Bull. Nov. Comp. Center, Num. Anal., 2 (1993), 37–44

Generation of dynamic Delaunay triangulations

A. Y. Kouznetsov

Non-stationary problems of finite element solutions require an efficient generation of sequential meshes with minor changes in density functions of points distribution. The Delaunay meshes sequential generation is studied in terms of successive insertions and removings of points following changes in density function.

Key words. finite element method, Delaunay tesselation, triangulation.

AMS(MOS) subject classification. 65N30

1. Introduction

Delaunay meshes (tessellations) [1,2] are widely used for solution of mathematical physics problems by finite element method. There are two main reasons of the popularity: some optimal geometrical properties [3,4], and easy implementations of the Delaunay tessellation generation for n-dimensional space [5,6].

In a simplest case mesh generation is a geometrical approximation of some region Ω for step h. When h is a function in Ω, a mesh is called *a refined mesh* related with the density function $h(\bar{x})$, $\bar{x} \in \Omega$. In case function $h(\bar{x})$ is defined by some solution process for Ω, the mesh is called *an adaptive mesh.*

Following [7], here we introduce a kind of mesh called herein as *a dynamic mesh.* The dynamic mesh is different from previously mentioned ones and is defined as a sequence of meshes with different density functions $h_i(\bar{x})$, $i = 1, \ldots, N$. For each $h_i(\bar{x})$ the related mesh DT_i:

$$DT_i = DT(P(h_i(\bar{x})))$$

can be refined or adaptive, and $P(h_i(\bar{x}))$ is a set of points in Ω, related with the density function $h_i(\bar{x})$.

The approach proposed below for *dynamic Delaunay meshes* consists of deriving DT_i from DT_{i-1} by successive local modifications (instertions and removings of points). The statement of proved Theorem 1 is most important, it shows the way to reduce the number of points in the Delaunay tessellation.

2. Delaunay tessellation and some properties

Let P be any arbitrary set of points in n-dimensional Euclidean space. Let P satisfy two following conditions:

P1) the value $r > 0$ exists, that for a ball $B(r,p)$ of radius r for any point p from P as a center, the ball $B(r,p)$ doesn't include any other point from P, i.e., P is a discrete set.

P2) the value $R > 0$ exists, that a ball $B(R,x)$ of radius R for any point x of n-dimensional space mentioned above, includes one point from P at least.

A series $\{1/n\}$ and $\{n^2\}$ for $n = 1, 2, \ldots$ are examples of set P in R^1 with violation of conditions P1) and P2) respectively.

Here below we will assume points from P not lying entirely in a hyperplane.

A n-polytope is a convex hull of disjoint m points ($m > n$, $n > 1$) that does not lie entirely in a hyperplane.

Definition 1. Two n-polytopes with vertices from P are disjoint if there are no common points for their interiors.

Definition 2. Tessellation $T(P)$ of n-dimensional space on polytopes is a set of not-intersected polytopes with vertices from P, that fill entire space, can be adjacent pairwise by whole $n-1$ dimension faces only and any point from P is a vertex of some polytope from $T(P)$.

Definition 3. A B-polytope is a convex polytope with vertices from P that a circumball exists having all vertices on its surface.

Definition 4. A Delaunay tessellation is a tessellation of B-polytopes, that circumball of any polytope does not include points of P in the interior.

The existence and uniqueness of the Delaunay tessellation is stated by

Lemma 1 [1,2]. *The Delaunay tessellation $DT(P)$ is entirely defined by set P, and vice versa – P is entirely defined by $DT(P)$.*

The following "tessellation main lemma" can be proved [1,2].

Lemma (Delaunay). *Tessellation $T(P)$ is a Delaunay tessellation $DT(P)$ if and only if:*

1) *all polytopes of tessellation are B-polytopes,*
2) *for any two polytopes, adjacent by common face of dimension $n-1$, the vertices of the polytopes do not belong to interior of circumball of adjacent polytope.*

The definitions above can be easily adopted for a case of finite set of points P in the Euclidean space. In the case, we will assume below to denote a convex hull $C(P)$ for points from P as entire space. So we'll mean a tessellation of a space as a tessellation of $C(P)$. It easily can be checked that condition P2) always will be valid, and condition P1) will be satisfied if P includes distinct points only.

Let us recall here two useful lemmas for spheres intersections [4].

Lemma 2. *Let E_1 and E_2 be two distinct n-spheres in R^n $(n \geq 2)$, intersecting in a set E_{12} consisting of more than a single point. Then E_{12} is an $(n-1)$-sphere contained in uniquely determined hyperplane h_{12}.*

Lemma 3. *With E_1 and E_2 – two spheres from Lemma 2, let B_1 and B_2 denote corresponding open balls. Then on one side of hyperplane h_{12}, B_1 will include B_2, and on another side it will be vice versa.*

Figure 1 illustrates it for $n = 2$.

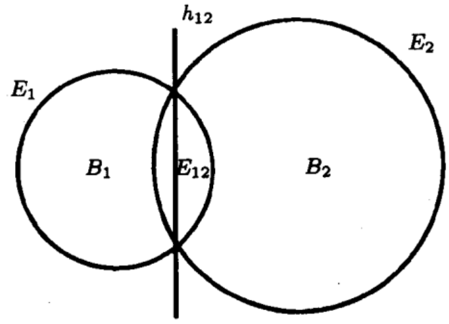

Figure 1. Illustration for Lemma 3

3. Delaunay tessellation generation algorithm

To simplify explanation, let us use the following notations for operations with sets:

$$\begin{aligned} A+B=B+A &\quad \text{means} \quad A\cup B, \\ A-B &\quad \text{means} \quad A\backslash B, \\ A+q &\quad \text{means} \quad A+\{q\}, \\ A-q &\quad \text{means} \quad A-\{q\}. \end{aligned}$$

Watson's algorithm [5] for generation of the n-dimensional Delaunay tessellations is widely useed for practical implementations (for instance, [6]).

Let P be a finite set of points to be used for generation of Delaunay tessellation $DT(P)$ and let Π be an arbitrary B-polytope, that contains all points of P in its interior. The polytope Π is used as initial tessellation for generation of $DT(P)$. Watson's algorithm proceeds in successive insertion of points from P into $DT(P)$, starting from the first one. Any time a new point is to be inserted in $DT(P)$, local modification of $DT(P)$ around the point needs to be done.

Definition 5. Modification set $M(q)$ for tessellation $DT(P)$ is a set of polytopes from $DT(P)$, that contains point q in related circumballs.

Definition 6. Modification area $\widetilde{M}(q)$ is a polytope formed by a union of B-polytopes from $M(q)$.

For a set of points P with no $n+1$ points on the same n-sphere for n-dimensional space it was proved in [5] that

1) there are no already inserted points in the interior of modification area,
2) any vertex of $\widetilde{M}(q)$ can be connected with point q by straight line lying within the modification area,
3) resulting tessellation of modification area $\widetilde{M}(q)$ is a Delaunay tessellation $DT(\widetilde{M}(q))$,
4) replacement of polytopes from $M(q)$ in $DT(P)$ by polytopes from $DT(\widetilde{M}(q))$ will form a set of polytopes

$$DT(P+q)=DT(P)-M(q)+DT(\widetilde{M}(q))$$

that is a Delaunay tessellation for $P+q$ set of points.

Let the function

$$F_q : DT(P) \to DT(P + q)$$

denote modification of the Delaunay tessellation for instertion of point q. So the Delaunay tessellation generation according to this approach is a sequential implementation of the function for points from the set P.

4. Removing points from Delaunay tessellation

Let $DT(P)$ be a Delaunay tessellation for some set of points P. Let's now consider the problem of removing some point q from P and related changes in $DT(P)$, i.e., the problem of reducing Delaunay tessellation $DT(P - q)$ from $DT(P)$.

If $N(q)$ is a set of B-polytopes from $DT(P)$ that includes point q as a vertex, we introduce

Definition 7. The set $N(q)$ is called a neighbours set for point q (Figure 2). As we did it with $\widetilde{M}(q)$ let us also introduce a neighbours area $\widetilde{N}(q)$.

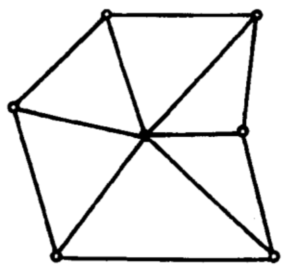

Figure 2. Neighbours set

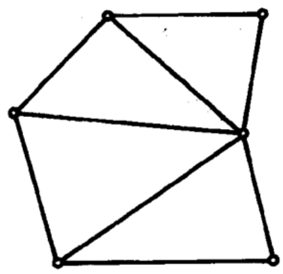

Figure 3. Neighbours area

The Delaunay tessellation of the neighbours area $\widetilde{N}(q)$ can be denoted as $DT(\widetilde{N}(q))$ (Figure 3).

Theorem 1. *For any point of the set P a tessellation*

$$DT(P - q) = DT(P) - N(q) + DT(\widetilde{N}(q))$$

of $P - q$ set of points is a Delaunay tessellation.

Proof. As far as any polytope t from $DT(P) - N(q)$ belongs to $DT(P)$, it is easy to see that t will satisfy the conditions of Delaunay Lemma for $P - q$ set.

As far as $DT(\widetilde{N}(q))$ is the Delaunay tessellation of the neighbours area the conditions of Delaunay Lemma are valid for polytops from $DT(\widetilde{N}(q))$ and points from P that are the nodes of the neighbours area $\widetilde{N}(q)$.

So, now we need to verify the condition 2) of Delaunay Lemma for any polytope w from $DT(\widetilde{N}(q))$ and any node Q from $P - \widetilde{N}(q)$ that belongs to adjacent polytope W from $DT(P) - DT(\widetilde{N}(q))$ (Figure 4).

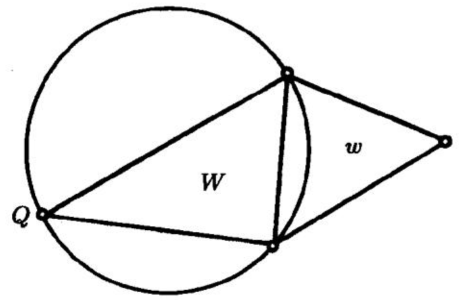

Figure 4. Adjacent polytopes

It easy follows from Lemma 3. □

Lemma 4 follows from the definitions of modification set $M(q)$ for $DT(P)$ and the neighbours set $N(q)$ for $DT(P+q)$:

Lemma 4. *The following equations are valid*

$$DT(\widetilde{M}(q)) = N(q),$$
$$DT(\widetilde{N}(q)) = M(q).$$

Let a mapping

$$F_q^{-1} : DT(P) \to DT(P - q)$$

correspond to transformation of the initial Delaunay tessellation for point q to be removed. Then it follows from Lemma 4 and definitions for F_q and F_q^{-1}:

$$(F_q^{-1} \cdot F_q) \cdot DT(P) = DT(P).$$

For a fixed and enumerated set of points Q, addition of points from Q into tessellation $DT(P)$ we'll denote as $F(Q) \cdot DT(P)$:

$$F(Q) = \prod_{i=n}^{1} F_i = F_n \cdot F_{n-1} \cdot \ldots \cdot F_1,$$

where $F_i = F_{q_i}, q_i \in Q$. In the same way let us denote $F^{-1}(Q)$:

$$F^{-1}(Q) = \prod_{i=1}^{n} F_i^{-1} = F_1^{-1} \cdot F_2^{-1} \cdot \ldots \cdot F_n^{-1}.$$

Then

$$(F^{-1}(Q) \cdot F(Q)) \cdot DT(P) = DT(P).$$

As far as we defined $F(Q)$ and $F^{-1}(Q)$ as insertion and removing points from set Q and $DT(P)$, so following Lemma 1 it is clear to see that previous equations are valid. It also follows that the order of points in Q is not important for $F(Q)$ and $F^{-1}(Q)$, i.e.,

$$F_i \cdot F_j = F_j \cdot F_i$$

and

$$F_i^{-1} \cdot F_j^{-1} = F_j^{-1} \cdot F_i^{-1}.$$

5. Dynamic triangulations generation

Generation of dynamic triangulation for domain Ω, i.e., a sequence of triangulations with different density functions can be done by successive use of transformations F and F^{-1} for each step of changing triangulation. It is also possible to change domain Ω in a time of changing triangulation with related changes to be done for transformation F.

In [5] a Delaunay tessellation speed estimation as $O(N \cdot \log(N))$ was proved for N to be a number of nodes in tessellation.

It follows from points removing process above and Theorem 1 that without estimations of pre-processing time the estimation for removing K points from a Delaunay tessellation will be $O(K)$.

Thus, if transformation from DT_i to DT_{i-1} will require removing $K1$ points and addition of $K2$ points, then generation of DT_i can be implemented at $O(K1 + K2 \cdot \log(K2))$ time, so this is in practice much faster than DT_i generation from scratch without use of information from DT_{i-1}.

The approach proposed above shows an efficient way of generation of dynamic Delaunay tessellations (triangulations) and can be widely used for solving non-stationary problems of mathematical physics by finite element method.

References

[1] B.N. Delaunay, Sur la sphere vide, Izvestia Akad. Nauk SSSR: Seria Matemat. i Estestv. Nauki, 6, 1934, 793–800 (in French).

[2] B.N. Delaunay, Petersbourg number theory school, USSR Academy of Sciences Publishing, Moscow – Leningrad, 1947, 268–318 (in Russian).

[3] R. Sibson, Locally equiangular triangulations, Comp.J., Vol. 21, No. 3, 1978, 243–245.

[4] C.L. Lawson, Properties of n-dimensional triangulations, Comp. Aided Geom. Design, Vol. 3, 1986, 231–246.

[5] D.F. Watson, Computing n-dimensional Delaunay tessellation with application to Voronoi polytopes, Comp.J. Vol. 24, No. 2, 1981, 167–172.

[6] A.Y. Kouznetsov, Algorithms for $2D$ generation of Delaunay triangulation, Preprint 909, Vychisl. Tsentr SO AN SSSR, Novosibirsk, 1990 (in Russian).

[7] A.Y. Kouznetsov, Generation of dynamic Delaunay triangulation, Variational Methods for Numerical Analysis Problems, Vychisl. Tsentr SO AN SSSR, Novosibirsk, 1991, 76–83 (in Russian).