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The generator of RK-scheme GRKSCHEM*

Yu.I. Kuznetsov

The generator of M-stage modified RK-scheme, $2 < M < 21$, for solving the ODE $y' = f(t, y)$ with initial conditions and the Volterra integral equation (by the RK-method of advanced accuracy) is proposed. Modified and classical RK-methods differ for a triangular matrix B only are used.

1. Generator

```
call GRKSCHEM(A, B1, B, M, MB, METHOD, L, Um, Const)
```

Arguments:

`A` – the array of nodes in the interval $[0, 1]$. It is an input only at `MB` = `M`. in this case its elements must be different. In other cases `A` is also output.

`B1` – the weights of RK-method (output).

`B` – the quasi-weights of the quasi-solution (output).

`M` – the number of stages (input); $2 <$ `M` < 21.

`MB` – the approximation order; `MB` = `M`, `M` + 1 if `METHOD` > 1 and `MB` = 2`M`, 2`M` − 1, 2`M` − 2 if `METHOD` = 0, 1 (input).

`METHOD` – determines the RK-method parameter (input):

- If `METHOD` = 0 or 1, the RK-scheme of order `MB` of the RK-method of advanced accuracy is formed; in this case the matrix `B` is non-singular (`METHOD` = 0) or singular (`METHOD` = 1);
- If `METHOD` > 1, the nilpotent RK-scheme is formed. In this case `Um` is `M`-multiple eigenvalue for the matrix `B`. If `MB` = `M` + 1, the subroutine gives `Um` = `Um`*. If `MB` = `M`, then `Um` is input. If $c =$ `Um`/`Um`* is outside the interval $[a, 1.5]$, then `Um` is corrected; $a = 0.1$ except the case of `METHOD` = 2, where $a = 0$;
- `METHOD` = 2: the RK-scheme of order `MB` of the modified DIRK-method (the MDIRK-method) is formed; for the MDIRK-method `Um` = λ_1; in this case the condition $C(1)$ holds; the array `A` is input

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only if MB = M; Um = λ_1 and if Um = 0, then the MDIRK-method is explicit;

- METHOD = 3: RK-scheme of order MB of the collocation method is formed; the condition C(M) holds;
- METHOD = 4: RK-scheme of order MB of the transformed method is formed; the condition C(M − 1) holds;
- METHOD = 5: RK-scheme of order MB of the new transformed method is formed; the condition C(M − 1) holds;
- METHOD = 6: SIRK-scheme of order MB with arbitrary nodes is formed; in this case the condition C(M − 1) holds; the array A is input.

L – the parameter of singularity in the RK-scheme of advanced accuracy: if L = 0, then the nodes matrix diag(A) is nonsingular and it is singular if L = 1 (input only if METHOD = 0 or 1).

Um – M-multiple eigenvalue for the matrix B of the nilpotent RK-method (METHOD > 1); Um is input for MB = M.

Const – the parameter of the transformed and the new transformed methods (METHOD = 4 or 5); by choosing this parameter one can form RK-scheme with $\lambda_M > 1$ for a forward-jump strategy.

Remarks. Subroutine GRKSCHEM writes into the file SCHEME the data about the generated RK-scheme and some control data also:

1. The string "Approximation error = E_{M+1}" for METHOD = 0 or 1;
2. The string "Error of condition $C(1) = \delta$" if the difference of the sum of the elements of the i-th row of the matrix B and λ_i, $i = 1(1)m$, is computed with error $> 10^{-8}$;
3. The string "Error of weights = δ" if the difference δ of the sum of the elements of array B1 and 1 is computed with error $> 10^{-8}$.

2. Algorithms

In this section, we rename M to m and MB to M, respectively.

A discretization of an ordinary differential equation

$$\frac{\partial y}{\partial t} = f(t,y), \quad 0 \le t \le T, \quad y(0) = y_0,$$

is the system of nonlinear algebraic equations

$$\eta_0 = y_n, \quad \eta_i = y_n + \tau \sum_{j=1}^{m} \beta_{ij} f_j, \quad i = 1(1)m+1, \quad y_{n+1} = \eta_{m+1}$$

with the notations: $f_j = f(\xi_j, \eta_j)$,

$$\xi_i = t_n + \lambda_i \tau, \quad i = 0(1)m+1,$$
$$\lambda_0 = 0, \quad \lambda_{m+1} = 1, \quad \lambda_j \le \lambda_{j+1},$$

which can be presented in the vector form

$$\eta = y_n e + \tau B f, \quad y_{n+1} = y_n + \tau b_{m+1} f,$$
$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \dots & \dots & & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mm}, \end{pmatrix}$$
$$b_i = (\beta_{i1}, \dots, \beta_{im}), \quad i = 1(1)m+1,$$
$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m), \quad h = \operatorname{diag}(1, 1/2, \dots, 1/m),$$
$$\eta = (\eta_1, \dots, \eta_m)^T, \quad f = (f_1, \dots, f_m)^T,$$
$$g = (g_1, \dots, g_m)^T, \quad e = (1, \dots, 1)^T.$$

The modified RK-method generates M-stage RK-scheme of order $\ge M$, because it is based on the relation of weak approximation:

$$b_{m+1} B^{j-1} \frac{d^{j-1}}{dt_n^{j-1}} \left(\frac{d\eta}{dt_n} - f \right), \quad j = 1(1)N-1,$$

where the number N is the weak approximation order.

1. Approximation order of the nilpotent RK-method is determined by the choice of the parameter μ: if the index of nilpotent matrix $B - \mu E$ equals m, then the condition

$$L_{m+1}^{(1)} \left(\frac{1}{\mu} \right) = 0$$

is necessary and sufficient for the approximation order $M = m+1$.

For the triangular matrix B, the author has proposed the following algorithm [1-5]. At first it is necessary to define the values θ_{ij}:

$$\sum_{i=1}^{j} \theta_{ij} \lambda_i^k = g_k^j,$$

$k = 0(1)j-1$, $j = 1(1)m$, where

$$g_k^j = \frac{k!(m-j)!}{(m-j+k+1)!} \mu^{m-j} L_{m-j+k+1}^{(k+1)} \left(\frac{1}{\mu} \right),$$

with the $(k+1)$-th derivative of the Laguerre polynomials of order $m-j+k+1$ in the right-hand side. For a fixed j, these relations define a system of equations with the Vandermonde matrix V_j of order j,

$$V_j = \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{j-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{j-1} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_j & \dots & \lambda_j^{j-1} \end{pmatrix}.$$

In notations

$$c_{j,0} = 1, \quad c_{j,i} = c_{j-1,i} - \lambda_j c_{j-1,i-1}, \quad i = 1(1)j, \quad c_{j-1,j} = 0,$$

the relation

$$\lambda_j = \frac{\sum_{l=0}^{j-1} c_{j-1,l} g_{j-1}^{j+1}}{\sum_{l=0}^{j-1} c_{j-1,l} g_{j-l-1}^{j+1}}, \tag{1}$$

$j = 1(1)m-1$, is critical for the choice of λ_i.

Theorem 1. *For the triangular nilpotent matrix $B - \mu E$ with single λ_i, $i = 1(1)m$, to have the approximation order $M \geq m$, it is necessary and sufficient that*

1) *condition* (1) *be violated for all j;*

2) $L_m^{(1)}(1/\mu) \neq 0$.

So,

$$w_{m,j-i}^{m-j} = \left(\theta_{j-i,j} - \sum_{l=m-i}^{m-1} \beta_{m+1,l} w_{l,j-i}^{m-j}\right) \beta_{m+1,m},$$

$j = i + 1(1)m - 1$;

$$w_{m-l-1,j-i}^{m-j-l-1} = \left(w_{m-l,j-i}^{m-j-l} - \sum_{k=m-i-l-1}^{m-l-2} \beta_{m-l,k} w_{k,j-i}^{m-j-l-1}\right) \Big/ \beta_{m-l,m-l-1},$$

$j = i + 1(1)m - l - 2$, $l = 0(1)m - i - 3$, $i = 0(1)m - 2$, where, usually, it is assumed that the sum is equal to zero if the upper limit is less than the lower one.

Here w_{kl}^j is an element at the intersection of the k-th row and the l-th column of the matrix $(B - \mu E)^j$.

The solution is attained stage by stage. At first, the stage $i = 0$ is carried out:

$$w_{m,j}^{m-j} = \theta_{jj}/\beta_{m+1,m}, \quad j = 1(1)m - 1,$$

$$w_{m-l-1,j}^{m-j-l-1} = w_{m-l,j}^{m-j-l}/\beta_{m-l,m-l-1}, \quad j = 1(1)m - l - 2, \; l = 0(1)m - 3.$$

Meanwhile the k-th diagonals of the matrices $(B - \mu E)^k$, $k = 1(1)m - 1$ are consecutively calculated. Therefore, at zero stage we have already calculated elements $\beta_{j,j-1}$, $j = m + 1(-1)2$.

Consider the i-th stage, $i = 1(1)m-2$. At the preceding stages, the values w_{lk}^j are determined. At the i-th stage, the elements w_{lk}^j should be calculated, i.e., the elements of the $(j+i)$-th diagonal of the matrix $(B-\mu E)^j, j = 1(1)m-i-1$. In particular, the $(i+1)$-th diagonal of the matrix $B-\mu E$ is calculated. Therefore, after i stages the first $i+1$ diagonals of the matrix $B-\mu E$, which differ from B by the main diagonal μE only, are determined.

Note, also, that $b_{m+1} = (\theta_{1m}, \ldots, \theta_{mm})$.

The RK-method with the triangular matrix B has the bad approximation in the nodes $\lambda_i < 1$, only $C(1)$.

2. In the SIRK-method, the matrix B is not triangular. This nilpotent method approximates the solution of the ODE in the nodes $\lambda_i \neq 1$ much better, than the condition $C(m-1)$ or $C(m)$:

$$\sum_{j=1}^{m} \beta_{ij} \lambda_j^{k-1} = \frac{1}{k} \lambda_j^k, \quad i = 1(1)m, \quad k = m-1 \quad \text{or} \quad k = m,$$

hold in this case. Hence, the solution of the ODE in the nodes $\lambda_i \neq 1$ is approximated with the order $m-1$ or m.

In the SIRK-method, the matrices Λ and V must be nonsingular. The SIRK-method is determined by the polynomial

$$\pi_m(\lambda) = m!\,\mu^m L_m(\lambda/\mu) + cL_{m-1}(\lambda/\nu), \tag{2}$$

with the parameters μ, nu, and c, where mu is m-multiple eigenvalue of the matrix B.

The nodes λ_i, $i = 1(1)m$, are the roots of the polynomial $\pi(\lambda)$. Let

$$L_{m-1}(\lambda/\nu) = \sum_{i=1}^{m} z_i \lambda^{i-1}$$

and $z = (z_1, \ldots, x_m)^T$. Then

$$BV = (\Lambda V + V z e_m^T)h,$$

e_m is the last column of the identity matrix, and

$$b_{m+1} V = e^T h.$$

This general case is proved by the author [2, 4–5]. If $c = 0$, then we have the collocation RK-method. The condition $\nu = \mu$ gives the transformed RK-method [6].

The approximation order M is equal to $m+1$ if and only if the condition

$$L_{m+1}^{(1)}(1/\mu) = cL_m^{(1)}(1/\nu) = 0 \tag{3}$$

holds. Hence for the transformed RK-methods $M = m$, as the orthogonal polynomials $L^{(1)}_{m+1}(1/\mu)$ and $L^{(1)}_m(1/\mu)$ have no common roots.

In the new transformed RK-method, $\nu \neq \mu$ and conditions (3) hold at $c \neq 0$. In this case $M = m + 1$.

In the collocation RK-method ($c = 0$) $\lambda_i = \mu\mu_i$, where μ_i are defined, the condition

$$L_m(\mu_i) = 0, \quad i = 1(1)m. \tag{4}$$

For the transformed and the new transformed RK-methods we can use the following conclusion [5, 7]:

Theorem 2. *Let two polynomials $P_m(\lambda)$, $P_{m-1}(\lambda)$, of orders m and $m-1$, respectively, with real single roots μ_i, $i = 1(1)m$, ν_i, $i = 1(1)m-1$ ordered by ascendance be specified, i.e., $\mu_i < \mu_{i+1}$, $i = 1(1)m-1$, $\nu_i < \nu_{i+1}$, $i = 1(1)m-2$. Assume in addition that the roots ν_i separate the roots μ_i, i.e.,*

$$\mu_i < \nu_i < \mu_{i+1}.$$

Then the polynomial $R_m(\lambda)$ of order m,

$$R_m(\lambda) = P_m(\lambda) + cP_{m-1}(\lambda)$$

where c is an arbitrary real constant, has real single roots λ_i, $i = 1(1)m$, separated by the roots ν_i, i.e., $\lambda_i < \nu_i < \lambda_{i+1}$.

In the transformed RK-method ($\nu = \mu$), the value λ_i satisfies, by Theorem 2, the inequalities

$$\lambda_i < \mu\mu_i < \lambda_{i+1}, \quad i = 1(1)m-1.$$

For the orthogonal polynomials we have

$$\mu_i < \nu_i < \mu_{i+1}, \quad i = 1(1)m-1,$$

if together with (4) $L_{m-1}(\nu_i) = 0$, $i = 1(1)m-1$.

However, the choice of ν must be careful. For the new transformed RK-method let us put

$$p_{m-1}(\lambda/\mu) = L_{m-1}(\lambda/\nu) = L_{m-1}(\frac{\mu}{\nu}\lambda/\mu).$$

If $P_{m-1}(\rho_i) = 0$, $i = 1(1)m-1$, then $\nu\nu_i = \mu\rho_i$.

Hence, according to Theorem 2,

$$\mu_i < \frac{\mu}{\nu}\rho_i = \nu_i < \mu_{i+1}, \quad i = 1(1)m-1,$$

for any values ν, $\mu > 0$. But we must choose ν/μ so that the correlation

$$\mu_i < \rho_i = \frac{\nu}{\mu}\nu_i < \mu_{i+1}, \quad i = 1(1)m-1,$$

holds. In this case, by Theorem 2, we have

$$\lambda_i < \frac{\nu}{\mu}\nu_i < \lambda_{i+1}, \quad i = 1(1)m-1, \tag{5}$$

i.e., real and single nodes λ_i.

Let, also, $L_{m+1}^{(l)}(\mu_i^{(l)}) = 0$, $L_m^{(l)}(\nu_i^{(l)}) = 0$. If

$$\mu^{(l)} = 1/\mu_m^{(l)}, \quad \nu^{(l)} = 1/\nu_{m-1}^{(l)}, \tag{6}$$

then the relation $\nu^{(l)}/\mu^{(l)}$ for $1 < m < 31$ can be calculated. The numerical experiment confirms relation (5).

The choice of (6) is very good, as that the accuracy order equals to $m+1$. Taking $\nu = \nu^{(l)}\mu/\mu^{(l)}$, we may generate the new transformed RK-method of order m.

Now let us consider the changing range of the parameter c. From (2) it follows that $\lambda_m = 1$ for

$$c^* = -\frac{L_m(1/\mu^{(l)})}{L_{m-1}(1/\nu^{(l)})}.$$

Hence, if $c < c^*$, then $\lambda_m > 1$. The forward-jump RK-method uses the condition $c < c^*$. As $\lambda_m > 1$, then $\xi_m = t\lambda_m\tau \in [t_{n+1}, t_{n+2}]$. This case can be used for the control of the numerical solution.

The stability functions of the nilpotent RK-methods have the form:

$$R(\tau) = \frac{\sum_{j=0}^{m}(\mu\tau)^j L_m^{(m-j)}(1/\mu)}{(1-\mu\tau)^m}.$$

In particular, $R(-\infty) = L_m(1/\mu)$. Since the explicit method is a special case of the diagonally implicit one for $\mu = 0$, its stability function $R(\tau)$ is the polynomial:

$$R(\tau) = \sum_{j=0}^{m}\frac{1}{j!}\tau^j,$$

i.e., a segment of the Taylor exponent expansion.

3. The advanced accuracy RK-method can be determined through the stability function

$$R(\tau) = \frac{N_{m-\theta,M-m+\theta}(\tau)}{D_{m-\theta,M-m+\theta}(\tau)},$$

where

$$D_{m-\theta,M-m+\theta}(\tau) = \frac{1}{\binom{M}{m-\theta}} \sum_{k=0}^{m-\theta} \frac{(-\tau)^k}{k!} \binom{M-k}{M-m+\theta},$$

$$N_{m-\theta,M-m+\theta}(\tau) = \frac{1}{\binom{M}{m-\theta}} \sum_{k=0}^{M-m+\theta} \frac{\tau^k}{k!} \binom{M-k}{m-\theta}.$$

The parameter $\theta = 0$ or 1 determines nonsingularity or singularity of the matrix B.

Another parameter $\omega = 0$ or 1 determines nonsingularity or singularity of the matrix Λ. The characteristic polynomial of the matrix Λ is the following:

$$\begin{aligned} \mathcal{P}_m(\lambda) &= \frac{1}{\binom{M}{m-\omega}} \sum_{j=0}^{m-\omega} (-1)^j \binom{m}{j} \binom{M-j}{m-\omega-j} \lambda^{m-j} \\ &= \frac{m!}{M!} \frac{d^{M-m}}{d\lambda^{M-m}} (\lambda^{M-m+\omega}(\lambda-1)^{m-\omega}). \end{aligned}$$

For $\omega = 0$, $M = 2m$, these polynomials are the Legendre ones.

For calculating the nodes λ_i, $i = 1(1)m$, it is convenient to use the following algorithm [3, 5]:

Theorem 3. *The Euclidean algorithm applied to the polynomials* $\psi_m(\lambda) = \mathcal{P}_m(\lambda)$ *and* $\psi_{m-1}(\lambda) = \mathcal{P}_m^{(1)}(\lambda)/m$, *brings about the sequence*

$$\psi_k(\lambda) = \sum_{j=0}^{k} (-1)^{k-j} \lambda^j \binom{k}{j} \binom{k+j-u+2}{j-\omega+1} \Big/ \binom{2k-u+2}{k-\omega+1},$$

$k = m-1(-10)$, $u = 2m - M$, *satisfying the recursive relation*

$$\psi_0(\lambda) = 1, \quad \psi_1(\lambda) = \lambda - \frac{2-\omega}{4-u},$$

$$\psi_k(\lambda) = (\lambda - U_k)\psi_{k-1}(\lambda) - V_k\psi_{k-2}(\lambda), \quad k = 2(1)m,$$

thereto,

$$U_k = \frac{2k(k-u+1) + \omega(u-2)}{(2k-u)(2k-u+2)},$$

$$V_k = \frac{(k-1)(k-\omega)(k-u+1)(k-u+\omega)}{(2k-u)^2((2k-u)^2-1)}, \quad k = m-1(-11),$$

$$U_m = \frac{m-\omega}{M}, \quad V_m = \frac{(m-1)(m-\omega)(M-m+\omega)}{(M-1)M^2}.$$

The approximation quality is defined by the expression for $E_{M+1} = 1 - (M+1)b_{m+1}\Lambda^M e$ [3, 5]:

$$E_{M+1} = \frac{(-1)^{M-\omega}}{\binom{M}{m}\binom{M}{m-\omega}}, \tag{7}$$

due to which the expression for the local truncation error has the form

$$l_{n+1} = E_{M+1}\tau^{M+1}.$$

It is easily seen from this expression that the m-stage Radau schemes-1 and Radau scheme-2 enclose the true solution from both sides (under assumption of a constant sign of the $(M+1)$-th derivative of y on the segment $[t_n, t_{n+1}]$). The same property is shared by $m-1$ stage Gauss scheme and m-stage Lobatto scheme.

For a non-degenerate matrix B ($\theta = 0$) the equalities

$$B\Lambda^{k-1}e = \frac{1}{k}\Lambda^k e, \quad k = 1(1)m-\omega,$$

hold, i.e., the condition $C(m-\omega)$ is satisfied.

Therefore, if $\omega = 0$, the condition $C(m)$ is transformed into the equality $BV = \Lambda Vh$, which uniquely defines the matrix B. There are only two RK-schemes for which $\theta = \omega = 0$: $M = 2m$ is the Gaussian scheme and $M = 2m-1$ is the Radau scheme-1, both being uniquely defined.

For a non-degenerate matrix B ($\theta = 0$) and a degenerate matrix Λ ($\omega = 1$) the following hold

$$\beta_{i1} = \frac{1}{m(M-m+1)}, \quad i = 1(1)m+1. \tag{8}$$

There are two RK-schemes satisfying condition (8), which possess the property $\theta = 0$ (and the condition $C(m-1)$), $\omega = 1$: $M = 2m-1$ - the Radau scheme-2 and $M = 2m-2$ - the Lobatto scheme-1. (The 20-stage Radau scheme-2 was generated by subroutine GRKSCHEM [9] and successfully used for solving the integral Volterra equation.) The generation of the matrix B has the same form as for the transformed RK-schemes.

The last possible case consists in the choice of $\omega = \theta = 1$. In this case $M = 2m-2$. As well as in the Lobatto scheme-1, these RK-schemes use the end-points of the integration step $\lambda_1 = 0$, $\lambda_m = 1$. If $\theta = \omega = 1$ and $M = 2m-2$, the equality

$$B\Lambda^{k-1}e = \frac{1}{k}\Lambda^k e, \quad k = 1(1)m,$$

takes place, if and only if the following conditions are satisfied:

1. The matrix $\bar{B}$ of the size $m-1$, obtained from B by deleting the first row and the first column out, is non-degenerate.
2. The first row of the matrix B is formed by zeros.

The RK-scheme with $\theta = \omega = 1$, $M = 2m-2$, for which the condition $C(m)$ is satisfied, will be called the Lobatto scheme-2.

3. Examples

The choice of the RK-scheme is determined by the parameters m, M, ω, θ. In subroutine GRKSCHEM we rename M to MB, m to M, ω to 1 and θ to METHOD, μ to Um, Λ to A, $b1$ to B1, respectively.

Example 1 (The MDIRK-method of order MB = M). M = 3, METHOD = 2, Um = 0.100,

$$\mathtt{B} = \begin{bmatrix} 0.100 & 0.000 & 0.000 \\ 0.400 & 0.100 & 0.000 \\ 0.064 & 0.736 & 0.100 \end{bmatrix}.$$

Weights B1 = (0.260, 0.479, 0.260); nodes A = (0.100, 0.500, 0.900).

Example 2 (The MDIRK-method of order MB = M+1). M = 3, METHOD = 2, Um = 0.129,

$$\mathtt{B} = \begin{bmatrix} 0.129 & 0.000 & 0.000 \\ 0.371 & 0.129 & 0.000 \\ 0.258 & 0.485 & 0.129 \end{bmatrix}.$$

Weights B1 = (0.303, 0.395, 0.302); nodes A = (0.129, 0.500, 0.871).

Example 3 (Collocation method of order MB = M + 1). M = 3, METHOD = 3, Um = 0.129,

$$\mathtt{B} = \begin{bmatrix} 0.062 & -0.009 & 0.001 \\ 0.170 & 0.131 & -0.005 \\ 0.046 & 0.571 & 0.194 \end{bmatrix}.$$

Weights B1 = (0.108, 0.444, 0.448); nodes A = (0.054, 0.296, 0.811).

Example 4 (Transformed method of order M + 1). M = 3, METHOD = 4, Um = 0.129, Const = −0.50,

$$\mathtt{B} = \begin{bmatrix} 0.068 & -0.008 & 0.001 \\ 0.196 & 0.143 & 0.004 \\ -0.571 & 1.560 & -0.031 \end{bmatrix}.$$

Weights B1 = (0.031, 0.690, 0.279); nodes A = (0.060, 0.335, 0.958).

Example 5 (Transformed method of order M). M = 3, METHOD = 4, Um = 0.100, Const = −1.00,

$$\mathtt{B} = \begin{bmatrix} 0.056 & -0.007 & 0.000 \\ 0.163 & 0.119 & -0.003 \\ -0.662 & 1.560 & -0.026 \end{bmatrix}.$$

Weights B1 = (0.007, 0.618, 0.375); nodes A = (0.049, 0.279, 0.872).

Example 6 (Transformed method of order M + 1). M = 3, METHOD = 4, Um = 0.129, Const = −1.00,

$$B = \begin{bmatrix} 0.072 & -0.008 & 0.000 \\ 0.210 & 0.153 & -0.003 \\ -0.854 & 2.011 & -0.034 \end{bmatrix}.$$

Weights B1 = (−0.013, 0.835, 0.179); nodes A = (0.064, 0.360, 1.123).

Example 7 (The new transformed method of order M+1). M = 3, METHOD = 5, $MB = 3$, Um = 0.129, Const = −1.00,

$$B = \begin{bmatrix} 0.092 & -0.014 & 0.001 \\ 0.239 & 0.149 & -0.003 \\ -0.098 & 0.939 & -0.003 \end{bmatrix}.$$

Weights B1 = (0.191, 0.427, 0.382); nodes A = (0.080, 0.385, 0.839).

Example 8 (The new transformed method of order M+1). M = 3, METHOD = 5, MB = 3, Um = 0.129, Const = −5.00,

$$B = \begin{bmatrix} 0.122 & -0.018 & 0.003 \\ 0.348 & 0.306 & -0.029 \\ 0.007 & 1.245 & -0.105 \end{bmatrix}.$$

Weights B1 = (0.305, 0.632, 0.064); nodes A = (0.108, 0.624, 1.147).

Example 9 (SIRK with arbitrary nodes of order M). M = 3, METHOD = 6, Um = 0.013,

$$B = \begin{bmatrix} 0.120 & -0.027 & 0.007 \\ 0.287 & 0.240 & -0.026 \\ 0.253 & 0.506 & 0.141 \end{bmatrix}.$$

Weights B1 = (0.260, 0.479, 0.260); nodes A = (0.100, 0.500, 0.900).

Example 10 (SIRK with arbitrary nodes of order MB = M + 1). M = 3, METHOD = 6, Um = 0.129,

$$B = \begin{bmatrix} 0.120 & -0.027 & 0.007 \\ 0.287 & 0.240 & -0.026 \\ 0.253 & 0.506 & 0.141 \end{bmatrix}.$$

Weights B1 = (0.260, 0.479, 0.260); nodes A = (0.100, 0.500, 0.900).

Example 11 (Superprecise RK-method: The Gauss method). L = 0, METHOD = 0, M = 3, MB = 6, approximation error $0.3 \cdot 10^{-2}$,

$$\mathtt{B} = \begin{bmatrix} 0.139 & -0.036 & 0.010 \\ 0.300 & 0.222 & -0.023 \\ 0.268 & 0.480 & 0.139 \end{bmatrix}.$$

Weights B1 = (0.278, 0.444, 0.278); nodes A = (0.113, 0.500, 0.887).

Example 12 (Advanced accuracy: The Radau 1-method (Radau II)). L = 0, METHOD = 0, M = 3, MB = 5, approximation error = −0.01,

$$\mathtt{B} = \begin{bmatrix} 0.197 & -0.066 & 0.024 \\ 0.394 & 0.292 & -0.042 \\ 0.376 & 0.513 & 0.111 \end{bmatrix}.$$

Weights B1 = (0.376, 0.513, 0.111); nodes A = (0.155, 0.645, 1.000).

Example 13 (Advanced accuracy: The Radau 2-method (Radau I)). L = 1, METHOD = 0, M = 3, MB = 5, approximation error = 0.01,

$$\mathtt{B} = \begin{bmatrix} 0.111 & -0.192 & 0.081 \\ 0.111 & 0.292 & -0.048 \\ 0.111 & 0.537 & 0.197 \end{bmatrix}.$$

Weights B1 = (0.111, 0.513, 0.376); nodes A = (0.000, 0.355, 0.845).

Example 14 (Advanced accuracy: The Lobatto 1-method (Lobatto C)). L = 1, METHOD = 0, M = 3, MB = 4, approximation error = −0.04,

$$\mathtt{B} = \begin{bmatrix} 0.167 & -0.333 & 0.167 \\ 0.167 & 0.417 & -0.083 \\ 0.167 & 0.667 & 0.167 \end{bmatrix}.$$

Weights B1 = (0.167, 0.667, 0.167); nodes A = (0.000, 0.500, 1.000).

Example 15 (Advanced accuracy: The Lobatto 2-method (Lobatto A)). L = 1, METHOD = 1, M = 3, MB = 4, approximation error = −0.04,

$$\mathtt{B} = \begin{bmatrix} 0.000 & 0.000 & 0.000 \\ 0.208 & 0.333 & -0.042 \\ 0.167 & 0.667 & 0.167 \end{bmatrix}.$$

Weights B1 = (0.167, 0.667, 0.167); nodes A = (0.000, 0.500, 1.000).

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