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Weak convergence of randomized spectral models of Gaussian random vector fields

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Convergence of randomized spectral models of homogeneous vector fields is studied in the sense of convergence in distribution in a uniform metric of the Banach space of continuous functions. Under quite moderate restrictions on the parameters of the spectral model, weak convergence to a Gaussian field is shown if the spectral density $p(\lambda)$ of this field satisfies the condition $\int[\ln(1+|\lambda|)]^{1+a}p(\lambda)d\lambda < \infty$ at some $a > 0$.

Introduction

It is well-known (see, e.g., [1, 2]) that modeling of the homogeneous Gaussian fields with a continuous spectrum can be performed only approximately. As for the computational cost, the most economical models are the randomized spectral ones. Such simplest model was first considered in [3] for approximating the three-dimensional isotropic non-compressible field. An algorithm for modeling homogeneous scalar fields with a given spectrum on the basis of the stratified randomization of the spectral representation was introduced in [1]. Modeling of homogeneous, in the wide sense, vector fields with a given spectral tenzor was considered in [4–6]. The randomized spectral models of the Gaussian fields are used for solving a wide range of problems by the method of statistical modeling (see, e.g., [3, 5, 7–10]).

This study concerns the problem of convergence of randomized spectral apprroximations of the homogeneous Gaussian vector fields. Note that this problem has been studied only for the case of scalar fields (see, e.g., [2, 11–14]).

In the present study we investigate the conditions of weak convergence in $C(D)$ $(D = [0,1]^l)$ of the sequence of the approximations u_n, $n = 1, 2, \ldots$, of a Gaussian field u. These approximations are constructed on the basis of the randomized spectral models. The main result is as follows: under quite weak restrictions on the spectrum (condition (3) which is close to the necessary one for continuity with probablity one of the realizations of the Gaussian fields) and general enough conditions on the other parameters of the spectral model, the weak convergence $u_n \xrightarrow{D} u$, $n \to \infty$ is proved.

1. Convergence of randomized spectral models

Let $u(x)$, $x \in R^l$, be an m-dimensional homogeneous Gaussian random field with zero mean and spectral tenzor $F(\lambda) = \|F_{js}(\lambda)\|_{j,s=1}^m$. Hereafter we assume that $F(\lambda)$ is continuous with respect to $\lambda \in R^l$. Let $p(\lambda) = \sum_{j=1}^m F_{jj}(\lambda)$ be the trace of tenzor $F(\lambda)$, $p_1 = \int\limits_{R^l} p(\lambda)d\lambda$.

The sequence of random fields

$$\begin{aligned} u_n(x) = \frac{\sqrt{p_1}}{\sqrt{n}} \sum_{k=1}^n \Big\{ \big[\cos\langle\lambda_k, x\rangle Q'(\lambda_k) - \sin\langle\lambda_k, x\rangle Q''(\lambda_k)\big]\xi_k \\ + \big[\cos\langle\lambda_k, x\rangle Q''(\lambda_k) + \sin\langle\lambda_k, x\rangle Q'(\lambda_k)\big]\eta_k\Big\}, \\ n = 1, 2, \ldots, \quad x \in R^l, \end{aligned} \tag{1}$$

will be called randomized spectral model (RSM). Here $\{\lambda_k\}_{k=1}^\infty$ is a sequence of independent similarly distributed l-dimensional random variables (r.v.) with the distribution density $f(\lambda) = p(\lambda)/p_1$; $\{\xi_k\}_{k=1}^\infty$, $\{\eta_k\}_{k=1}^\infty$ are two sequences of pairwise independent m-dimensional r.v. with zero mean and unit covariational matrices; $\langle\cdot,\cdot\rangle$ is a scalar product in R^l; $Q'(\lambda)$, $Q''(\lambda)$ are two $m \times m$-matrices continuous with respect to λ, satisfying the following:

$$F(\lambda) = p(\lambda)Q(\lambda)Q^*(\lambda). \tag{2}$$

Here $Q(\lambda) = Q'(\lambda) + iQ''(\lambda)$, $Q^*(\lambda)$ is a matrix complex adjoint to $Q(\lambda)$. It is not difficult to varify that at every $n = 1, 2, \ldots$, the random fields (1) are, in the wide sense, homogeneous and have the same spectral tenzor equal to $F(\lambda)$. We will investigate the convergence of these fields to a Gaussian field $u(x)$ in the Banach space $C(D)$ of continuous m-dimensional vector functions $f : D \to R^m$ with the norm $\| f \|_{C(D)} = \sup\{\| f \|;\ x \in D\}$. Here $D = [0,1]^l$, $\|f\| = \max\{|f_j|,\ j = 1, \ldots, m\}$ is a uniform norm in R^m. The ordinary Euclidean norm generated by the scalar product $\langle\cdot,\cdot\rangle$ will be denoted by $|\cdot|$.

Theorem 1. *Let the spectral density $p(\lambda) = SpF(\lambda)$ satisfy the condition*

$$\int \ln^{1+\varepsilon}(1+|\lambda|)p(\lambda)d\lambda < \infty \tag{3}$$

at some $\varepsilon > 0$. Then with probability one the realizations of the field $u(x)$ are continuous and the fields $u_n(x)$ converge weakly in $C(D)$ to $u(x)$ as $n \to \infty$.

As in the proof of the analogous assertion for the scalar fields [13] (where $p(\lambda)$ is under a more serious restriction: $\int |\lambda|^2 p(\lambda)d\lambda < \infty$) we make use of the Jain-Marcus theorem [15].

Let (S, d) be a metric compact, $C(S)$ be the Banach space of continuous real-valued functions, ξ be a random element (r.e.) in $C(S)$, $E\xi(t) = 0$, $t \in S$, $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent realizations of the r.e. ξ. The central limit theorem (CLT) is said to be true for a r.v. ξ if the sequence

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(t)$$

converges weakly in $C(S)$ to a Gaussian element w.

Theorem 2 (Jain-Marcus, [15]). *For a r.e. ξ the CLT holds if:*

1) $\sup_{t \in S} M\xi^2(t) \le C < \infty$,
2) *there can be found a nonzero random variable $G = G(\omega)$, $MG^2(\omega) < \infty$, and metric ρ on S, continuous with respect to d, such that*

$$|\xi(s,\omega) - \xi(t,\omega)| \le G(\omega)\rho(s,t), \quad s,t \in S$$
$$\int_0^1 H_\rho^{1/2}(S;\varepsilon)d\varepsilon < \infty, \tag{4}$$

where $H_\rho(S;\varepsilon) = \ln N_\rho(S;\varepsilon)$, $N_\rho(S;\varepsilon)$ is a minimal number of ρ-balls, with the radius less or equal to ε, that cover the set S.

We start our proof of Theorem 1 by noting that the Jain-Marcus theorem can be generalized directly for the Banach space $C_m(S)$ of continuous vector functions $f : S \to R^m$. Indeed, let $S_1, \ldots, S_m$ be "m-copies" of the set S, $S_i = S \times \{i\}$, $i = 1, \ldots, m$; $\tilde{S} = \cup_{i=1}^m S_i$. Let us define the metric $\tilde{d}(\tilde{t}, \tilde{s})$ on $\tilde{S}$ by setting

$$\tilde{d}(\tilde{t},\tilde{s}) = \begin{cases} d(t,s), & \text{if } \tilde{t}=(t,i),\ \tilde{s}=(s,i),\ i=1,2,\ldots,m, \\ 1, & \text{otherwise.} \end{cases}$$

Since (S,d) is a compact, $(\tilde{S},\ \tilde{d})$ is also compact. Let us establish the one-to-one correspondence

$$f \in C_m(S) \longleftrightarrow \tilde{f} \in C(\tilde{S}),$$

between $C_m(S)$ and $C(\tilde{S})$, assuming $\tilde{f}(\tilde{t}) = f_i(t)$, if $\tilde{t}=(t,i), i=1,2,\ldots,m$. Apparently, the above correspondence is the isometric isomorphism of the Banach spaces of m-dimensional continuous vector functions $C_m(S)$ and continuous scalar functions $C(\tilde{S})$. Hence, if ξ is a r.e. from $C_m(S)$, then its image $\tilde{\xi}$ is a r.e. from $C(\tilde{S})$. The fulfillment of the CLT for one of these r.e.s implies its fulfillment for the other one.

Thus, the Jain-Marcus theorem remains true for the Banach space $C_m(S)$ if in conditions 1)–2) instead of $|\xi(t)|$ we consider the norm $\| \xi(t) \| = \max_{j=1,\ldots,m} |\xi_j(t)|$.

Proof of Theorem 1. The connection between the correlational and spectral tenzors (see, e.g., [16]) implies

$$E\big((u(x+r)-u(x))(u(x+r)-u(x))^*\big) = \int_{R^l} (e^{i\langle\lambda,x+r\rangle} - e^{i\langle\lambda,x\rangle}),$$

$$(e^{-i\langle\lambda,x+r\rangle} - e^{-i\langle\lambda,x\rangle})F(\lambda)d\lambda = 2\int_{R^l}(1-\cos\langle\lambda,r\rangle)F(\lambda)d\lambda.$$

By applying the trace operator Sp to the latter inequality we obtain

$$\begin{aligned} \sum_{j=1}^{m} E(u_j(x+r)-u_j(x))^2 &= 2\int_{R^l}(1-\cos\langle\lambda,r\rangle)p(\lambda)d\lambda \\ &= 4\int_{R^l}\sin^2\Big(\frac{\langle\lambda,r\rangle}{2}\Big)p(\lambda)d\lambda. \end{aligned} \tag{5}$$

It is known [17] that

$$|\sin\langle\lambda,r\rangle| \le |\lambda|^\beta |r|^\beta \frac{\ln^\theta(e+|\lambda|)}{|\ln|r||^\theta},$$

at $|r|<1$, for all $\beta\in[0,1]$, $\theta\in[0,1]$. Setting $\beta=0$, $\theta=\frac{1+\varepsilon}{2}$ $(0<\varepsilon<1)$, we obtain

$$|\sin\langle\lambda, r\rangle|^2 \le \frac{\ln^{1+\varepsilon}(e+|\lambda|)}{|\ln|r||^{1+\varepsilon}} \le \frac{[1+\ln(1+\frac{|\lambda|}{e})]^{1+\varepsilon}}{\ln^{1+\varepsilon}|r|^{-1}} \le 2\frac{[1+\ln^{1+\varepsilon}(1+|\lambda|)]}{\ln^{1+\varepsilon}|r|^{-1}}. \tag{6}$$

From (5)–(6) we obtain the estimate

$$\sum_{j=1}^{m} E(u_j(x+r)-u_j(x))^2 \le \frac{C_\varepsilon}{\ln^{1+\varepsilon}|r|^{-1}}, \tag{7}$$

where

$$C_\varepsilon = 8\int[1+\ln^{1+\varepsilon}(1+|\lambda|)]p(\lambda)d\lambda < \infty.$$

From this estimate and the sufficient condition of Dudley for the continuity of the Gaussian random fields [18] it follows that the realizations of the field $u(x)$ are also continuous.

Let us prove the CLT. Denote

$$\xi(x) = [\cos\langle\lambda, x\rangle Q'(\lambda) - \sin\langle\lambda, x\rangle Q''(\lambda)]\zeta + [\cos\langle\lambda, x\rangle Q''(\lambda) + \sin\langle\lambda, x\rangle Q'(\lambda)]\eta,$$

where λ is a random vector with the distribution density $p(\lambda)/p_k$, ζ, η are two independent of each other (and of λ) random m-dimensional vectors with zero mean and unit covariational matrix. It can easily be seen that

$$|\xi(x+r)-\xi(x)| \le |\cos\langle\lambda, x+r\rangle - \cos\langle\lambda, x\rangle|\cdot|Q'(\lambda)\zeta + Q''(\lambda)\eta| + |\sin\langle\lambda, x+r\rangle - \sin\langle\lambda, x\rangle|\cdot|-Q''(\lambda)\zeta + Q'(\lambda)\eta|. \tag{8}$$

From (2) it follows that

$$Sp\{Q'(\lambda)(Q'(\lambda))^T + Q''(\lambda)(Q''(\lambda))^T\} = 1, \tag{9}$$

and thus

$$|Q'(\lambda)\zeta| \le |\zeta|, \quad |Q''(\lambda)\eta| \le |\eta|, \quad |Q''(\lambda)\zeta| \le |\zeta|, \quad |Q'(\lambda)\eta| \le |\eta|.$$

Whence by (8) we have

$$|\xi(x+r)-\xi(x)| \le \big(|\cos\langle\lambda, x+r\rangle - \cos\langle\lambda, x\rangle| + |\sin\langle\lambda, x+r\rangle - \sin\langle\lambda, x\rangle|\big)\big(|\zeta|+|\eta|\big). \tag{10}$$

By making use of the obvious equalities

$$|\sin\langle\lambda, x+r\rangle - \sin\langle\lambda, x\rangle| \le 2\Big|\sin\frac{\langle\lambda, r\rangle}{2}\Big|,$$
$$|\cos\langle\lambda, x+r\rangle - \cos\langle\lambda, x\rangle| \le 2\Big|\sin\frac{\langle\lambda, r\rangle}{2}\Big|,$$

and taking into account (6) from (10) we obtain

$$|\xi(x+r) - \xi(x)| \le 4\sqrt{2}\frac{[1+\ln^{1+\varepsilon}(1+|\lambda|)]^{1/2}}{\ln^{\frac{1+\varepsilon}{2}}|r|^{-1}}(|\zeta|+|\eta|).$$

Then setting

$$G = 4\sqrt{2}(|\zeta|+|\eta|)\left[1+\ln^{1+\varepsilon}(1+|\lambda|)\right]^{1/2},$$
$$\rho(x,y) = \frac{1}{|\ln|x-y||^{\frac{1+\varepsilon}{2}}},$$

and using the Jain-Marcus theorem we have the statement of Theorem 1. □

Remark. The above theorem immediately implies the similar convergence assertion for more general models. Let, for instance,

$$F(\lambda) = \sum_{i=1}^{N} F^{(i)}(\lambda), \tag{11}$$

where $F^{(i)}(\lambda)$, $i = 1, 2, \ldots, N$ are the spectral tenzors (i.e., non-negative matrices). For the multiindex $\overline{n} = (n_1, \ldots, n_N)$ consider the fields

$$u_{\overline{n}}(x) = \sum_{i=1}^{N} u_{n_i}(x), \tag{12}$$

where every field $u_{n_i}(x)$ is constructed independently, similarly to fields (1) at $F(\lambda) = F^{(i)}(\lambda)$. The proved theorem implies that $u_{n_i} \xrightarrow{D} u^{(i)}$, $n_i \to \infty$, $i = 1, 2, \ldots, N$. Here $u^{(i)}$ is a Gaussian element in $C(D)$. Since the fields $u_{n_i}(x)$, $i = 1, \ldots, N$, are independent, at $\min_i n_i \to \infty$ the fields $u_{\overline{n}}(x)$ weakly converge in $C(D)$ to the Gaussian field $u(x) = \sum_{i=1}^{N} u^{(i)}(x)$. Let $\Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_N = R^l$ be a partition of the Euclidean space R^l into pairwise non-overlapping domains $\Lambda_1, \ldots, \Lambda_N$. Then setting

$$F^{(i)}(\lambda) = \begin{cases} F(\lambda), & \lambda \in \Lambda_i, \\ 0, & \lambda \overline{\in} \Lambda_i, \end{cases}$$

we obtain (11). The field (12) constructed with the above scheme is called stratified randomized spectral model with the fixed strata $\Lambda_1, \ldots, \Lambda_N$.

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