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Pseudo-orthogonal polynomials

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The classical problem of the transformation of the orthogonality weights of polynomials [1–3] by theory of the space $\boldsymbol{R}^n$ is discussed. The described system of polynomials – pseudo-orthogonal on the discrete set of n points – is a new result. The polynomials of this system, as the orthogonal ones, are connected by the three-term relations with a tridiagonal matrix which is irreducible but not the Jacobi one. Nevertheless, these polynomials have real single roots with a weak separation. Some orthogonality weights are negative. Analysis of the relation between the matrices of two orthogonal polynomial systems, providing the condition of existence of the pseudo-orthogonal polynomials, is another result.

1. The Jacobi matrix

$$T = \begin{bmatrix} b_1 & 1 & & \\ g_1 & b_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & g_{n-1} & b_n \end{bmatrix}, \qquad g_k > 0, \tag{1}$$

and the system of the orthogonal polynomials $P_k(\lambda)$, $k = 0, \dots, n-1$, on the discrete set of points λ_i,

$$\sum_{i=1}^{n} c_i P_k(\lambda_i) P_l(\lambda_i) = d_k \delta_{kl},$$

$c_i > 0$, $i = 1, \dots, n$, are connected with the positive definite Hankel matrix $H = (h_{i+j-2})$, $i, j = 1, \dots, n$,

$$h_k = \sum_{i=1}^{n} c_i \lambda_i^k, \quad d_k = |H_k|/|H_{k-1}|, \quad d_1 = h_0 = 1.$$

The following equations are valid:

$$TY^T = Y^T \Lambda, \tag{2}$$

$$Y = VP, \tag{3}$$

$$H = V^T C V = P^{-T} D P^{-1}, \tag{4}$$

$$Y^T C Y = D, \tag{5}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $D = \mathrm{diag}(d_1, \dots, d_n)$, $C = \mathrm{diag}(c_1, \dots, c_n)$, $V = (\lambda_i^{j-1})$, $i, j = 1, \dots, n$ ($Ve_k = \Lambda^{k-1}e$, e_k is the k-th column of the identity matrix E, $e = (1, \dots, 1)^T$),

$$P_k(\lambda) = \sum_{j=0}^{k} p_j^k \lambda^j, \qquad p_k^k = 1,$$

$P = (p_i^j)$, $i, j = 0, \dots, n-1$, is the upper triangular matrix.

If $\lambda_j^{(k)}$ are the ordered roots of the polynomial $P_k(\lambda)$, $P_k(\lambda_j^{(k)}) = 0$, $j = 1, \dots, k$, the following property of the roots separation holds

$$\lambda_{j+1}^{(k)} < \lambda_j^{(k-1)} < \lambda_j^{(k)}, \quad j = 1, \dots, k-1, \tag{6}$$

$\lambda_j^{(n)} = \lambda_j$. The situation, when some entries g_k of the matrix T are negative, is an object of our interest. In this case, the roots of the polynomials $P_j(\lambda)$ may be complex one. However, and that is the main result of this work, if the weights c_i with ordering (6) have one alternate of sign only, then $\lambda_j^{(k)}$, $j = 1, \dots, k$, are real. Here, separation (6) is also valid but in the weak form.

2. The positive definiteness of the matrix H is provided by the conditions $c_i > 0$, $i = 1, \dots, n$. Let us introduce the conditions $\lambda_i > 0$, $i = 1, \dots, n$ and transform the orthogonality weights:

$$\tilde{c}_i = a(\lambda_i - b)c_i, \quad i = 1, \dots, n. \tag{7}$$

If $\tilde{c}_i$ are positive, then we may construct the new system of the orthogonal polynomials $\tilde{P}_k(\lambda)$ on the set λ_i, $i = 1, \dots, n$. Let the matrices and all elements connected with this system be denoted by the symbol '$\sim$'. Almost all the correlations are valid for the new system. For example, the equations

$$\tilde{H} = V^T \tilde{C} V = aV^T(T - bE)CV$$

follow from $\tilde{C} = a(\Lambda - bE)C$. The representation $\tilde{H} = aP^{-T}(T - bE)DP^{-1}$ is also valid.

The classical theory of the orthogonal polynomials demands the positive definiteness of the matrix H and restricts the parameter b [4]:

$$b < \lambda_n \quad \text{or} \quad b > \lambda_1, \tag{8}$$

$$a = \frac{1}{h_1 - b}. \tag{9}$$

The transport from the continuum to the discrete set of points removes the restriction (8). But first let us obtain some correlations.

To find the matrices $\tilde{P}$ and $\tilde{D}$, let us take the triangular decomposition of the symmetric matrix $a(T - bE)D$:

$$a(T - bE)D = L^T \Delta L, \tag{10}$$

where $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_n)$ and

$$L = E + \sum_{k=1}^{n-1} l_k e_k e_{k+1}^T$$

is the upper bi-diagonal matrix. Hence,

$$\tilde{H} = P^{-T} L^T \Delta L P^{-1} = \tilde{P}^{-T} \tilde{D} \tilde{P}^{-1}.$$

The diagonal entries of the matrices L and P are equal to one, therefore $\tilde{P} = PL^{-1}$, $\tilde{D} = \Delta$. By definition (3), $\tilde{Y} = YL^{-1}$. Now we should determine the matrix $\tilde{T}$. Let us consider the representation

$$a(\tilde{T} - bE) = \Delta L D^{-1} L^T.$$

Lemma 1. *The entries of the matrix $\tilde{T}$ have the representation*

$$\begin{aligned} \tilde{g}_k &= \frac{\delta_{k+1}}{\delta_k}, \quad k = 1, \dots, n-1, \\ \tilde{b}_k &= b_k + l_k - l_{k-1}, \quad k = 1, \dots, n, \end{aligned}$$

and, in addition, $l_0 = l_n = 0$,

$$\begin{aligned} \delta_k &= -a d_k \frac{P_k(b)}{P_{k-1}(b)}, \quad k = 1, \dots, n, \\ l_k &= -g_k \frac{P_{k-1}(b)}{P_k(b)}, \quad k = 1, \dots, n-1. \end{aligned}$$

The connection of the polynomials $P_k(\lambda)$ and $\tilde{P}_k(\lambda)$ is given by the following lemma:

Lemma 2. *The relations*

$$\begin{aligned} &P_k(\lambda) = \tilde{P}_k(\lambda) + l_k \tilde{P}_{k-1}(\lambda), \quad l_n = 0, \\ &\frac{P_k(b)}{P_{k-1}(b)} P_{k-1}(\lambda) + P_k(\lambda) = (\lambda - b)\tilde{P}_{k-1}(\lambda), \\ &\tilde{P}_{k-1}(\lambda) = \frac{d_k}{P_{k-1}(b)} \sum_{j=1}^{k} \frac{1}{d_j} P_{j-1}(b) P_{j-1}(\lambda), \quad k = 1, \dots, n, \end{aligned}$$

are valid.

3. Now we can investigate the dependence of the polynomial system $\tilde{P}_k(\lambda)$, $k = 0, \dots, n-1$, from b. Let us introduce the intervals:

$$I_j^{(k)} = \left(\lambda_{j+1}^{(k)}, \lambda_j^{(k)}\right), \qquad \tilde{I}_j^{(k)} = \left(\tilde{\lambda}_{j+1}^{(k)}, \tilde{\lambda}_j^{(k)}\right), \qquad j = 0, \dots, k,$$

where $(\lambda_{k+1}^{(k)}, \lambda_0^{(k)})$ is some interval containing all the roots of the polynomial $P_k(\lambda)$ and the point b. The interval $(\tilde{\lambda}_{k+1}^{(k)}, \tilde{\lambda}_0^{(k)})$ has the same meaning. By the definition, $I_j^{(n)} = I_j$. Thus, such a value l, provided

$$b \in I_l, \quad l = 0, \dots, n, \tag{11}$$

always exists. It follows from (7) that the sequence $\tilde{c}_i$, $i = 1, \dots, n$, has no sign alternate if $b \in I_0$ or $b \in I_n$, and one sign alternate, otherwise. The polynomials $\tilde{P}_k(\lambda)$ are orthogonal if the sign alternate in the sequence $\tilde{c}_i$ is missing.

Lemma 3. *Under the condition*

$$P_k(b) \neq 0, \quad k = 1, \dots, n, \tag{12}$$

the matrix $\tilde{H}$ is strongly non-singular (i.e., all the main successive minors are non-zero). Its triangular decomposition (4) *takes place in this case, only.*

Theorem 1. *Let $b \in I_m^{(k)}$, $m = 0, \dots, k$, and conditions* (12) *hold. Then one of two groups of inequalities:*

$$\tilde{\lambda}_{j+1}^{(k)} < \tilde{\lambda}_j^{(k-1)} < \tilde{\lambda}_j^{(k)}, \quad j = 1, \dots, m-1, \tag{13}$$

$$\tilde{\lambda}_{j+2}^{(k)} < \tilde{\lambda}_j^{(k-1)} < \tilde{\lambda}_{j+1}^{(k)}, \quad j = m, \dots, k-1, \tag{14}$$

if

$$P_{k-1}(b) P_k(b) < 0; \tag{15}$$

or

$$\tilde{\lambda}_j^{(k)} < \tilde{\lambda}_j^{(k-1)} < \tilde{\lambda}_{j-1}^{(k)}, \quad j = 1, \dots, m, \tag{16}$$

$$\tilde{\lambda}_{j+1}^{(k)} < \tilde{\lambda}_j^{(k-1)} < \tilde{\lambda}_j^{(k)}, \quad j = m+1, \dots, k-1, \tag{17}$$

if

$$P_{k-1}(b) P_k(b) > 0 \tag{18}$$

is valid. Moreover, $b \in \tilde{I}_{m-1}^{(k-1)}$ in condition (15) *and $b \in \tilde{I}_m^{(k-1)}$ in condition* (18).

Definition 1. Let relations (13), (14) and (16), (17) be called the weak separation of the roots of the polynomials $\tilde{P}_{k-1}(\lambda)$, $\tilde{P}_k(\lambda)$.

Corollary 1. *From the weak separation of zeroes it follows, in particular, that all zeroes of the polynomials* $\tilde{P}_k(\lambda)$ *are real and simple.*

Definition 2. Let $l \neq 0$, $l \neq n$, and conditions (11), (12) hold. Then the polynomials $\tilde{P}_k(\lambda)$, $k = 0, \ldots, n-1$, are called pseudo-orthogonal on the set of points λ_j with the weights $\tilde{c}_j$, $j = 1, \ldots, n$.

Theorem 2. *Let* $b \in I_m^{(k)}$, $m = 0, \ldots, k$. *Then*

$$\begin{aligned} &\tilde{\lambda}_{j+1}^{(k)} < \lambda_j^{(k)} < \tilde{\lambda}_j^{(k)}, \quad j = 1, \ldots, m-1, \\ &\tilde{\lambda}_{m+1}^{(k)} < \lambda_{m+1}^{(k)} < \lambda_m^{(k)} < \tilde{\lambda}_m^{(k)}, \\ &\tilde{\lambda}_j^{(k)} < \lambda_j^{(k)} < \tilde{\lambda}_{j-1}^{(k)}, \quad j = m+2, \ldots, k. \end{aligned}$$

Theorem 3. *If under ordering* (6) *in the sequence* $\tilde{c}_i \neq 0$, $i = 1, \ldots, n$, *there is one sign alternate only, the system of polynomials* $\tilde{P}_k(\lambda)$, $k = 0, \ldots, n-1$, *is pseudo-orthogonal.*

Remark. It follows from Theorems 1 and 3 that pseudo-orthogonality (or orthogonality) of polynomials takes place if and only if there is only one sign alternate (or there is no sign alternate) in the sequence $c_1, \ldots, c_n$ with ordering (6). In this case, there is only one interval from $I_j^{(k)}$, $j = 1, \ldots, k-1$ (or there is no such interval at all), which does not have a root of the polynomial $P_{k-1}(\lambda)$.

Thus, one-to-one correspondence between the polynomials $P_k(\lambda)$ and $\tilde{P}_k(\lambda)$, $k = 0, \ldots, n-1$, holds if there is at most one sign alternate in the sequence $\tilde{c}_i$, $i = 1, \ldots, n$. This fact is stated in the following lemma:

Lemma 4. *The inequality*

$$P_k(b)\tilde{P}_k(b) > 0, \quad k = 0, \ldots, n,$$

holds.

Corollary 2. *In Theorem 1, conditions* (15) *and* (18) *can be represented in the equivalent form*

$$\tilde{P}_{k-1}(b)\tilde{P}_k(b) < 0 \quad \text{and} \quad \tilde{P}_{k-1}(b)\tilde{P}_k(b) > 0.$$

References

[1] Sege G. The Orthogonal Polynomials. – Moscow: Fizmatgiz, 1962 (in Russian).

[2] Belov B.I. The elements of algebraic combinatorics (orthogonality and codes): Second Doctoral Thesis. – Irkutsk: Sib. Power Inst. Sibirean Branch, Ac. Sci. USSR, 1990 (in Russian).

[3] Levenshtain V.I. Boundaries for metric spaces envelopes and some applications // Problems of cybernetics. – Moscow: Nauka, 1983. – № 40. – P. 43–110 (in Russian).

[4] Kuznetsov Yu.I. The connection of positive definite matrices and the Hessenberg matrices // Sib. Math. J. – Nauka, 1986. – № 2. – P. 94–100 (in Russian).