = \int_0^\infty f(t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht).$$

It should be noted that we must select an integer parameter $\alpha \geq 1$ to satisfy the initial data (47).

Problem (51), (52) is reduced to a system of the linear algebraic equations with the help of the finite difference approximation with respect to the coordinate z by analogy with Section 2 and Section 3. As a result, after finding $\bar{W}_s^m(k_n, z)$, $s = 1, \dots, 6$, it is sufficient to substitute them in the inverse formulas (42)–(45) and (50) for obtaining the solution of the original problem (39)–(41).

5. Some aspects of convergence of the method. Examples of calculation of seismic fields

For the solution of systems (21), (38), and (51) we make use of fast methods of solution of a linear algebraic system of equations with a great number of the right-hand sides, for example, the Cholesky method. In this case, the matrix of the system is only once transformed for all the right-hand sides, due to its independence of number m. The right-hand side of the system has a recurrent dependence on the parameter m. Note, that if we take advantage of the Fourier transform along the temporal coordinate, we will obtain a matrix, dependent on the temporal frequency. It would considerably increase our computer costs. Thus, we can consider the spectral Laguerre method for approximation of temporal derivatives as an alternative to the Fourier method.

The inversion formulas of the analytical finite integral cosine-sine Fourier and Laguerre transforms contain the sums with an infinite number of terms. Therefore, when carrying out numerical calculations one should know the sufficient number of terms in a series to be summed up in order to obtain a solution with the desired accuracy. Thus, the number of terms in a series, necessary for performing the finite integral cosine-sine Fourier transform depends on the spatial wavelength in a medium, which in turn depends on the spectrum of a temporal signal $f(t)$ in the source. According to the Laguerre transform the number of harmonics needed for the waveform reconstruction of the solution, depends both on $f(t)$ and the last moment of time before which the solution can be reconstructed. In the latter case, it is possible to define the number of harmonics in the following manner. At the first step, for the function $f(t)$ we find the expansion coefficients

$$f_m = \int_0^\infty f(t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht), \tag{53}$$

where l_m^α are the Laguerre functions defined in Section 1. Using the analytical formula of solution of the 1D acoustic problem [4], it appears possible to find the expansion coefficients (according to Laguerre) of the function $f(t+T_0)$ in the following form:

$$\tilde{f}_n = \sum_{m=0}^{n} f_m L_{n-m}^0(hT_0) - \sum_{m=0}^{n-1} f_m L_{n-m-1}^0(hT_0). \tag{54}$$

Here L_n^0 are the orthonormal Laguerre polynomials of zero order. By analyzing the convergence of these coefficients one can define the necessary number of harmonics for the waveform reconstruction at the fixed time T_0. The function diagram of the decomposition coefficients $\tilde{f}_n = \tilde{f}(n)$ for $f(t)$ of the form (56) for various times T_0 is shown in Figure 1 ($h = 20$, $\alpha = 2$).

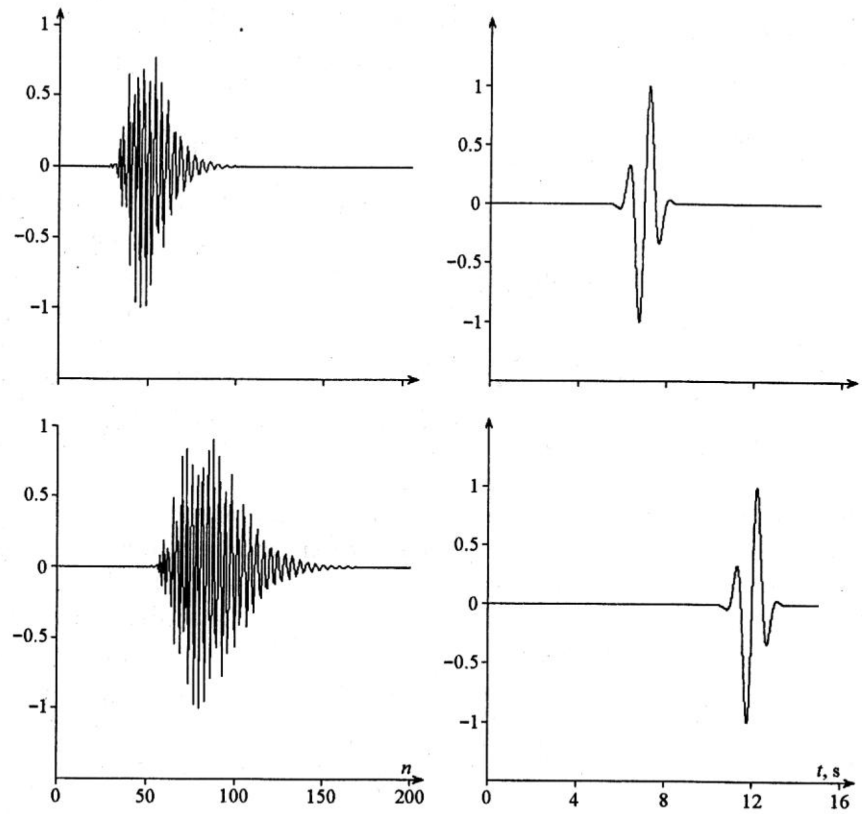

Figure 1. The function diagram of the decomposition coefficients $f_n = f(n)$ for the function $f(t + T_0)$ for $T_0 = 5$ s (the upper) and 10 s (the lower)

As is seen from the figure, the higher is the arrival time of a signal, the more shifted to the right are the coefficients f_n. Hence, if are sufficient number of harmonics for a certain T_0 have been found, the number of harmonics for all the lesser moments of the time T_0 will be automatically sufficient.

Let us analyze the effect of the Laguerre parameters α and h on the accuracy of calculation when solving our problems. As is evident from Sections 2–4, the solutions of the original problems reduce to systems of the linear algebraic equations. It is a well-known fact that the accuracy of numerical algorithms for solution of a system of the linear algebraic equations is essentially dependent on conditioning of a matrix of the system. One should keep in mind that the decrease of conditioning considerably improves the solution accuracy. The experience gained in solving seismic problems indicates to the fact that the conditioning of the resulted system is strongly affected by the physical characteristics of the model medium. The conditioning becomes worse, especially, in the case of the drastically contrast media. However, in the proposed algorithm, the conditioning of the system of equations (21), (38), and (51) after approximation of the spatial derivatives with respect to z is to a greater extent dependent on the Laguerre parameter h. This effect is attained because of the location of the parameter $h/2$ on the diagonal of the matrix A_n. The numerical results obtained show that even for models of the drastically contrast media, the conditioning of the matrix of our algorithm does not exceed one order and tends to unity with increasing h.

Now let us dwell on the role of the parameter α in the numerical implementation of the proposed method. Based on the following asymptotic expression for the Laguerre function

$$l_n^\alpha(ht) = \sqrt{\frac{1}{\pi}}(ht)^{-1/4}n^{\alpha/2-1/4}\left[\cos\left(2\sqrt{nht} - \frac{(2\alpha+1)\pi}{4}\right) + O\left(\frac{1}{\sqrt{n}}\right)\right], \quad (55)$$

$ht > 0$, one can see that the expansion components in the inversion formulas (15), (35) and (50) have the asymptotics $(ht)^{\alpha/2-1/4}$ in the variable t at $n \to \infty$. As values of the components of the field are limited for any moment of time, one can arrive at the conclusion that for all $\alpha > 1/2$, the greater the value of α, the greater the damping.

Having fixed the parameter h, we can increase the parameter α. With increasing this parameter up to a certain value, the accuracy of computation increases as well. However, in this case some problems connected with a limited machine accuracy may arise. This difficulty can be overcome by using the computation with double precision and, or some other computational technique. For example, when computing an exponent with a large index, it seems reasonable to present it as a number of exponents with smaller indices, and so on. Thus, depending on the selected wavelet $f(t)$ and the duration of the time interval on which we want to calculate a synthetic seismogram, it is possible to determine the optimal parameters h and α.

The most serious error of the calculations is connected with approximation of derivatives with respect to the variable z. In order to diminish such an error, it seems reasonable to use a higher order approximation when solving the elastic problems discussed above. In addition, it would be appropriate to introduce non-regular difference grids with allowance for specific features of the drastically contrast layers of a medium. Thus, for each concrete model, the optimal discretization step depends both on smoothness of solution of the field under reconstruction and on the physical parameters of the proposed model.

The results of numerical calculations of problem (7)–(9) for the model of the medium with drastically contrast interfaces are presented in Figure 2. This model contains three homogeneous isotropic layers: *air–water–solid.* The figures represent snapshots of U_z vector component of the wave field for the times $T = 3$, 6, and 9 s (see Figure 2, the upper) and $T = 20$ s

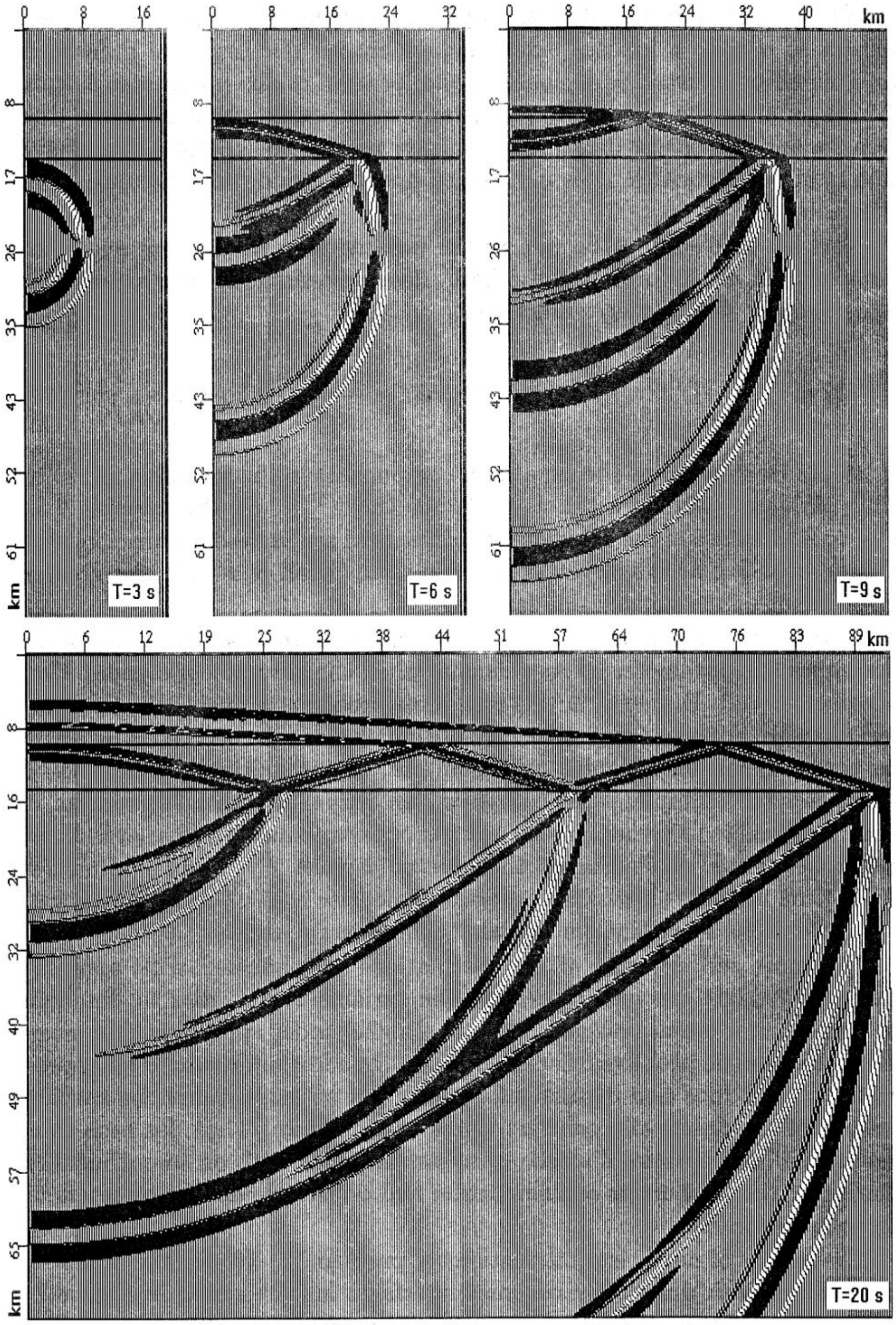

Figure 2. Snapshots of the wave field vector component U_z: the upper – for $T = 3$, 6 and 9 s; the lower – for $T = 20$ s

(see Figure 2, the lower). Physical parameters of this medium model are the following:

1) the upper layer (air) – $V_p = 0.336$ km/s, $V_s = 0$ km/s, $\rho = 0.001$ g/cm^3;
2) the middle layer (water) – $V_p = 1.5$ km/s, $V_s = 0$ km/s, $\rho = 1$ g/cm^3;
3) the lower layer (solid) – $V_p = 5$ km/s, $V_s = 3$ km/s, $\rho = 2.5$ g/cm^3.

The layers interfaces are shown by the solid horizontal line. Depth z of the upper and the lower boundaries is 10 and 15 km, respectively. Location of a source of the explosive type is defined by the coordinate points $x_0 = 0$ km, $z_0 = 25$ km. We define the time variation of the source $f(t)$ as function of the form:

$$f(t) = \exp\left[-\frac{(2\pi f_0(t-t_0))^2}{\gamma^2}\right]\sin(2\pi f_0(t-t_0)), \tag{56}$$

where $\gamma = 4$, $f_0 = 1$, $t_0 = 1.5$ s.

Figures 3 present the results of calculation of the wave field for problem (39)–(41). This model contains two homogeneous isotropic layers. Physical parameters of this medium model are the following:

1) the upper layer – $V_p = 1.5$ km/s, $V_s = 1$ km/s, $\rho = 1$ g/cm^3;
2) the lower layer – $V_p = 3$ km/s, $V_s = 2$ km/s, $\rho = 2$ g/cm^3.

The upper field represents a snapshot of the cross-section of the wave field in the plane (r, z) at $\varphi = \text{const}$. The distribution of values of the component u_z for $T = 10$ s is shown. The lower graph represents seismotraces for the displacement vector component u_z on the free surface for the given medium model. Location of a source of the vertical force type is defined by the coordinate points $x_0 = 0$ km, $z_0 = 0$ km.

Analysis of the results obtained shows a good stability of the proposed algorithms even for the drastically contrast models of media. In this case, an error of the numerical calculation of the wave field does not exceed 1% at distances of 100 wavelengths.

Conclusion

We have presented the new approach based on the integral Laguerre transforms for approximations of temporal derivatives for the time dependent problems.

The technique can be applied to a number of methods including the finite difference method, the finite element method or the spectral method. The approach presents an alternative to the frequency domain modeling based on the integral Fourier transforms.

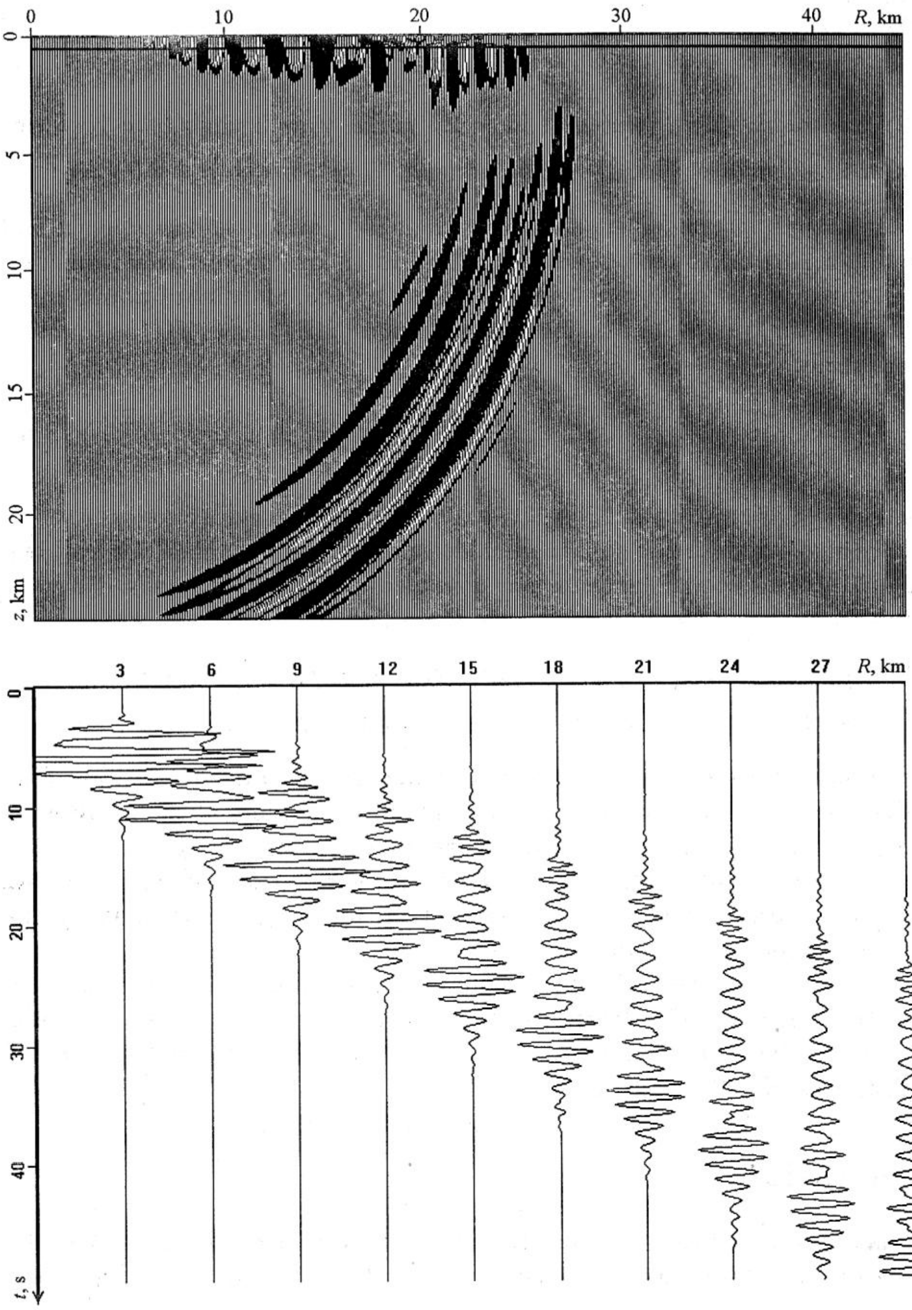

Figure 3. The distribution of values of the component u_z for $T = 10$ s (the upper) and the seismotraces for the displacement vector component u_z on the free surface (the lower)

The method discussed here can be applied for the calculation of spatial derivatives in the first order equations. Its application would make it possible to automatically satisfy the far-field conditions. In this case, there is no need to introduce the absorbing boundary conditions.

At present, this approach, based on the integral Laguerre transforms, has been developed for the modeling of the attenuation effects in the elastic medium. In this case, the difficulties connected with representation of integral terms in the time-dependent elastic wave equation are easily overcome.

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