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On uniform convergence of Scharfetter's scheme

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The transfer of charge carries in a semiconductor device for stationary conditions is described by elliptic differential equations with oscilating coefficients. The uniform convergence of one-dimensional Scharfetter's scheme on the whole interval is shown in this paper.

Key words. semiconductor devices, Scharfetter's scheme, uniform convergence.

1. Introduction

One-dimensional equation of the type

$$\frac{d}{dx}\exp(-\varphi(x))\frac{d}{dx}\exp(\varphi(x))p(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

$p(0) = p^0$, $p(1) = p^1$, f, $\varphi \in C^2([0,1])$ is considered in this paper. It describes the transfer of charge carries (holes, for example) in a semiconductor device for stationary conditions. This equation is a part of the fundamental system of equations describing processes in semiconductor devices [1,2]:

$$\begin{cases} \lambda^2 \dfrac{d^2\varphi}{dx^2} = n - p - N(x,\lambda), \\ J_n = -n\cdot\dfrac{d\varphi}{dx} + \dfrac{dn}{dx}, \quad J_p = -p\cdot\dfrac{d\varphi}{dx} - \dfrac{dp}{dx}, \\ \dfrac{dJ_n}{dx} = \beta_n R(n,p), \qquad \dfrac{dJ_p}{dx} = -\beta_p R(n,p), \\ R(n,p) = \dfrac{n\cdot p - 1}{\tau_n(p+p_1) + \tau_p(n+n_1)}, \end{cases} \tag{2}$$

$$0 \le x \le 1,$$
$$\varphi(0) = \varphi^0, \quad \varphi(1) = \varphi^1, \quad n(0) = n(1) = n^1,$$
$$p(0) = p^0, \quad p(1) = p^1, \quad N(x,\lambda) = N(x)\cdot\lambda^{-1}.$$

Here we make use of the following notation:

$$\lambda = \frac{1}{N_0}, \quad N_0 = \max_{0 \le x \le 1} N(x),$$

l, β_n, β_p, τ_n, τ_p, n_1, p_1 – positive constants; φ – electrostatical potential; p, n – densities of hole and electron concentrations; J_p, J_n – densities of hole and electron flows respectively; $N(x)$ – density of admixture concentration in a semiconductor having large gradients in some inner layer with the center at the point $x_0 \in (0,1)$ and width proportional to the small parameter λ of system (2). Following [1], we assume that the function $N(x)$ outside the inner layer possesses a limited derivative of necessary order, but in the inner layer it satisfies the inequality

$$N(\tilde{x}) \le c_1 \exp\Big(- c_2 \Big|\frac{\tilde{x}}{\lambda}\Big|\Big), \quad \tilde{x} = x - x_0, \quad |\tilde{x}| \le c_0 \lambda,$$

where c_0, c_1, c_2 are positive constants. Let c, c_0, d_i, c_i, k_i, μ, $\tilde{\mu}$, α_i, γ_i, k_i, $i \ge 0$ be the mean different positive constants, independent of λ. Then it is stated that for the potential the following inequalities are executed:

$$\begin{gathered}
\Big|\frac{d^i\varphi(x)}{dx^i}\Big| \le c, \quad x \in \Omega_1, \quad \Omega_1 = \Big\{\Big[0, x_0 - \frac{\varepsilon}{2}\Big) \bigcup \Big(x_0 + \frac{\varepsilon}{2}, 1\Big]\Big\}; \\
\Big|\frac{d^i\varphi(\tilde{x})}{dx^i}\Big| \le D_1 \lambda^{-i} \exp\Big(- D_2 \Big|\frac{\tilde{x}}{\lambda}\Big|\Big), \quad \tilde{x} = x - x_0, \quad \tilde{x} \le \varepsilon, \\
i \le 2,
\end{gathered} \tag{3}$$

where ε is the width of the inner layer for the potential $\varphi(x)$, proportional to the quantity $\lambda|\ln\lambda|$ (later on we take $\varepsilon = 2 \cdot \lambda|\ln\lambda|$).

To get a finite difference approximation for equation (1) we use the integral identity according to G.I. Marchuk:

$$\begin{aligned}
(Lp)_i \equiv \frac{1}{\hbar_i}\Big[\Big(\int_{x_i}^{x_{i+1}} \exp(\varphi(x))dx\Big)^{-1} \\
\times\Big(p_{i+1} \cdot \exp\varphi_{i+1} - p_i \cdot \exp\varphi_i + \int_{x_i}^{x_{i+1}} \exp(\varphi(\xi)) \int_{\xi}^{x_{i+\frac{1}{2}}} f(t)dtd\xi\Big) \\
- \Big(\int_{x_{i-1}}^{x_i} \exp(\varphi(x))dx\Big)^{-1}
\end{aligned} \tag{4}$$

$$\times\Big(p_i \exp\varphi_i - p_{i-1}\exp\varphi_{i-1} - \int\limits_{x_{i-1}}^{x_i} \exp(\varphi(\xi)) \int\limits_{x_{i-\frac{1}{2}}}^{\xi} f(t)dtd\xi\Big)\Big]$$

$$= \frac{1}{\hbar_i} \int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(t)dt, \quad i = \overline{1, N-1},$$

where $\{x_i\}$, $i = \overline{0, N}$, is a differential net, $x_0 = 0$, $x_N = 1$,

$$h_i = x_{i+1} - x_i, \quad x_{i\pm\frac{1}{2}} = \frac{(x_i + x_{i\pm1})}{2}, \quad \hbar_i = \frac{(h_{i-1} + h_i)}{2},$$
$$\hbar_0 = \min_{i=\overline{0,N}}\{h_i\}, \quad h = \max_{i=\overline{0,N}}\{h_i\},$$

and the uniform norm

$$\|g\|_\infty = \max_{i=\overline{0,N}}\{|g_i|\}$$

are introduced in the space of net functions

$$g = \{g_i\}_{i=\overline{0,N}}, \quad g_i = g(x_i))_{i=\overline{0,N}}.$$

Applying to double integrals in numerator quadrature formulas of the rectangular and calculating integrals in the denominator assuming piece-wise-linearity of the function $\varphi(x)$, we get finite ifference Scharfetter's scheme [3]:

$$(L_h p)_i \equiv \frac{1}{\hbar_i}\Big[\frac{p_{i+1}\exp(\varphi_{i+1}) - p_i\exp(\varphi_i)}{\exp(\varphi_{i+1}) - \exp(\varphi_i)} \cdot \frac{\varphi_{i+1} - \varphi_i}{h_i}$$
$$- \frac{p_i\exp(\varphi_i) - p_{i-1}\exp(\varphi_{i-1})}{\exp(\varphi_i) - \exp(\varphi_{i-1})} \cdot \frac{\varphi_i - \varphi_{i-1}}{h_{i-1}}\Big] = f_i, \qquad (5)$$
$$i = \overline{1, N-1}.$$

Schemes similar to this one, the so-called schemes of exponential fitting, are suggested in literature [4,5] for elliptic and parabolic differential equations at a small parameter with higher derivative.

In our case (1) is the equation with oscillating coefficients $\frac{d\varphi}{dx}$, $\frac{d^2\varphi}{dx^2}$:

$$\frac{d^2p}{dx^2} + \frac{dp}{dx} \cdot \frac{d\varphi}{dx} + p \cdot \frac{d^2\varphi}{dx^2} = f(x),$$

where the function $\varphi(x)$ satisfies conditions (3).

We define the error of a difference solution as vector z with components $z_i = ((p)_i - p_i)_{i=\overline{0,N}}$, where p is the solution of differential equation (1), $(p)_i = p(x_i)_{i=\overline{0,N}}$, and $\{p_i\}_{i=\overline{0,N}}$ is the solution of difference system (5). P.A. Markowich and others have analyzed discrete Scharfetter's scheme for equation (1) in case $f(x) = 0$. It was shown [1] that for the case $h \geq \epsilon$

$$\|z\|_\infty \leq c(h+\epsilon).$$

In this paper uniform convergence of Scharfetter's scheme is shown not only outside the inner layer, but on the whole interval [0,1] for the case of sufficiently smooth right side of equation (1).

2. Construction of exact solution

For simplicity we consider, that there is the only inner layer in the interior of domain [0,1], otherwise we distinquish some domains each possessing the only inner layer. So, the proofs are similar for the final number of inner layers. We consider that

$$\max_{x\in\Omega_1}\left|\frac{d^i\varphi(x)}{dx^i}\right| \leq c_i, \quad \max_{0\leq x\leq 1}\left|\frac{d^i f(x)}{dx^i}\right| \leq k_i, \quad i \leq 2,$$
$$\tilde{c_0} = \min_{0\leq x\leq 1}|\varphi(x)|.$$

Let the introduced net be quasi-uniform, i.e., $\frac{h}{\hbar_0} \leq \mu$ for $h \to 0$. One-dimensional differential equation (1) has the following solution:

$$p(x) = p^0\exp\big(-\varphi(x)+\varphi^0\big) + \exp\left(-\varphi(x)\right)\int_0^x \exp\left(\varphi(t)\right)J(t)dt,$$
$$J(t) = \int_0^t f(\xi)d\xi + J_0, \tag{6}$$
$$J_0 = \frac{\exp(\varphi^1)p^1 - \exp(\varphi^0)p^0 - \int_0^1\exp(\varphi(x))\int_0^x f(t)dtdx}{\int_0^1\exp(\varphi(x))dx}.$$

In particular, the solution of homogeneous equation (1) appears as follows:

$$p(x) = p^0\exp(-\varphi(x)+\varphi^0) + J_0\exp{-\varphi(x)}\int_0^x\exp(\varphi(t))dt,$$
$$J_0 = \frac{\exp(\varphi)p - \exp(\varphi^0)p^0}{\int_0^1\exp(\varphi(x))dx}.$$

Algebraic system of three-point equation (5) for $f(x) = 0$ has the exact solution of the form:

$$p_i = \exp(-\varphi_i + \varphi^0)p^0 + \exp(-\varphi_i)J_0^p \sum_{j=0}^{i-1} \frac{\exp\varphi_{j+1} - \exp\varphi_j}{\varphi_{j+1} - \varphi j} \cdot h_j,$$

$$J_0^p = \left(\exp\varphi^1 p^1 - \exp\varphi^0 p^0\right) \Big/ \Big(\sum_{j=0}^{N-1} \frac{\exp\varphi_{j+1} - \exp\varphi_j}{\varphi_{j+1} - \varphi_j} \cdot h_j \Big),$$

$$i = \overline{1, N}.$$

This solution is easy to obtain from equation (5), if we denote a constant flow with $f(x)$ as J_0^p and rewrite the discrete equation as correlation:

$$p_i \exp(\varphi_i) = p^0 \exp(\varphi^0) + J_0^p \sum_{j=0}^{i-1} \frac{\exp(\varphi_{j+1}) - \exp(\varphi_j)}{\varphi_{j+1} - \varphi_j} h_j, \quad i = \overline{1, N}.$$

For $i = N$ from this equality we find value J_0^p. Similarly, we determine the exact solution for $f(x) \neq 0$, replacing equation (5) by the recurrent expression:

$$J_i^p - J_{i-1}^p = \hbar_i f_i,$$

$$J_i^p = \frac{\exp(\varphi_{i+1})p_{i+1}}{\exp(\varphi_{i+1}) - \exp(\varphi_i)} \cdot \frac{\varphi_{i+1} - \varphi_i}{h_i} = \frac{\exp(\varphi_{i+1})p_{i+1}}{I_i},$$

$$I_i = \frac{\exp(\varphi_{i+1}) - \exp(\varphi_i)}{\varphi_{i+1} - \varphi_i} h_i, \quad i = \overline{0, N-1}.$$

Denoting the quantity of flow J_0^* on the left at the point $x_0 = 0$ we obtain

$$\exp(\varphi_i)p_i = \exp(\varphi^0)p^0 + J_0^* \sum_{k=0}^{i-1} I_k + \sum_{k=0}^{i-1} I_k \sum_{j=0}^{k} \hbar_j f_j,$$

$$J_0^* = \frac{\exp(\varphi^1)p^1 - \exp(\varphi^0)p^0 - \sum_{k=0}^{N-1} I_k \sum_{j=0}^{k} \hbar_j f_j}{\sum_{k=0}^{N-1} I_k},$$

$$i = \overline{1, N-1}.$$

In more detail we can write down the solution of difference scheme (5) as follows:

$$p_i = \exp(-\varphi_i + \varphi^0)p^0 + \exp(-\varphi_i)\sum_{k=0}^{i-1} I_k \sum_{j=0}^{k} \hbar_j f_j$$
$$+ \exp(-\varphi_i)\left[\frac{\exp(\varphi^1)p^1 - \exp(\varphi^0)p^0 - \sum_{k=0}^{N-1} I_k \sum_{j=0}^{k} \hbar_j f_j}{\sum_{k=0}^{N-1} I_k} \cdot \sum_{k=0}^{i-1} I_k\right],$$
$$i = \overline{1, N-1}.$$

3. Obtaining uniform evaluation for difference solution error

Using the form of the exact solutions of differential equation (1) and difference system (5) we will show that Sharfetter's scheme is uniformly convergent with an accuracy lower than unity:

$$||z||_\infty \leq ch|\ln h|. \tag{8}$$

Let

$$\triangle = \{\delta_i\}_{i=\overline{0,N-1}}, \quad \delta_i = [x_i, x_{i+1}], \quad I = \{i\}_{i=\overline{0,N-1}}.$$

In future we will need to represent spaces $\triangle$ and I as sums of some subspaces with definite properties

$$I_A = \{i \in I; \delta_i \in \triangle_A\}, \quad \triangle_A = \{\delta_i \subset \Omega_1\},$$
$$I_C = \{i \in I; \delta_i \in \triangle_C\}, \quad \triangle_C = \{\delta_i \subset \Omega_2 = [x_0 - \hbar_0, x_0 + \hbar_0]\},$$
$$I_B = \{i \in I; \delta_i \in \triangle_B\}, \quad \triangle_B = \{\delta_i \subset \Omega_3 = [0,1] \setminus \{\Omega_1 \cup \Omega_2\}\}.$$

It is obvious that $I = I_A + I_C + I_B$, where I_B is empty space in the case $h \geq \epsilon$ (this case was discussed in [1]). Taking into account condition (3), we can write the inequality:

$$\left|\frac{d^k\varphi(x)}{dx^k}\right| \leq c^k, \quad k \leq 2, \quad x \in \delta_i, \quad \delta_i \in \triangle_A;$$
$$\left|\frac{d^k\varphi(x)}{dx^k}\right| \leq D_1\lambda^{-k}\exp\left(-\frac{D_2\hbar_0}{\lambda}\right), \quad k \leq 2, \quad x \in \delta_i, \quad \delta_i \in \triangle_C;$$
$$\sum_{i \in I_C} |\delta_i| \leq \gamma h,$$

where γ equals two or three depending on location of the interior layer center for the point x_0 with respect to nodes of the net. For proving inequality (8), we need some auxiliary statements.

Statement 1. *Let*

$$\Theta_i = \frac{h_i^3}{\lambda^2}\exp\Big(-\frac{D_2}{\hbar_0}\Big), \quad i \in I, \quad \Theta = \max_{i\in I}|\Theta_i|.$$

Then

$$\sum_{i\in I_B}\Theta_i \le \tilde{\mu}h|\ln h|,$$

where $\tilde{\mu}$ is a positive constant.

PROOF. To prove this we use the following consideration. The amount of elements in the set $\triangle_B$, hence and in I_B, does not exceed value $\varepsilon/\hbar_0$, what gives

$$\begin{aligned}\sum_{i\in I_B}\Theta_i &\le \Theta\frac{\lambda|\ln\lambda|}{\hbar_0},\\ \Theta\frac{\lambda|\ln\lambda|}{\hbar_0} &\le \frac{h^3}{\lambda^2}\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big)\cdot\frac{\lambda|\ln\lambda|}{\hbar_0}\\ &\le \mu^3\hbar_0^2\frac{|\ln\lambda|}{\lambda}\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big).\end{aligned}$$

We consider two cases:

1. $\hbar_0/\lambda \le 1$. This assumption gives a chain of inequalities:

$$\begin{aligned}&1 \le \frac{1}{\lambda} \le \frac{1}{\hbar_0}; \quad \ln\frac{1}{\lambda} \le \frac{1}{\hbar_0},\\ &\frac{\hbar_0^2}{\lambda}\ln\frac{1}{\lambda}\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big) \le \frac{1}{D_2}\hbar_0|\ln\hbar_0|.\end{aligned}$$

2. $\hbar_0/\lambda \ge 1$. We transform the expression:

$$\begin{aligned}\frac{\hbar_0^2}{\lambda}\ln\frac{1}{\lambda}\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big) &= \frac{\hbar_0^2}{\lambda}\Big(\ln\frac{\hbar_0}{\lambda}+\ln\frac{1}{\hbar_0}\Big)\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big)\\ &= \hbar_0\Big(\frac{\hbar_0}{\lambda}\ln\frac{\hbar_0}{\lambda}+\frac{\hbar_0}{\lambda}\ln\frac{1}{\hbar_0}\Big)\exp\Big(-D_2\frac{\hbar_0}{\lambda}\Big)\\ &\le \tilde{\mu}_0\hbar_0|\ln\hbar_0|, \quad \tilde{\mu} \ge 0.\end{aligned}$$

Both inequalities can be united together:

$$\sum_{i\in I_B}\Theta_i \le \tilde{\mu}h|\ln h|. \qquad \square$$

Statement 2. *We introduce auxiliary functions:*

$$\begin{aligned}
\chi_1^i &= \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(t)dt - f_i,\\
\chi_2^i &= \int_{x_i}^{x_{i+1}} \exp(\varphi(\xi))\int_{\xi}^{x_{i+\frac{1}{2}}} f(t)dtd\xi,\\
\chi_3^i &= \int_{x_i}^{x_{i+1}} \exp(\varphi(x))dx - \frac{\exp(\varphi_{i+1}) - \exp(\varphi_i)}{\varphi_{i+1}-\varphi_i}\cdot h_i,\\
&\qquad i = \overline{1, N-1}.
\end{aligned}$$

For them the following estimates take place:

$$\begin{aligned}
\|\chi_1\|_\infty &\le R_1 h,\\
\|\chi_2\|_\infty &\le R_2 h^2,\\
\|\chi_3\|_\infty &\le \begin{cases} R_3 h^3, & i\in I_A,\\ R_4\dfrac{h^3}{\lambda^2}\exp\Big(-D_2\dfrac{\hbar_0}{\lambda}\Big), & i\in I_B,\\ R_5 h, & i \in I_C. \end{cases}
\end{aligned}$$

PROOF. Let us get the first inequality using the quadrature rectangular formula in the expression for χ_1. We evaluate the expression for χ_2 directly from its form:

$$|\chi_2^i| \le \exp C_0\cdot h^2\cdot\frac{k_0}{2} \le R_2 h^2.$$

We evaluate the value of fuction χ_3. Write $\varphi(x)$ in the form of Lagrange's polinomial with the second order residual term for $x\in\delta_i$:

$$\begin{aligned}
\varphi(x) &= \varphi_i + (x-x_i)\frac{\varphi_{i+1}-\varphi_i}{h_i} + (x-x_i)(x-x_{i+1})\frac{\varphi''(\xi_0)}{2},\\
\int_{x_i}^{x_{i+1}} \exp(\varphi(x)) &= \exp(\varphi_i)\int_{x_i}^{x_{i+1}}(x-x_i)\frac{\varphi_{i+1}-\varphi_i}{h_i}dx\\
&\quad + \int_{x_i}^{x_{i+1}}\Big(\exp(\varphi(x)) - \exp(x-x_i)\frac{\varphi_{i+1}-\varphi_i}{h_i}\Big)dx,\\
&\quad \xi_0\in\delta_i,
\end{aligned}$$

We can rewrite the latter addendum by theorem on the average as follows:

$$\int_{x_i}^{x_{i+1}} \Big(\exp\varphi(x) - \exp(x - x_i)\frac{\varphi_{i+1} - \varphi_i}{h_i}\Big)dx$$
$$\times h_i\Big(\exp\varphi(\xi) - \exp(\xi - x_i)\frac{\varphi i + 1 - \varphi_i}{h_i}\Big)$$
$$= h_i \exp(\xi - x_i)\frac{\varphi i + 1 - \varphi_i}{h_i}\Big(\exp\varphi(\xi) - (\xi - x_i)\frac{\varphi i + 1 - \varphi_i}{h_i}\Big),$$
$$\exp\Big(\varphi(\xi) - (\xi - x_i)\frac{\varphi_{i+1} - \varphi_i}{h_i}\Big) = \exp\Big((\xi - x_i)(\xi - x_{i+1})\frac{\varphi''(\xi_0)}{2}\Big).$$

Let

$$\alpha = (\xi - x_i)(\xi - x_{i+1})\frac{\varphi''(\xi_0)}{2}.$$

For ξ_0, $\xi \in \delta_i$ there exists such $\alpha_0 \leq 0$, independent of the value λ, h, that

$$|\alpha| \leq \alpha_0.$$

Making use of the inequality:

$$\exp|\alpha| \leq 1 + |\alpha| \cdot \exp\alpha_0,$$

where $0 \leq \alpha \leq \alpha_0$, we get the final estimation:

$$\begin{aligned} |\chi_3^i| &\leq h_i \exp(x - x_i)\frac{\varphi_{i+1} - \varphi_i}{h_i} \cdot (\exp|\alpha| - 1) \\ &\leq h_i \exp c_0 - \tilde{c_0} \exp\alpha \cdot |\xi - x_i| \cdot |\xi - x_{i+1}|\frac{|\varphi''(\xi_0)|}{2} \\ &\leq R_6 h_i^3 |\varphi''(\xi_0)|, \quad \xi_0, \xi \in \delta_i. \end{aligned}$$

If $i \in I_A$, then from limitedness of $\varphi''(\xi_0)$ follows:

$$|\chi_3^i| \leq R_3 h^3.$$

For $i \in I_B$ we use an estimation for the second derivative for the potential on the multitude $\triangle_B$:

$$|\chi_3^i| \leq R_4 \frac{h^3}{\lambda^2} \exp\Big(- D_2\frac{\hbar_0}{\lambda}\Big).$$

From definition of the function χ_3 for $i \in I_C$ we get

$$|\chi_3^i| \leq h \exp c_0 \cdot \Big(1 + \frac{\exp(c_0 - \tilde{c_0}) - 1}{(c_0 - \tilde{c_0})}\Big) \leq R_5 h. \qquad \square$$

Let us prove the basic inequality (8) in two steps. At first we will show that it is true for the case of constant flows, i.e., for $f(x) \equiv 0$.

Statement 3. *If z is a vector of difference scheme error (5) in the case $f(x) \equiv 0$, then*

$$\|z\|_\infty \le ch|lnh|.$$

PROOF. We will write out the value of the error vector component:

$$|z_i| = |(p)_i - p_i| = \exp(-\varphi_i) \cdot \left|\exp(\varphi^1)p - \exp(\varphi^0)p^0\right|$$
$$\times \left| \frac{\int_0^{x_i} \exp(\varphi(t))dt}{\int_0^1 \exp(\varphi(x))dx} - \frac{\sum_{j=0}^{i-1} h_j \frac{\exp(\varphi_{j+1}) - \exp(\varphi_i)}{\varphi_{i+1} - \varphi_i}}{\sum_{j=0}^{N-1} h_j \frac{\exp(\varphi_{j+1}) - \exp(\varphi_i)}{\varphi_{i+1} - \varphi_i}} \right|,$$
$$i = \overline{1, N-1}.$$

Denote

$$I_i = h_i \frac{\exp(\varphi_{i+1}) - \exp(\varphi_i)}{\varphi_{i+1} - \varphi_i}, \quad I_i^T = \int_{x_i}^{x_{i+1}} \exp(\varphi(x))dx,$$
$$A_i = \sum_{j=0}^{i-1} I_j^T = \int_0^{x_i} \exp(\varphi(x))dx,$$
$$B_i = \sum_{j=0}^{i-1} I_j,$$
$$A = A_N, \quad B = B_N.$$

and note that

$$A \ge \gamma_1, \quad B \ge \gamma_3, \quad A_i \le \gamma_2, \quad B_i \le \gamma_4, \quad i = \overline{1, N},$$
$$\gamma_1 = \exp \tilde{c_0}, \quad \gamma_2 = \exp c_0, \quad \gamma_3 = 1, \quad \gamma_4 = \exp c_0 \frac{\exp(c_0 - \tilde{c_0}) - 1}{c_0 - \tilde{c_0}}.$$

We rewrite the expression for z_i in the form

$$|z_i| = \exp(-\varphi_i)\left|\exp(\varphi^1)p^1 - \exp(\varphi^0)p^0\right| \cdot \left|\frac{A_i}{A} - \frac{B_i}{B}\right|$$
$$= \exp(-\varphi_i)\left|\exp(\varphi^1)p^1 - \exp(\varphi^0)p^0\right| \cdot \left|\frac{(B - A_i)A_i + A(A_i - B_i)}{AB}\right|,$$
$$i = \overline{1, N}.$$

We estimate the value $(A_i - B_i)$, $i = \overline{1,N}$ using Statements 1 and 2:

$$\begin{aligned} |A_i - B_i| &\leq \sum_{i \in I_A} |I_i^T - I_i| + \sum_{i \in I_C} |I_i^T - I_i^T| + \sum_{i \in I_B} |I_i^T - I_i| \\ &\leq R_3 h^2 + R_5 h + R_7 h |\ln h|, \quad i = \overline{1,N}. \end{aligned}$$

We obtain that

$$|A_i - B_i| \leq R_8 h |\ln h|, \quad i = \overline{1,N}.$$

Now we can estimate the error vector component:

$$\begin{aligned} |z_i| &\leq \exp(-\tilde{c_0}) \cdot 2 \cdot \exp c_0 \max(p^0, p^1) \gamma_1^{-1} \gamma_3^{-1} \cdot 2\gamma_2 R_8 h |\ln h| \\ &\leq R_9 h |\ln h|, \quad i = \overline{1,N}. \end{aligned}$$

□

A change-over to the case $f(x) \neq 0$ is the following step of the proof of Statement 8:

$$\begin{aligned} z_i &= (p)_i - p_i \\ &= \exp(-\varphi_i + \varphi^0) p^0 + \exp(-\varphi_i) \int_0^{x_i} \exp\varphi(t) \int_0^t f(\xi) d\xi dt \\ &\quad + \exp(-\varphi_i) \cdot J_0 \cdot \int_0^{x_i} \exp\varphi(t) dt - \exp(-\varphi_i + \varphi^0) \\ &\quad - \exp(-\varphi_i) \sum_{j=0}^{i-1} I_j \sum_{k=0}^{j} h_k f_k - \exp(-\varphi_i) J_0^* \sum_{j=0}^{i-1} I_j \qquad (9) \\ &= \exp(-\varphi_i) \left[(\exp\varphi^1 p^1 - \exp\varphi^0 p^0) \left(- \frac{\int_0^{x_i} \exp\varphi(t) dt}{\int_0^1 \exp\varphi(x) dx} \right) - \frac{\sum_{j=0}^{i-1} I_j}{\sum_{j=0}^{N-1} I_j} \right] \\ &\quad + \exp(-\varphi_i) \left[\int_0^{x_i} \exp\varphi(t) \int_0^t f(\xi) d\xi dt - \sum_{j=0}^{i-1} I_j \sum_{k=0}^{j} \hbar_k f_k \right] \\ &\quad + \exp(-\varphi_i) \left[\frac{\sum_{k=0}^{N-1} I_k \sum_{j=0}^{k} \hbar_j f_j}{\sum_{k=0}^{N-1} I_k} \cdot \sum_{j=0}^{i-1} I_j \right. \\ &\qquad \left. - \frac{\int_0^1 \exp\varphi(x) \int_0^x f(t) dt dx}{\int_0^1 \exp\varphi(x) dx} \cdot \int_0^x \exp\varphi(t) dt \right], \\ &\quad i = \overline{1,N}. \end{aligned}$$

The first addendum of this sum was estimated in Statement 3.

To evaluate the two latter addendums in the square brackets we transform double integrals presented in (9):

$$\int_0^{x_i} \exp(\varphi(t)) \int_0^t f(\xi) d\xi dt$$
$$= \sum_{k=0}^{i-1} \left[\int_{x_k}^{x_{k+1}} \exp(\varphi(x)) \int_{x_{k+\frac{1}{2}}}^{x} f(t) dt dx \right.$$
$$\left. + \int_{x_k}^{x_{k+1}} \exp(\varphi(x)) dx \cdot \sum_{j=0}^{k} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(t) dt \right]$$
$$= \sum_{k=0}^{i-1} \chi_2^k + \sum_{k=0}^{i-1} \left(\int_{x_k}^{x_{k+1}} \exp(\varphi(x)) dx \sum_{j=0}^{k} \hbar_j f_j \right)$$
$$+ \sum_{x_k}^{i-1} \left(\int_{x_k}^{x_{k+1}} \exp(\varphi(x)) dx \cdot \sum_{j=0}^{k} \chi_1^j f_j \right)$$
$$= \sum_{k=0}^{i-1} \chi_2^k + \sum_{k=0}^{i-1} \left(\int_{x_k}^{x_{k+1}} \exp(\varphi(x)) dx \cdot \sum_{j=0}^{k} \hbar_j \chi_1^j \right)$$
$$+ \sum_{k=0}^{i-1} \chi_3^k \sum_{j=0}^{k} \hbar_j f_j + \sum_{k=0}^{i-1} I_k \sum_{j=0}^{k} \hbar_j f_j,$$
$$i = \overline{1, N}.$$

Now it is easy to obtain

Statement 4. *For*

$$F_i = \int_0^{x_i} \exp \varphi(x) \int_0^x f(t) dt dx - \sum_{j=0}^{i-1} I_j \sum_{k=0}^{j} \hbar_k f_k, \quad i = \overline{1, N},$$

the following estimate is correct:

$$|F_i| \le R_0 h |\ln h|, \quad i = \overline{1, N}.$$

PROOF. We represent F_i as the sum

$$F_i = F_i^1 + F_i^2 + F_i^3,$$

where

$$F_i^1 = \sum_{k=0}^{i-1} \chi_2^k,$$

$$F_i^2 = \sum_{k=0}^{i-1} \int_{x_k}^{x_{k+1}} \exp\varphi(x)dx \cdot \sum_{j=0}^{k} \chi_1^j \hbar_j,$$

$$F_i^3 = \sum_{k=0}^{i-1} \chi_3^k \sum_{j=0}^{k} \hbar_j f_j, \quad i = \overline{1, N}.$$

From Statements 1 and 2 it is expected that

$$|F_i^1| \le R_2 h.$$

We estimate F_i^2 and F_i^3:

$$|F_i^2| \le \left|\sum_{k=0}^{N-1} \chi_1^k \hbar_k\right| \cdot \left|\sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \exp\varphi(x)dx\right| \le R_1 h \exp c_0 \le R_{10} h,$$

$$|F_i^3| \le \left|\sum_{k=0}^{N-1} \chi_3^k\right| \cdot \left|\sum_{j=0}^{N-1} \hbar_j f_j\right|$$

$$\le k_0\left(\sum_{i\in I_A} |\chi_3^i| + \sum_{i\in I_C} |\chi_3^i| + \sum_{i\in I_B} |\chi_3^i|\right)$$

$$\le k_0(R_3 h^2 + R_5 h + R_{11} h|\ln h|) \le R_{12} h|\ln h|, \quad i = \overline{1, N}.$$

Uniting three latter inequalities we obtain

$$|F_i| \le R_0 h|\ln h|. \qquad \square$$

At present we start to estimate two latter addendums of (9). For this we introduce the notation:

$$C_i = \sum_{k=0}^{i-1} I_k \sum_{j=0}^{k} \hbar_j f_j, \quad C = C_N,$$

$$D_i = \int_0^{x_i} \exp\varphi(x) \int_0^x f(t)dtdx, \quad D = D_N.$$

Expression (9) may be rewritten as:

$$z_i = \exp(-\varphi_i)\left[(\exp\varphi^1 p^1 - \exp\varphi^0 p^0)\left(\frac{A_i}{A} - \frac{B_i}{B}\right)\right]$$
$$+ \exp(-\varphi_i)\left[\frac{C}{B}B_i - \frac{D}{A}A_i\right] + \exp(-\varphi_i)[D_i - C_i],$$
$$i = \overline{1, N}.$$

It is easy to show that the values $\{C_i, D_i\}_{i=\overline{1,N}}$, are bounded by some positive constants:

$$|C_i| \le \gamma_5, \quad |D_i| \le \gamma_6, \quad i = \overline{1, N}.$$

The second addendum in square brackets is estimated by the following:

$$\left|\frac{C}{B}B_i - \frac{D}{A}A_i\right| = \left|\frac{A-B}{AB}\cdot CB_i + \frac{CB_i - DA_i}{A}\right|$$
$$\le \gamma_5\gamma_4\gamma_1^{-1}\gamma_3^{-1}R_8 h|\ln h| + \gamma_1^{-1}(R_8 + R_0)h|\ln h|$$
$$\le R_{13}h|\ln h|.$$

Bearing in mind that $F_i = D_i - C_i$, we have the estimate $|D_i - C_i| \le R_0 h|\ln h|$, $i = \overline{1, N}$. From the latter considerations the inequality follows:

$$\|z\|_\infty \le ch|\ln h|,$$

where c is a positive constant independent of h and ε.

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