Bull. Nov. Comp. Center, , (2013), 1–9

The subgrid modeling for Maxwell's equations with multiscale isotropic random conductivity and permittivity

E.P. Kurochkina, O.N. Soboleva

Abstract. The effective coefficients in the Maxwell's equations are calculated for a multiscale isotropic medium using a subgrid modeling approach. The correlated fields of conductivity and permittivity are mathematically represented by a Kolmogorov multiplicative continuous cascades with a lognormal probability distribution. The wavelength is assumed to be large as compared with the scale of heterogeneities of the medium.

1. Introduction

Wave propagation in complex media is a problem of great interest in many fields, for example, in hydrodynamics and electromagnetics. In order to compute the flow rate or the electromagnetic fields in an arbitrary medium, one must solve hydrodynamic or Maxwell's equations numerically for the given coefficients. The large-scale medium heterogeneities as compared with wavelength are taken into account in these models with the help of some boundary conditions (see, e.g., [1,2]). The problems for a complex medium require high the computational costs due to variations of coefficients on all the scales. In addition, the spatial distributions of small-scale heterogeneities are not exactly known. It has been experimentally shown that the irregularity of electric conductivity, permeability, porosity, density increases as the scale of measurement decreases. It is customary to assume these parameters to be random fields characterized by the joint probability distribution functions. The small-scale heterogeneities are taken into account by the effective parameters. Simple equations are found on the scales that can be numerically resolved. The solution to these equations must be close to the solution of the initial problem. This is the well-known procedure of subgrid modeling, homogenization, coarse grids [3–6]. There are some robust methods of finding the effective coefficients in theory of steady filtration [4–7].

To apply the above-mentioned methods we need a "scale regular" medium. It has been experimentally shown that many natural media are "scale regular" in the sense that their parameters, for example, permeability, porosity, density, electric conductivity can be described by fractals and multiplicative cascades [8–10].

In the present paper, the electric conductivity and permittivity are approximated by a multiplicative continuous cascade. We obtain formulas of effective coefficients in the Maxwell's equations when the following condition $\sigma(\boldsymbol{x})/(\omega\varepsilon(\boldsymbol{x})) \ll 1$ is satisfied.

2. Governing equations and a model of coefficients

We consider the time-harmonic Maxwell's equations for general media with an impressed current source $\boldsymbol{F}$ in a 3D-medium, which are

$$\operatorname{rot} \boldsymbol{H}(\boldsymbol{x}) = (-i\omega\varepsilon(\boldsymbol{x}) + \sigma(\boldsymbol{x}))\boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{F}, \quad \operatorname{rot} \boldsymbol{E} = i\omega\mu\boldsymbol{H}, \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the vectors of electric and magnetic field strengths respectively; $\varepsilon(\boldsymbol{x})$ is permittivity, μ is the magnetic permeability; $\sigma(\boldsymbol{x})$ is the electric conductivity; ω is the cyclic frequency; and $\boldsymbol{x}$ is the vector of spatial coordinates. The magnetic permeability is assumed to be equal to the magnetic permeability of vacuum. When $\frac{\sigma(\boldsymbol{x})}{\omega\varepsilon(\boldsymbol{x})} \gg 1$, the conduction current predominates over the displacement current and the permittivity of the medium $\varepsilon(\boldsymbol{x})$ has an weak effect on electric and magnetic field strength; the wave amplitude and phase of the fields depend mainly on the electric conductivity $\sigma(\boldsymbol{x})$ and the permittivity may be taken no account. At high frequencies or when resistivities of the medium are high, electric and magnetic field strength depend on the dielectric permittivity. In this case, we have

$$\frac{\sigma(\boldsymbol{x})}{\omega\varepsilon(\boldsymbol{x})} \ll 1. \tag{2}$$

For a medium, the radiation conditions must be satisfied; that is to say, the solution of system (1) must radiate away from the current source and dissipate as $|\boldsymbol{r}|$ goes to infinity. The wavelength is assumed to be large as compared with the maximum scale of heterogeneities of the medium L.

For modeling the coefficients $\sigma(\boldsymbol{x})$, $\varepsilon(\boldsymbol{x})$, we use approach described in [11]. Let, for example, the field of electric conductivity be known. This means that the field is measured on a small scale l_0 at each point $\boldsymbol{x}$, $\sigma(\boldsymbol{x})_{l_0} = \sigma(\boldsymbol{x})$. To pass to a coarser scale grid, it is not sufficient to smooth the field $\sigma(\boldsymbol{x})_{l_0}$ on a scale l, $l > l_0$. The field thus smoothed is not a physical parameter that can describe the physical process, governed by equations (1), on the scales (l, L), where L is the maximum scale of heterogeneities. This is due to the fact that the fluctuations of electric conductivity on the scale interval (l_0, l) correlate with the fluctuations of the electric field strength $\boldsymbol{E}$ induced by the electric conductivity. In this paper, to find an electric conductivity that can describe the physical process on the scales (l, L), system (1) will be used. Following Kolmogorov [12], consider a dimensionless field ψ, which is equal to the ratio of two fields obtained by smoothing the field $\sigma(\boldsymbol{x})_{l_0}$ on two different scales l', l. Let $\sigma(\boldsymbol{x})_l$ denote the parameter $\sigma(\boldsymbol{x})_{l_0}$ smoothed on the scale l. Then $\psi(x, l, l') = \sigma(x)_{l'}/\sigma(x)_l$, $l' < l$. We expand the field ψ

into a power series in $l - l'$, and retaining the first order terms of the series, at $l' \to l$, obtain the following equation:

$$\frac{\partial \ln \sigma(\boldsymbol{x})_l}{\partial \ln l} = \varphi(\boldsymbol{x}, l), \tag{3}$$

where $\varphi(\boldsymbol{x}, l') = (\partial\psi(\boldsymbol{x}, l', l'y)/\partial y)|_{y=1}$. Actually the small-scale fluctuations of the field φ can be observed on some finite interval of scales $l_0 < l < L$. The solution of equation (3) is as follows

$$\sigma_{l_0}(\boldsymbol{x}) = \sigma_0 \exp\left(-\int_{l_0}^{L} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right), \tag{4}$$

where σ_0 is a constant. The field φ determines the statistical properties of the electric conductivity. According to the limit theorem for sums of independent random variables [13] if the variance of $\varphi(\boldsymbol{x}, l)$ is finite, the integral in (4) tends to a field with a normal distribution as the ratio L/l_0 increases. If the variance of $\varphi(\boldsymbol{x}, l)$ is infinite and there exists a non-degenerate limit of the integral in (4), the integral tends to a field with a stable distribution. In the this paper it is assumed that the field $\varphi(\boldsymbol{x}, l)$ is isotropic with a normal distribution and a statistically homogeneous correlation function:

$$\langle\varphi(\boldsymbol{x}, l)\ \varphi(\boldsymbol{y}, l')\rangle - \langle\varphi(\boldsymbol{x}, l)\rangle\langle\varphi(\boldsymbol{y}, l')\rangle = \Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l')\,\delta(\ln l - \ln l'). \tag{5}$$

Here the angle brackets denote ensemble averaging. It follows from (5) that the fluctuations of $\varphi(\boldsymbol{x}, l)$ on different scales do not correlate. This assumption is standard in the scaling models [12]. This is due to the fact that the statistical dependence is small if the scales of fluctuations are different. To derive subgrid formulas to calculate effective coefficients, this assumption may be ignored. However, this assumption is important for the numerical simulation of the field σ.

For a scale-invariant medium, the following relation holds for any positive K

$$\Phi^{\varphi\varphi}(|\boldsymbol{x}-\boldsymbol{y}|, l, l') = \Phi^{\varphi\varphi}(K|\boldsymbol{x}-\boldsymbol{y}|, Kl, Kl').$$

In a scale invariant medium, the correlation function does not depend on the scale at $\boldsymbol{x} = \boldsymbol{y}$, and the following estimation is obtained [11]:

$$\langle\sigma_{l_0}(\boldsymbol{x})\sigma_{l_0}(\boldsymbol{x}+\boldsymbol{r})\rangle \sim C\left(\frac{r}{L}\right)^{-\Phi_0^{\varphi\varphi}}, \quad l_0 < l_\varepsilon < r < L, \tag{6}$$

where $C = \sigma_0^2 e^{-\Phi_0\gamma/2}$, γ is the Euler constant. For $r \gg L$, we have

$$\langle\sigma_{l_0}(\boldsymbol{x})\sigma(\boldsymbol{x}+\boldsymbol{r}, l_0)\rangle \to \sigma_0^2. \tag{7}$$

If for any l the equality $\langle\sigma_l(\boldsymbol{x})\rangle = \sigma_0$ is valid, then it follows from (4), (5) that

$$\Phi_0^{\varphi\varphi}(l) = 2\langle\varphi\rangle, \tag{8}$$

where $\Phi_0^{\varphi\varphi}(l) = \Phi^{\varphi\varphi}(0, l)$. As the minimum scale l_0 tends to zero, the electric conductivity field described by (4) becomes a multifractal. We obtain an irregular field on a Cantor-type set to be nonzero.

The permittivity coefficient $\varepsilon(\boldsymbol{x})$ is constructed by analogy to the conductivity coefficient:

$$\varepsilon_{l_0}(\boldsymbol{x}) = \varepsilon_0 \exp\Big(-\int_{l_0}^{L} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big). \tag{9}$$

The function $\chi(\boldsymbol{x}, l)$ is assumed to have the normal distribution and to be delta-correlated in the logarithm of the scale. We can write

$$\begin{aligned}\Phi^{\chi\chi}(\boldsymbol{x}, \boldsymbol{x}, l, l') &= \langle\chi(\boldsymbol{x}, l)\chi(\boldsymbol{x}, l')\rangle - \langle\chi(\boldsymbol{x}, l)\rangle\langle\chi(\boldsymbol{x}, l')\rangle\\ &= \Phi_0^{\chi\chi}\delta(\ln l - \ln l').\end{aligned} \tag{10}$$

The permittivity field satisfies the equality $\langle\chi_l(\boldsymbol{x})\rangle = \chi_0$ for any l. Then it follows from (9), (10) that

$$\Phi_0^{\chi\chi}(l) = 2\langle\chi\rangle. \tag{11}$$

The correlation between the permittivity and conductivity fields is determined by the correlation of the fields $\chi(\boldsymbol{x}, l')$ and $(\varphi(\boldsymbol{x}, l')$:

$$\begin{aligned}\Phi^{\varphi\chi}(\boldsymbol{x}, \boldsymbol{y}, l, l') &= \langle\varphi(\boldsymbol{x}, l)\chi(\boldsymbol{y}, l')\rangle - \langle\varphi(\boldsymbol{x}, l)\rangle\langle\chi(\boldsymbol{y}, l')\rangle\\ &= \Phi^{\varphi\chi}(|\boldsymbol{x} - \boldsymbol{y}|, l, l')\delta(\ln l - \ln l').\end{aligned} \tag{12}$$

3. Subgrid model

The electric conductivity and permittivity functions $\sigma(\boldsymbol{x}) = \sigma(\boldsymbol{x})_{l_0}$, $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})_{l_0}$ are divided into two components with respect to the scale l. The large-scale (ongrid) components $\sigma(\boldsymbol{x}, l)$, $\varepsilon(\boldsymbol{x}, l)$ are obtained, respectively, by statistical averaging over all $\varphi(\boldsymbol{x}, l_1)$ and $\chi(\boldsymbol{x}, l_1)$ with $l_0 < l_1 < l$, $l - l_0 = dl$, where dl is small. The small-scale (subgrid) components are equal to $\sigma'(\boldsymbol{x}) = \sigma(\boldsymbol{x}) - \sigma(\boldsymbol{x}, l)$, $\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) - \varepsilon(\boldsymbol{x}, l)$:

$$\begin{aligned}\varepsilon(\boldsymbol{x}, l) &= \varepsilon_0 \exp\Big(-\int_{l}^{L} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\langle\exp\Big(-\int_{l_0}^{l} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\rangle\\ \varepsilon'(\boldsymbol{x}) &= \varepsilon(\boldsymbol{x}, l)\left[\frac{\exp\Big(-\int_{l_0}^{l} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)}{\Big\langle\exp\Big(-\int_{l_0}^{l} \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\rangle} - 1\right], \quad \langle\varepsilon'(\boldsymbol{x})\rangle = 0,\\ \sigma(\boldsymbol{x}, l) &= \sigma_0 \exp\Big(-\int_{l}^{L} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\langle\exp\Big(-\int_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\rangle\\ \sigma'(\boldsymbol{x}) &= \sigma(\boldsymbol{x}, l)\left[\frac{\exp\Big(-\int_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)}{\Big\langle\exp\Big(-\int_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\Big\rangle} - 1\right], \quad \langle\sigma'(\boldsymbol{x})\rangle = 0.\end{aligned} \tag{13}$$

Hence

$$\begin{aligned} \varepsilon(\boldsymbol{x}, l) &\simeq \Big(1 - \langle\chi\rangle \frac{dl}{l} + \frac{1}{2}\Phi_0^{\chi\chi}(l)\frac{dl}{l}\Big)\,\varepsilon_l(\boldsymbol{x}), \\ \sigma(\boldsymbol{x}, l) &\simeq \Big(1 - \langle\varphi\rangle \frac{dl}{l} + \frac{1}{2}\Phi_0(l)\frac{dl}{l}\Big)\,\sigma_l(\boldsymbol{x}). \end{aligned} \tag{14}$$

The large-scale (ongrid) components of electric and magnetic field strengths $\boldsymbol{E}(\boldsymbol{x}, l), \boldsymbol{H}(\boldsymbol{x}, l)$ are obtained by averaging the solutions to system (1), in which the large-scale component of conductivity $\sigma(\boldsymbol{x}, l)$ is fixed and the small component $\sigma'(\boldsymbol{x})$ is a random variable. The subgrid components of the electric and magnetic field strengths are equal to $\boldsymbol{H}'(\boldsymbol{x}) = \boldsymbol{H}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x}, l)$, $\boldsymbol{E}'(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{x}, l)$. Substituting the relations for $\boldsymbol{E}(\boldsymbol{x}), \boldsymbol{H}(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ into system (1) and averaging over small-scale components, we have

$$\begin{aligned} \operatorname{rot} \boldsymbol{H}(\boldsymbol{x}, l) &= (-i\omega\varepsilon(\boldsymbol{x}, l) + \sigma(\boldsymbol{x}, l))\boldsymbol{E}(\boldsymbol{x}, l) + \langle(-i\omega\varepsilon' + \sigma')\boldsymbol{E}'\rangle + \boldsymbol{F}, \\ \operatorname{rot} \boldsymbol{E}(\boldsymbol{x}, l) &= \mu i\omega \boldsymbol{H}(\boldsymbol{x}, l). \end{aligned} \tag{15}$$

The subgrid term $\langle(-i\omega\varepsilon' + \sigma')\boldsymbol{E}'\rangle$ in system (15) is unknown. This term cannot be neglected without some preliminary estimation, since the correlation between the electric conductivity and the electric field strength may be significant. The form of this term in (15) determines a subgrid model. The subgrid term is estimated using perturbation theory. Subtracting system (15) from system (1) and taking into account only the first order terms, we obtain the subgrid equations:

$$\begin{aligned} \operatorname{rot} \boldsymbol{H}' &= (-i\omega\varepsilon(\boldsymbol{x}, l) + \sigma(\boldsymbol{x}, l))\boldsymbol{E}' + (-i\omega\varepsilon'(\boldsymbol{x}) + \sigma'(\boldsymbol{x}))\boldsymbol{E}(\boldsymbol{x}, l), \\ \operatorname{rot} \boldsymbol{E}' &= \mu i\omega \boldsymbol{H}'. \end{aligned} \tag{16}$$

The variable $\boldsymbol{E}(\boldsymbol{x}, l)$ on the right-hand side of (16) is assumed to be known. Solving system (16) for the components of the electric field strength, we have [14]:

$$\begin{aligned} E'_\alpha(\boldsymbol{x}) = &\frac{1}{4\pi} i\omega\mu \int \frac{1}{r} e^{ikr} (-i\omega\varepsilon'(\boldsymbol{x}') + \sigma'(\boldsymbol{x}'))E_\alpha(\boldsymbol{x}', l)d\boldsymbol{x}' + \\ &\frac{1}{4\pi(-i\omega\varepsilon(\boldsymbol{x}, l) + \sigma(\boldsymbol{x}, l))} \times \\ &\qquad \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta} \int \frac{1}{r} e^{ikr} (-i\omega\varepsilon'(\boldsymbol{x}') + \sigma'(\boldsymbol{x}'))E_\beta(\boldsymbol{x}', l)d\boldsymbol{x}', \end{aligned} \tag{17}$$

where $r = |\boldsymbol{x} - \boldsymbol{x}'|$, $k^2 = \omega\mu(\omega\varepsilon(\boldsymbol{x}, l) + i\sigma(\boldsymbol{x}, l))$. We take the square root such that $\operatorname{Re} k > 0$, $\operatorname{Im} k > 0$.

Using (17) the subgrid term can be written down as

$$
\begin{aligned}
&\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))E'_\alpha(\boldsymbol{x})\rangle = \\
&\quad \frac{1}{4\pi}i\omega\mu\iint\frac{1}{r}e^{ikr}\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))(-i\omega\varepsilon'(\boldsymbol{x}')+\sigma'(\boldsymbol{x}'))\rangle E_\alpha(\boldsymbol{x}',l)d\boldsymbol{x}'+ \\
&\quad \Bigg\langle\frac{-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x})}{4\pi(-i\omega\varepsilon(\boldsymbol{x},l)+\sigma(\boldsymbol{x},l))}\times \\
&\qquad \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\iint\frac{1}{r}e^{ikr}(-i\omega\varepsilon'(\boldsymbol{x}')+\sigma'(\boldsymbol{x}'))E_\beta(\boldsymbol{x}',l)d\boldsymbol{x}'\Bigg\rangle. \qquad (18)
\end{aligned}
$$

Since a small change in the scale of σ produces considerable fluctuations in the field (which is typical of fractal fields), the field $\sigma(\boldsymbol{x},l)$ and its derivatives are believed to change slower than σ' and its derivatives. Similar assumptions are made for $\boldsymbol{E}(\boldsymbol{x},l)$ and $\boldsymbol{H}(\boldsymbol{x},l)$. Therefore $\boldsymbol{E}(\boldsymbol{x},l)$, $\sigma(\boldsymbol{x},l)$ and their derivatives can be factored outside the integral sign in (18). Integrating (18) by parts we have

$$
\begin{aligned}
&\big\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))E'_\alpha(\boldsymbol{x})\big\rangle = \qquad (19)\\
&\quad \frac{1}{4\pi}i\omega\mu\int\frac{1}{r}e^{ikr}\times \\
&\qquad \big(-\omega^2\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle - 2i\omega\langle\varepsilon'(\boldsymbol{x}')\sigma'(\boldsymbol{x})\rangle + \langle\sigma'(\boldsymbol{x})\sigma'(\boldsymbol{x}')\rangle)d\boldsymbol{x}'E_\alpha(\boldsymbol{x},l)+ \\
&\quad \frac{1}{4\pi(-i\omega\varepsilon(\boldsymbol{x},l)+\sigma(\boldsymbol{x},l))}\int\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\frac{1}{r}e^{ikr}\times \\
&\qquad \big(-\omega^2\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle - 2i\omega\langle\varepsilon'(\boldsymbol{x}')\sigma'(\boldsymbol{x})\rangle + \langle\sigma'(\boldsymbol{x})\sigma'(\boldsymbol{x}')\rangle\big)d\boldsymbol{x}'E_\beta(\boldsymbol{x},l).
\end{aligned}
$$

Here the summation of repeated indices is implied. As follows from formulas (6),(12), (13) for a lognormal probability distribution of σ and ε at small dl we have

$$
\begin{aligned}
\big\langle\sigma'(\boldsymbol{x})\sigma'(\boldsymbol{x}')\big\rangle &\approx \sigma^2(\boldsymbol{x},l)\Phi^{\sigma\sigma}(r)\frac{dl_1}{l_1}, \\
\big\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\big\rangle &\approx \varepsilon^2(\boldsymbol{x},l)\Phi^{\chi\chi}(r)\frac{dl_1}{l_1}, \qquad (20)\\
\big\langle\sigma'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\big\rangle &\approx \varepsilon(\boldsymbol{x},l)\sigma(\boldsymbol{x},l)\Phi^{\chi\sigma}(r)\frac{dl_1}{l_1},
\end{aligned}
$$

$$
\begin{aligned}
&\big\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))(-i\omega\varepsilon'(\boldsymbol{x}')+\sigma'(\boldsymbol{x}'))\big\rangle \approx \\
&\quad -\omega^2\varepsilon^2(\boldsymbol{x},l)\left[\Phi^{\chi\chi}(r) - 2i\frac{\sigma(\boldsymbol{x},l)}{\omega\varepsilon(\boldsymbol{x},l)}\Phi^{\chi\varphi}(r) - \frac{\sigma^2(\boldsymbol{x},l)}{\omega^2\varepsilon^2(\boldsymbol{x},l)}\Phi^{\sigma\sigma}(r)\right]\frac{dl_1}{l_1}. \qquad (21)
\end{aligned}
$$

Using formula $1/\left[4\pi(-i\omega\varepsilon(\boldsymbol{x},l)+\sigma(\boldsymbol{x},l))\right] \approx -(1-\frac{i\sigma(\boldsymbol{x},l)}{\omega\varepsilon(\boldsymbol{x},l)})/(4\pi i\omega\varepsilon(\boldsymbol{x},l))$ since $\frac{\sigma(\boldsymbol{x})}{\omega\varepsilon(\boldsymbol{x})} \ll 1$, we obtain

$$\big\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))E'_\alpha(\boldsymbol{x})\big\rangle \approx$$
$$-\frac{1}{4\pi}i\omega^3\mu\varepsilon(\boldsymbol{x},l)\int\frac{1}{r}e^{ikr}\Phi^{\chi\chi}(r)d\boldsymbol{x}'\frac{dl}{l}\varepsilon(\boldsymbol{x},l)E_i(\boldsymbol{x},l)+$$
$$\frac{1}{2\pi}\omega^2\varepsilon(\boldsymbol{x},l)\mu\int\frac{1}{r}e^{ikr}\Phi^{\chi\varphi}(r)d\boldsymbol{x}'\frac{dl}{l}\sigma(\boldsymbol{x},l)E_i(\boldsymbol{x},l)+$$
$$\frac{1}{4\pi}i\omega\mu\sigma(\boldsymbol{x},l)\int\frac{1}{r}e^{ikr}\Phi^{\varphi\varphi}(r)d\boldsymbol{x}'\frac{dl}{l}\sigma(\boldsymbol{x},l)E_i(\boldsymbol{x},l)-$$
$$\frac{i\omega\varepsilon(\boldsymbol{x},l)}{4\pi}\int\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\frac{1}{r}e^{ikr}(1-\frac{i\sigma(\boldsymbol{x},l)}{\omega\varepsilon(\boldsymbol{x},l)})\times$$
$$\Big(\Phi^{\chi\chi}(r)+2i\frac{\sigma(\boldsymbol{x},l)}{\omega\varepsilon(\boldsymbol{x},l)}\Phi^{\chi\varphi}(r)-\frac{\sigma^2(\boldsymbol{x},l)\Phi^{\varphi\varphi}(r)}{\omega^2\varepsilon^2(\boldsymbol{x},l)}\Big)d\boldsymbol{x}'\frac{dl}{l}E_\beta(\boldsymbol{x},l). \quad (22)$$

In formula (22), the Cartesian coordinates are changed for spherical coordinates. Integrating n_jn_m, where $n_m=x_m/r$, over the complete solid angle, we arrive at the formula $\int n_jn_m d\vartheta=\frac{4\pi}{3}\delta_{jm}$. Using this formula, neglecting terms of second order of smallness of $\sigma(\boldsymbol{x},l)/\omega\varepsilon(\boldsymbol{x},l)$ and integrating (22) by parts, we have

$$\big\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))E'_\alpha(\boldsymbol{x})\big\rangle \approx$$
$$-\frac{1}{3}(2\mu\omega^2\varepsilon(\boldsymbol{x},l)-i\omega\mu\sigma(\boldsymbol{x},l))\int_0^\infty re^{ikr}\Phi^{\chi\chi}(r)dr\frac{dl}{l}i\omega\varepsilon(\boldsymbol{x},l)E_\alpha(\boldsymbol{x},l)+$$
$$\frac{2}{3}(2\mu\omega^2\varepsilon(\boldsymbol{x},l)-i\omega\mu\sigma(\boldsymbol{x},l))\int_0^\infty re^{ikr}\Phi^{\chi\sigma}(r)dr\frac{dl}{l}\sigma(\boldsymbol{x},l)E_\alpha(\boldsymbol{x},l)+$$
$$i\omega\mu\sigma(\boldsymbol{x},l)\int_0^\infty re^{ikr}\Phi^{\sigma\sigma}(r)dr\frac{dl}{l}\sigma(\boldsymbol{x},l)E_\alpha(\boldsymbol{x},l)+$$
$$\frac{1}{3}\Phi_0^{\chi\chi}\frac{dl}{l}i\omega\varepsilon(\boldsymbol{x},l)E_i(\boldsymbol{x},l)+\Big(\frac{1}{3}\Phi_0^{\chi\chi}-\frac{2}{3}\Phi_0^{\chi\sigma}\Big)\frac{dl}{l}\sigma(\boldsymbol{x},l)E_i(\boldsymbol{x},l). \quad (23)$$

If $\omega\mu L^2|(i\omega\varepsilon(\boldsymbol{x},l)+\sigma(\boldsymbol{x},l))|\ll 1$, the integrals in (23) are small [15]. This inequality is not restrictive for the problems of electromagnetic logging if L is much smaller than the wavelength. Hence, the integrals in (23) can be neglected. We have

$$\big\langle(-i\omega\varepsilon'(\boldsymbol{x})+\sigma'(\boldsymbol{x}))E'_\alpha(\boldsymbol{x})\big\rangle \approx$$
$$-\frac{1}{3}\Phi_0^{\chi\chi}(-i\omega\varepsilon(\boldsymbol{x},l)E_\alpha(\boldsymbol{x},l))\frac{dl}{l}-\Big(\frac{2}{3}\Phi_0^{\chi\sigma}-\frac{1}{3}\Phi_0^{\chi\chi}\Big)\frac{dl}{l}\sigma(\boldsymbol{x},l)E_\alpha(\boldsymbol{x},l). \quad (24)$$

Substituting (24) into (15), we derive

$$\begin{aligned}
\operatorname{rot} \boldsymbol{H}(\boldsymbol{x}, l) &= -i\omega\varepsilon_{l0} \exp\Big(-\int_l^L \chi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\boldsymbol{E}(\boldsymbol{x}, l) + \\
&\qquad \sigma_{l0} \exp\Big(-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big)\boldsymbol{E}(\boldsymbol{x}, l), \qquad (25)\\
\operatorname{rot} \boldsymbol{E}(\boldsymbol{x}, l) &= i\omega\mu\boldsymbol{H}(\boldsymbol{x}, l),\\
\varepsilon_{l0} &= \Big[1 - \frac{\Phi_0^{\chi\chi}}{3}\frac{dl}{l}\Big]\Big[1 + \Big(\frac{\Phi_0^{\chi\chi}}{2} - \langle\chi\rangle\Big)\frac{dl}{l}\Big]\varepsilon_0,\\
\sigma_{l0} &= \Big[1 - \Big(\frac{2}{3}\Phi_0^{\chi\varphi} - \frac{1}{3}\Phi_0^{\chi\chi}\Big)\frac{dl}{l}\Big]\Big[1 + \Big(\frac{\Phi_0^{\varphi\varphi}}{2} - \langle\varphi\rangle\Big)\frac{dl}{l}\Big]\sigma_0.
\end{aligned}$$

It follows from (25) that the new coefficients σ_{l0} and ε_{l0} are equal to

$$\begin{aligned}
\varepsilon_{l0} &= \varepsilon_0 + \Big(\frac{1}{6}\Phi_0^{\chi\chi} - \langle\chi\rangle\Big)\varepsilon_0\frac{dl}{l},\\
\sigma_{l0} &= \sigma_0 + \Big(-\frac{2}{3}\Phi_0^{\chi\varphi} + \frac{1}{3}\Phi_0^{\chi\chi} + \frac{1}{2}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle\Big)\sigma_0\frac{dl}{l}
\end{aligned}$$

with second order of accuracy in dl/l. As $dl \to 0$ we obtain the equation

$$\begin{aligned}
\frac{d\ln\varepsilon_{0l}}{d\ln l} &= \frac{1}{6}\Phi_0^{\chi\chi} - \langle\chi\rangle\,,\\
\frac{d\ln\sigma_{0l}}{d\ln l} &= -\frac{2}{3}\Phi_0^{\chi\varphi} + \frac{1}{3}\Phi_0^{\chi\chi} + \frac{1}{2}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle\,.
\end{aligned} \qquad (26)$$

For a scale-invariant medium, effective equations have the following simple form

$$\begin{aligned}
\operatorname{rot} \boldsymbol{H}(\boldsymbol{x}, l) &= -i\omega\Big(\frac{l}{L}\Big)^{\langle\chi\rangle - \frac{1}{6}\Phi_0^{\chi\chi}}\varepsilon_l(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, l) + \\
&\qquad \Big(\frac{l}{L}\Big)^{\langle\varphi\rangle + \frac{2}{3}\Phi_0^{\chi\varphi} - \frac{1}{3}\Phi_0^{\chi\chi} - \frac{1}{2}\Phi_0^{\varphi\varphi}}\sigma_l(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, l), \qquad (27)\\
\operatorname{rot} \boldsymbol{E}(\boldsymbol{x}, l) &= i\omega\mu\boldsymbol{H}(\boldsymbol{x}, l).
\end{aligned}$$

4. Conclusion

We have presented the effective coefficients for the Maxwell's equations if parameters in equations are described by extremely irregular fields which are close to multifractals. We obtain multifractals if the minimum scale l_0 in formulas (4), (9) tend to zero. As the minimum scale is finite, any singularities are absent, therefore we use only the theory of differential equations and the theory of stochastic processes. For a scale-invariant medium, effective coefficients have power dependence on the scale of smoothing. The exponents of power dependencies have been calculated.

References

[1] Mikhailenko B.G., Soboleva O.N. Mathematical modeling of seismomagnetic effects arising in the seismic wave motion in Earthes's constant magnetic field // Applied Mathematics Letters. — 1997. — Vol. 10, No. 3. — P. 47–55.

[2] Epov M.I., Shurina E.P., Nechaev O.V. 3D forward modeling of vector field for induction logging // Russian Geology and Geophysics. — 2007. — Vol. 48. — P. 770–774.

[3] Sahimi M. Flow phenomena in rocks: from continuum models, to fractals, percolation, cellular automata, and simulated annealing // Reviews of Modern Physics. — 1993. — Vol. 65. — P. 1393–1534.

[4] Dagan G. Higher-order correction of effective permeability of heterogeneous isotropic formations of lognormal conductivity distribution // Transport in Porous Media. — 1993. — Vol. 12. — P. 279–290.

[5] Yukalov V.I. Self-similar approximations for strongly interacting systems // Physica A. — 1990. — Vol. 167. — P. 833–860.

[6] Yukalov V.I., Gluzman S. Self-similar bootstrap of divergent series // Phys. Rev. E. — 1977. — Vol. 55. — P. 6552–6570.

[7] Gluzman S., Sornette D. Self-similar approximates of the Permeability in heterogeneous porous media from moment equation expansions // Transport in Porous Media. — 2008. — Vol. 71. — P. 75–97.

[8] Biswal B., Manwart C., Hilfer R. Three-dimensional local porosity analysis of porous media // Physica A. — 1998. — Vol. 255. — P. 221–241.

[9] Krylov S.S., Lyubchich V.A., The apparent resistivity scaling and fractal structure of an iron formation // Izvestiya. Physics of the Solid Earth. — 2002. — Vol. 38. — P. 1006–1012.

[10] Bekele A., Hudnall H.W., Daigle J.J., Prudente A., Wolcott M. Scale dependent variability of soil electrical conductivity by indirect measures of soil properties // J. Terramechanics. — 2005. — Vol. 42. — P. 339–351.

[11] Kuz'min G.A., Soboleva O.N. Subgrid modeling of filtration in porous self-similar media // J. Appl. Mech. Tech. Phys. — 2002. — Vol. 43. — P. 583–592.

[12] Kolmogorov A.N. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number // J. Fluid Mech. — 1962. — Vol. 13. — P. 82–85.

[13] Gnedenko B.V., Kolmogorov A.N. Limit distributions for sums of independent random variables / English trans. K.L. Chung. — Cambridge: Addison-Wesley, 1954.

[14] Koshljakov N.S., Smirnov M.M., Gliner E.B. Differential equations of mathematical physics. — Moscow: Fizmatgiz, 1962 (In Russian).

[15] Rytov S.M., Kravtsov Yu.A., Tatarskii V.I. Principles of Statistical Radiophysics 3, 4. — Berlin: Springer, 1987.